Congratulations for 70 th Birthday! KEN

I wish Happy and Active Days Ahead for You

Exact Resurgent Trans-series and Multi-Bion Contributions to All Orders

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PTEP 2015, 033B02 (2015) [arXiv:1409.3444] ; J.Phys.Conf.Ser.**597**, 012060 (2015);

[arXiv:1412.0861] JHEP 1406, 164 (2014) [arXiv:1404.7225]

Contents

1 Resurgence and Bion

1.1 Borel Sum of Divergent Series and Resurgence

Perturbation series

Partition function of ϕ^4 field theory in Euclidean d-dimension

$$Z(g^2) = \int D\phi(x)\; e^{-S_E}, \qquad S_E = \int d^d x \left(\frac{1}{2}(\partial_\mu \phi)^2 + m^2 \frac{\phi^2}{2} + g^2 \frac{\phi^4}{4} \right)$$

Perturbation series in g^2 ($m = 1$): $Z(g^2) =$ sum of Feynman diagrams

$d \longrightarrow 0$: **Number of Feynman diagrams** (with weight and sign)

$$Z(g^2) = \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}}\; e^{-S_E}, \qquad S_E = \frac{1}{2}\phi^2 + g^2 \frac{\phi^4}{4}$$

$Z(g^2)$ is well-defined for $g^2 > 0$, ($m = 1$)

Perturbation : Formal power series defined by

$$Z(g^2) = \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}}\; e^{-\frac{\phi^2}{2}} e^{-g^2 \frac{\phi^4}{4}} = \sum_{K=0}^{\infty} (g^2)^K Z_K$$

$$Z_K = \frac{1}{K!} \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}}\; e^{-\frac{\phi^2}{2}} \left(\frac{-\phi^4}{4} \right)^K$$

$$= \frac{1}{K!}\frac{(-1)^K}{\sqrt{\pi}}\Gamma\left(2K+\frac{1}{2}\right) \sim \frac{(-4)^K}{\sqrt{2}\pi}(K-1)!$$

Perturbation series is **Factorially divergent** and **Alternating**

Borel sum:

A method to make sense of the sum of **Factorially divergent series**

Factorially divergent series (Gevrey-I) is defined by (constant C, A)

$$P(g^2) = \sum_{K=0}^{\infty} a_K (g^2)^K, \quad |a_K| \le C K! \left(\frac{1}{A}\right)^K$$

Def: **Borel transform** $BP(t)$ $\longrightarrow$ finite radius of convergence

$$BP(t) = \sum_{K=0}^{\infty} \frac{a_K}{K!} t^K$$

Def: **Borel resummation** $\mathbb{P}(g^2)$

$$\mathbb{P}(g^2) = \int_0^{\infty} dt e^{-t} BP(g^2 t)$$

If this integral is well-defined, the series is called **Borel-summable**

Alternating factorially divergent series ($A > 0$)

$$P(g^2) = C \sum_{K=0}^{\infty} K! \left(\frac{-g^2}{A}\right)^K$$

Borel transform becomes

$$BP(t) = C \sum_{K=0}^{\infty} \left(\frac{-t}{A}\right)^K = \frac{CA}{A+t},$$

Borel resummation becomes

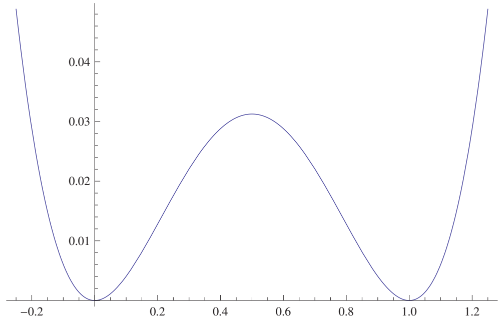
$$\mathbb{P}(g^2) = \int_0^{\infty} dt e^{-t} BP(g^2 t) = \int_0^{\infty} dt e^{-t} \frac{CA}{A + g^2 t}$$

$BP(g^2 t)$ is **Borel summable** (no singularity on the positive real axis)

1.2 Instantons and Bions

Quantum mechanics with degenerate minima

$$H = \frac{p^2}{2} + V(q), \qquad V(q) = \frac{q^2}{2}(1 - gq)^2$$



Double well potential

Path-integral representation of ground state energy

$$E(g^2) = \lim_{\beta\to\infty} \frac{-1}{\beta} \log \mathrm{tr}(e^{-\beta H}), \quad \mathrm{tr}(e^{-\beta H}) = \int Dq(t) e^{-S}$$

$$S = \int d\tau \left[\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q) \right], \quad V(q) = \frac{q^2}{2} - gq^3 + g^2\frac{q^4}{2}$$

A perturbative vacuum : $q = 0$

Expansion in powers of g : pertubation series around the $q = 0$ vacuum

$-gq^3$ is more important ($(gq^3)^2 \gg g^2q^4$ for $|q| \gg 1$)

Large order behavior of perturbation series

$$E_{\mathrm{pert}}(g^2) = \sum_{K=0}^{\infty} (g^2)^K E_K, \quad E_K \sim -\frac{3}{\pi} 3^K K!$$

Borel transform becomes

$$BE_{\rm pert}(t) \equiv \sum_{K=0}^{\infty} \frac{(t)^K}{K!} E_K \sim -\frac{3}{\pi} \sum_{K=0}^{\infty} (3t)^K = -\frac{3}{\pi} \frac{1}{1-3t},$$

Borel resummation is ill-defined for $g^2 > 0$ (**Borel non-summable**)

$$\mathbb{E}_{\rm pert}(g^2) = \int_0^\infty dt e^{-t} BE_{\rm pert}(g^2 t) = -\frac{3}{\pi} \int_0^\infty dt e^{-t} \frac{1}{1-3g^2 t}$$

a **Pole** at $t = 1/(3g^2)$ on the positive real axis of **Borel plane**

Well-defined at $g^2 < 0 \longrightarrow$ **Analytic** continuation to $g^2 > 0$ gives

$${\rm Im}\mathbb{E}_{\rm pert}(g^2) \sim \mp 3 e^{\frac{-1}{3g^2}}$$ **imaginary ambiguity** (path-dependent)

There should be **nonperturbative** contributions cancelling this ambuguity

Nonperturbative saddle points as solutions of Euclidean Action

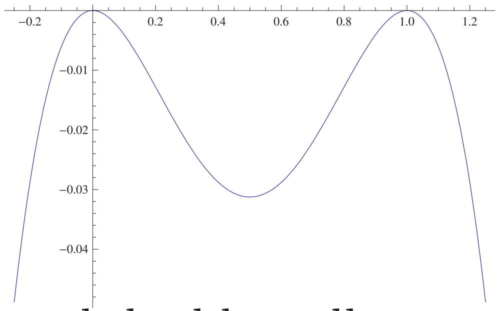
Instantons as nonperturbative saddle points $S_{\rm I} = \frac{1}{6g^2}$

Bion : A pair of Instanton and Anti-instanton (**not exact solution**)

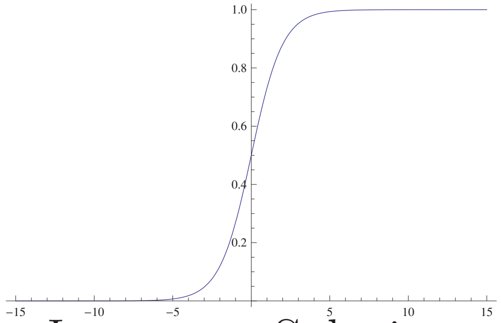
Separation is a quasi-moduli : integration over the separation is required

Analytic continuation $\longrightarrow$ (nonperturbative) **imaginary ambiguity**

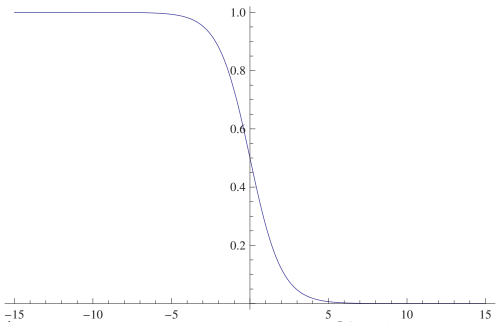
nonperturbative and **perturbative** ambiguities cancel $\longrightarrow$ **Resurgence**



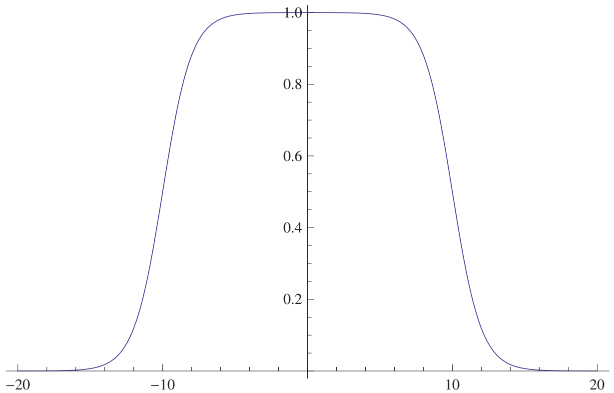
Inverted double well potential



Instanton Solution



Anti-instanton Solution



Bion

2 Exact ground-state energy of $\mathbb{C}P^1$ QM

(Lorentzian) $\mathbb{C}P^1$ QM with fermions : $\mu = m|\varphi|^2/(1+|\varphi|^2)$

$$L = \frac{1}{g^2}\left[G\Big(|\partial_t\varphi|^2 - |m\varphi|^2 + i\bar{\psi}\mathcal{D}_t\psi\Big) - \epsilon\,\frac{\partial^2\mu}{\partial\varphi\partial\bar{\varphi}}\,\psi\bar{\psi}\right]$$

$$G = \frac{\partial^2}{\partial\varphi\partial\bar{\varphi}}\log(1+\varphi\bar{\varphi}), \quad \mathcal{D}_t = \partial_t + \partial_t\varphi\,\frac{\partial}{\partial\varphi}\log G$$

SUSY for $\epsilon = 1$,

States are classified by Fermion number $F \equiv G\psi\bar{\psi} = 0, 1$

Lagangian for $F = 0$ sector (containing ground state)

$$L = \frac{|\partial_t\varphi|^2}{g^2(1+|\varphi|^2)^2} - V\,, \quad V = \frac{1}{g^2}\frac{m^2|\varphi|^2}{(1+|\varphi|^2)^2} - \epsilon m\frac{1-|\varphi|^2}{1+|\varphi|^2}$$

At $\epsilon = 1$, SUSY ground state $\Psi_0 = \langle\varphi|0\rangle = \exp(-\mu/g^2)$ is obtained

$$H_{\epsilon=1}\Psi_0 = \left[-g^2(1+|\varphi|^2)^2\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\bar{\varphi}} + V_{\epsilon=1}\right]\Psi_0 = 0$$

Expansion around SUSY: nontrivial and calculable resurgence structure

$$E = \delta\epsilon\,E^{(1)} + \delta\epsilon^2\,E^{(2)} + \cdots\,, \quad \Psi = \Psi_0 + \delta\epsilon\,\delta\Psi, \quad \delta\epsilon \equiv \epsilon - 1$$

$$E^{(1)} = \frac{\langle 0|\delta H|0\rangle}{\langle 0|0\rangle}, \quad E^{(2)} = -\frac{\langle \delta\Psi|H_{\epsilon=1}|\delta\Psi\rangle}{\langle 0|0\rangle}, \cdots$$

We obtain **exact results** as

$$E^{(1)} = g^2 - m \coth \frac{m}{g^2}$$

$$E_0^{(2)} = g^2 - \frac{m \coth \frac{m}{g^2}}{2\sinh^3 \frac{m}{g^2}} \left[E_i\left(-\frac{2m}{g^2}\right) + \bar{E}_i\left(\frac{2m}{g^2}\right) - 2\gamma - 2\log\frac{2m}{g^2} \right]$$

Exponential integral functions are defined as ($x > 0$)

$$E_i(-x) = -\int_x^\infty dt e^{-t}\frac{1}{t}, \quad \bar{E}_i(x) = -\int_{-x}^\infty dt e^{-t}\frac{\mathcal{P}}{t}$$

Power series $E^{(i)} = \sum_{p=0}^\infty E_p^{(i)}$ in e^{-2m/g^2} are convergent

Power series in g^2 is asymptotic $\longrightarrow$ **Borel resummation** gives

$$E_0^{(1)} = -m + g^2, \quad E_p^{(1)} = -2me^{-\frac{2m}{g^2}}, \ (p \geq 1)$$

$$E_0^{(2)} = g^2 + 2m \int_0^\infty dt \frac{e^{-t}}{t - \frac{2m}{g^2 \pm i0}}$$

$$
\begin{aligned}
E_p^{(2)} = &\Bigg[2m\int_0^\infty dt\, e^{-t}\Big(\frac{(p+1)^2}{t-\frac{2m}{g^2\pm i0}}+\frac{(p-1)^2}{t+\frac{2m}{g^2}}\Big)\\
&+4mp^2\left(\gamma+\log\frac{2m}{g^2}\pm\frac{\pi i}{2}\right)\Bigg]e^{-\frac{2m}{g^2}}, \quad (p\geq 1)
\end{aligned}
$$

3 Single-Bion Solutions

Energy $\boldsymbol{E}$, angular momentum $\boldsymbol{l}$ conservation

$$
E \equiv \frac{1}{g^2}\frac{\partial_\tau\varphi\partial_\tau\bar{\varphi}}{(1+\varphi\bar{\varphi})^2}-V(\varphi\bar{\varphi}), \quad l \equiv \frac{i}{g^2}\frac{\partial_\tau\varphi\bar{\varphi}-\partial_\tau\bar{\varphi}\varphi}{(1+\varphi\bar{\varphi})^2}
$$

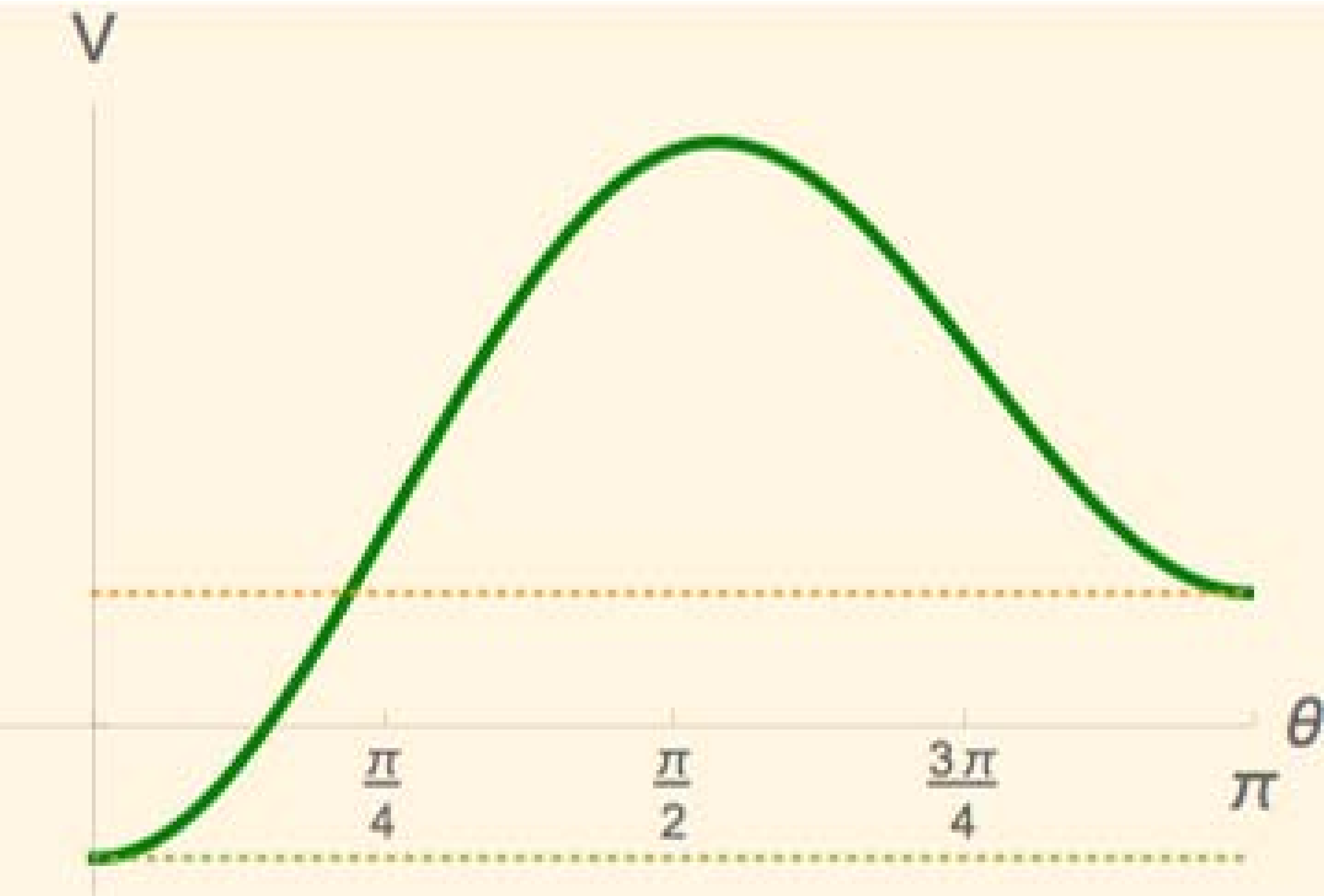
Finite action $\longrightarrow$ boundary condition at $\tau\to\pm\infty$

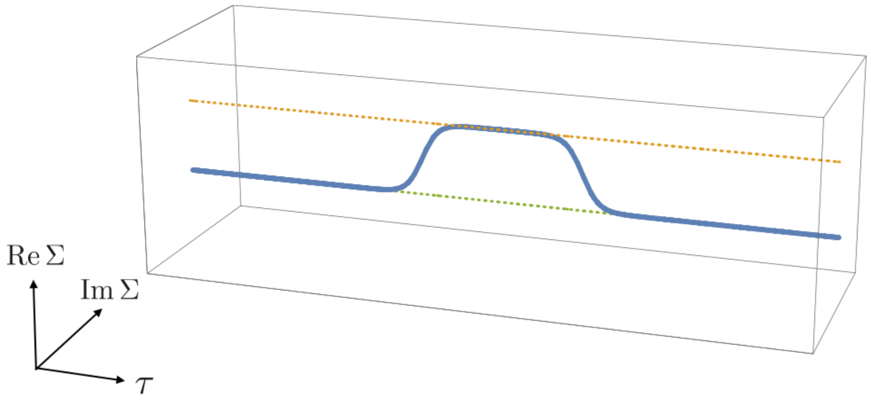
$$
\lim_{\tau\to\pm\infty}\varphi=\lim_{\tau\to\pm\infty}\bar{\varphi}=0 \;\longrightarrow\; l=0,\; E=E|_{\varphi=0}=\epsilon m
$$

Exact single **Bion** solution

$$
\varphi = e^{i\phi_0}\sqrt{\frac{\omega^2}{\omega^2-m^2}}\frac{1}{i\sinh\omega(\tau-\tau_0)}, \quad \omega\equiv m\sqrt{1+\frac{2\epsilon g^2}{m}},
$$

$$
\varphi^{-1} = e^{\omega(\tau-\tau_+)-i\phi_+}+e^{-\omega(\tau-\tau_-)-i\phi_-}
$$


V
θ
π/4
π/2
3π/4
π



Kink profiles for $\Sigma(\tau) = \frac{m\varphi\tilde{\varphi}}{1+\varphi\tilde{\varphi}}$ for the single bion

$$\tau_\pm = \tau_0 \pm \frac{1}{2\omega} \log \frac{4\omega^2}{\omega^2 - m^2}, \qquad \phi_\pm = \phi_0 \mp \frac{\pi}{2}$$

2 real moduli parameters : τ_0: translational moduli, ϕ_0: $U(1)$ moduli

Value of action S for the single bion solution

$$S = \frac{2\omega}{g^2} + 2\epsilon \log \frac{\omega + m}{\omega - m}$$

Real bion gives a nonperturbative correction to ground state energy

A.Behtash, G.V.Dunne, T.Schafer, T.Sulejmanpasic and M.Unsal, Phys.Rev.Lett.**116**, 011601 (2016); arXiv:1510.03435 [hep-th]; E.Witten, [arXiv:1001.2933 [hep-th]] $\cdots$

T.Fujimori, S.Kamata, T.Misumi, M.Nitta and N.Sakai, Phys.Rev.**D94**, 105002 (2016); Phys.Rev.**D95**, 105001 (2017)

4 Multi-Bion Solutions

Complexified theory: $\varphi \equiv \varphi_R^{\mathbb{C}} + i\varphi_I^{\mathbb{C}}$ and $\tilde{\varphi} \equiv \varphi_R^{\mathbb{C}} - i\varphi_I^{\mathbb{C}}$ are independent

$$S_E = \int_0^\beta d\tau \left[\frac{\partial_\tau \varphi \partial_\tau \tilde{\varphi}}{g^2(1+\varphi\tilde{\varphi})^2} + V(\varphi\tilde{\varphi})\right]$$

Contributions from Saddle points in finite interval: $\varphi(\tau+\beta) = \varphi(\tau)$

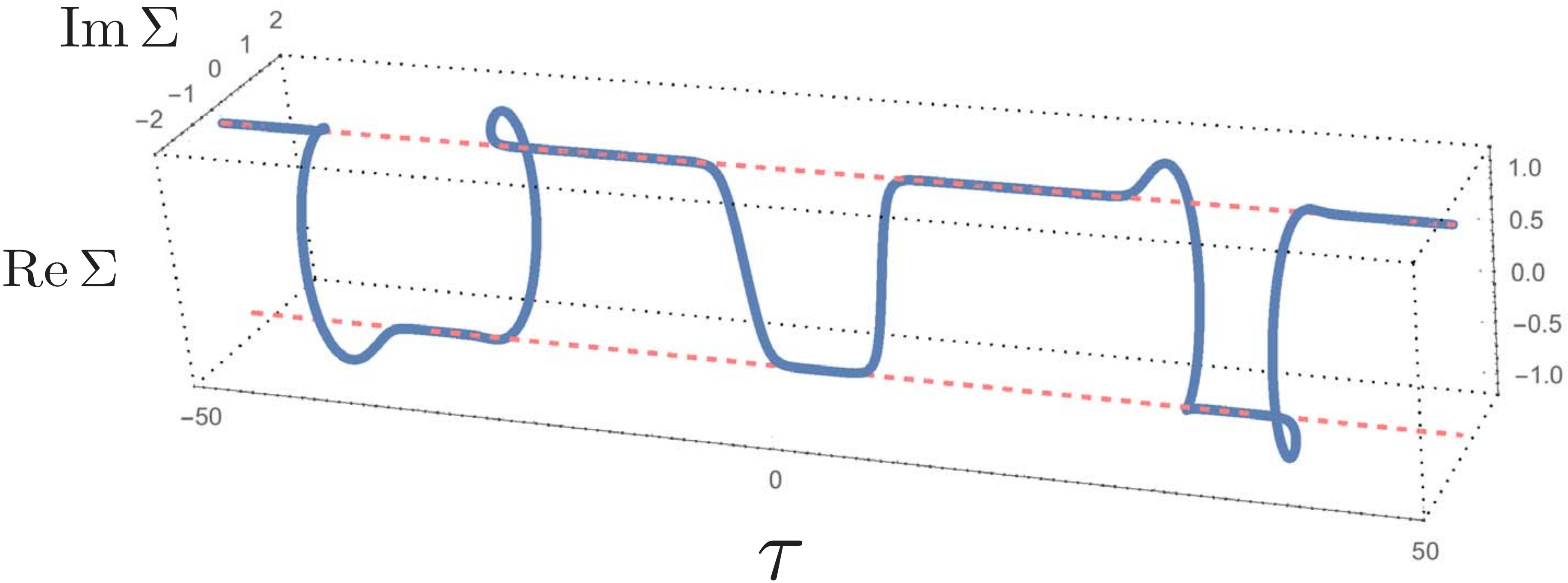
$$Z(\beta) = \int \mathcal{D}\varphi \, \exp(-S_E[\varphi]) = \sum_{\sigma\in\mathfrak{S}} e^{-S_\sigma}\left[(\det\Delta_\sigma)^{-\frac{1}{2}} + \mathcal{O}(g)\right]$$

Complexified symmetry ($a, b \in \mathbb{C}$) $\longrightarrow$ Conserved charges

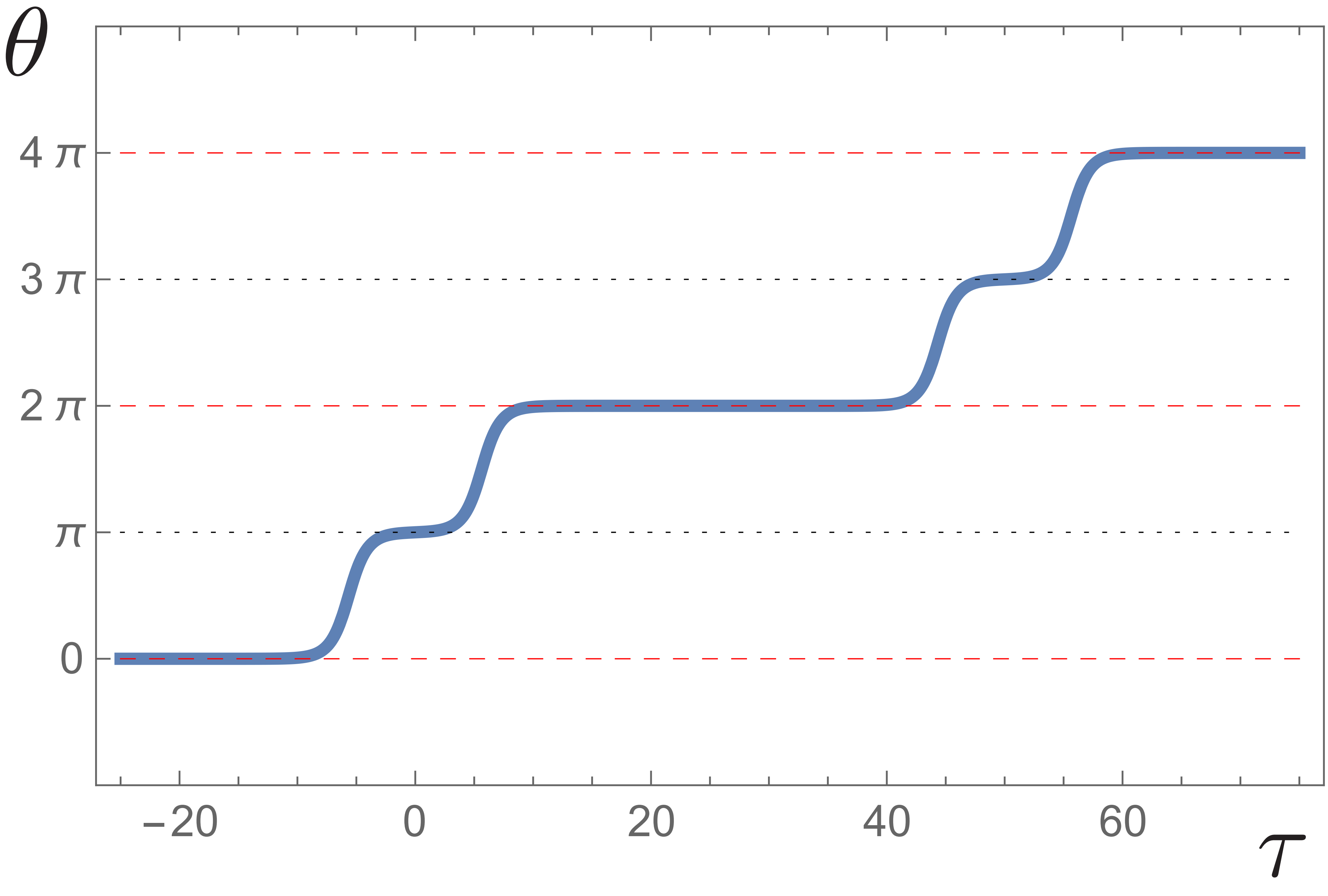
Time translation $\tau \longrightarrow \tau + a$, Phase rotation $(\varphi, \tilde{\varphi}) \longrightarrow (e^{ib}\varphi, e^{-ib}\tilde{\varphi})$

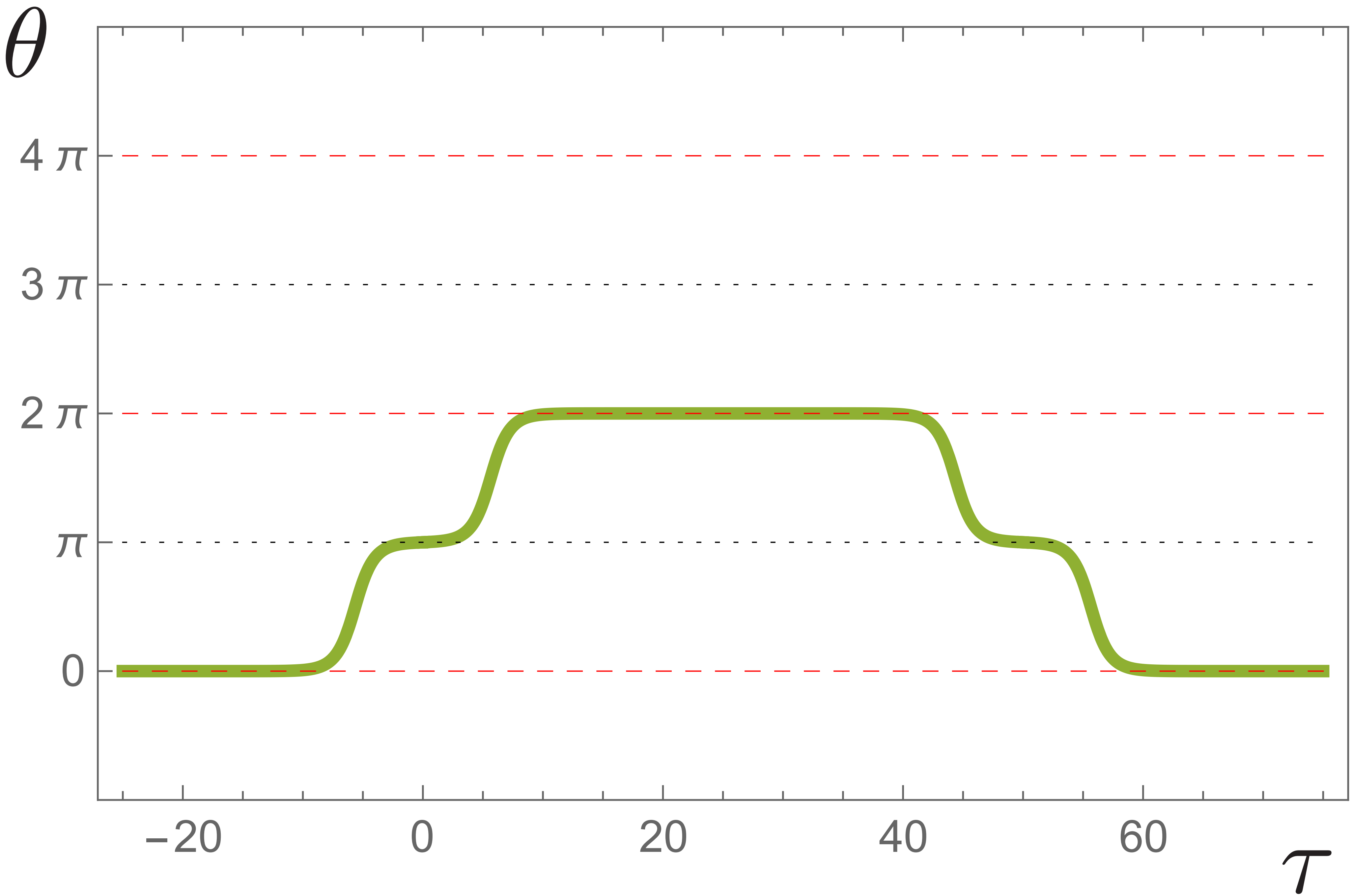
Solutions are given by **elliptic function** cs with complex moduli (τ_c, ϕ_c)

$$\varphi = e^{i\phi_c}\frac{f(\tau-\tau_c)}{\sin\alpha}, \quad \tilde{\varphi} = e^{-i\phi_c}\frac{f(\tau-\tau_c)}{\sin\alpha}$$



Im Σ
2
1
0
−1
−2
Re Σ
1.0
0.5
0.0
−0.5
−1.0
−50
0
50
τ


θ
4 π
3 π
2 π
π
0
−20
0
20
40
60
τ


θ
4 π
3 π
2 π
π
0
−20
0
20
40
60
τ

$$f(\tau) = \mathrm{cs}(\Omega\tau, k) \equiv \mathrm{cn}(\Omega\tau, k)/\mathrm{sn}(\Omega\tau, k)$$

cs has periods $2K(k)$ and $4iK'$ ($K' \equiv K(\sqrt{1-k^2})$) and satisfies

$$(\partial_\tau f)^2 = \Omega^2(f^2+1)(f^2+1-k^2)$$

The solutions are characterized by two integers (p, q) for the period

$$\beta = \frac{(2pK + 4iqK')}{\Omega}$$

(α, Ω, k) are given in terms of β, and asymptotic forms for large β are

$$k \approx 1 - 8\, e^{-\frac{\omega\beta - 2\pi i q}{p}}, \quad \Omega \approx \omega\left(1 + 8\frac{\omega^2+m^2}{\omega^2-m^2}e^{-\frac{\omega\beta-2\pi i q}{p}}\right)$$

$$\cos\alpha \approx \frac{m}{\omega}\left(1 - \frac{8m^2}{\omega^2-m^2}e^{-\frac{\omega\beta-2\pi i q}{p}}\right), \quad 0 \le q < p$$

$$S \approx pS_{\mathrm{bion}} + 2\pi i \epsilon l, \quad S_{\mathrm{bion}} = \frac{2m}{g^2} + 2\epsilon \log\frac{\omega+m}{\omega-m}$$

Position of n-th instanton and antiinstaton

$$\tau_n^{\pm} = \tau_c + \frac{n-1}{\omega p}(\omega\beta - 2\pi i q) \pm \frac{1}{2\omega}\log\frac{4\omega^2}{\omega^2-m^2}$$

5 One-Loop Determinant and Lefschetz Thimble

For $\mathbf{0} \leq \boldsymbol{\epsilon} \leq \mathbf{1}$, instanton-antiinstanton **separation** becomes **large** :

we have normalizable **quasi-moduli** (almost flat direction)

One-Loop Determinant for non-zero modes $\mathbf{det''\Delta}$

$\approx$ product of determinant of constituent (anti-)instantons

Relative position $\boldsymbol{\tau_r}$ and relative phase $\boldsymbol{\phi_r}$

$$Z_{\text{bion}} \approx \int d\tau_0 d\phi_0 \int d\tau_r d\phi_r \, \mathrm{det}''\Delta \exp\left(-V_{\text{eff}}\right)$$

Deform $\boldsymbol{\tau_r}, \boldsymbol{\phi_r}$ in complex plane

Determine integration paths (**thimbles**) and their weight

(by intersection of **dual thimbles** with the original path)

Gradient Flow and Lefschetz Thimble

Prototype of Quasi-Moduli integral

$$I = \int_{\mathcal{C}} dy \exp\left[-V(y)\right], \quad V(y) \equiv a e^{-y} + b y, \quad \mathrm{Re}\, b > 0$$

Instanton-instanton : $\boldsymbol{a > 0}$, Instanton-Antiinstanton : $\boldsymbol{a < 0}$

Gradient flow equation

$$\frac{\partial y}{\partial t} = \overline{\frac{\partial V}{\partial y}} = -\bar{a}e^{-\bar{y}} + \bar{b}$$

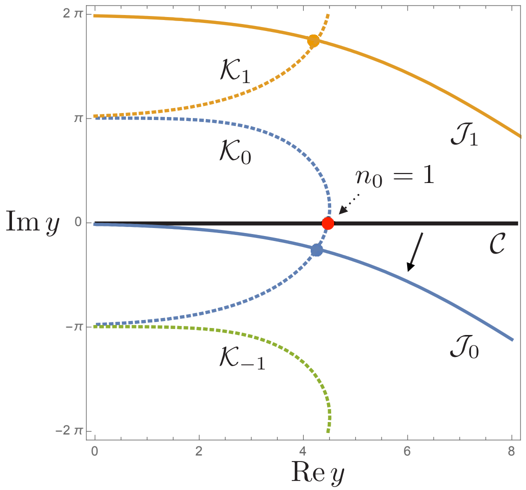
$\partial y/\partial t = 0$: Saddle point y_s

Thimble $y(t)$ (steepest descent contour): $\lim_{t\to-\infty} y(t) = y_s$

Dual Thimble $y(t)$ (deformable direction): $\lim_{t\to+\infty} y(t) = y_s$

If the dual thimble intersects with the original contour

$\longrightarrow$ integration contour can be deformed to the thimble



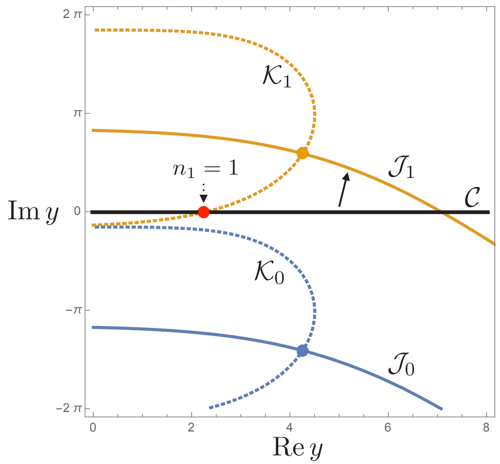
The Lefschetz thimles $\mathcal{J}_q$ and their duals $\mathcal{K}_q$. No Stokes phenomenon at $\arg a = 0$.

$\boldsymbol{a > 0}$ case :

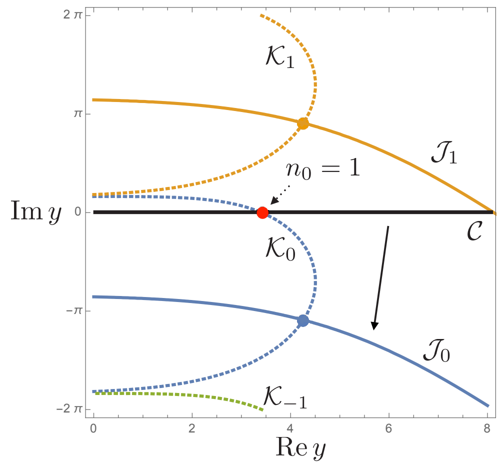
$$\boldsymbol{I \;=\; \int_{\mathcal{J}_0} dy\, \exp\left[-V(y)\right] \;=\; a^{-b}\Gamma(b)}$$

$\boldsymbol{a < 0}$ case requires $\boldsymbol{\theta \equiv -\pi - \arg a = \pm 0 \neq 0}$ (Stokes phenomenon)

$$\boldsymbol{I \;=\; \left\{ \begin{array}{l} \int_{\mathcal{J}_1} dy\, \exp\left[-V(y)\right] \\ \int_{\mathcal{J}_0} dy\, \exp\left[-V(y)\right] \end{array} \right. \;=\; |a|^{-b} \exp(\mp \pi i b)\, \Gamma(b)}$$



$\boldsymbol{\theta > 0}$



$\boldsymbol{\theta < 0}$

Stokes phenomenon at $\boldsymbol{\arg a = -\pi}$ ($\boldsymbol{\theta = -\pi - \arg a}$). The original integration contour $\boldsymbol{\mathcal{C}}$ intersects with $\boldsymbol{\mathcal{K}_1}$ ($\boldsymbol{\mathcal{K}_0}$) for $\boldsymbol{\theta > 0}$ ($\boldsymbol{\theta < 0}$) and hence $\boldsymbol{\mathcal{C}}$ is deformed to $\boldsymbol{\mathcal{J}_1}$ ($\boldsymbol{\mathcal{J}_0}$).

6 Multi-Bion contributions

Effective potential for well-separated kinks

$$S_E \;\rightarrow\; V_{\text{eff}} = -m\epsilon\beta + \sum_{i=1}^{2p}\left(\frac{m}{g^2} + V_i\right)$$

$$\frac{V_i}{m} = \epsilon_i(\tau_i - \tau_{i-1}) - \frac{4}{g^2}e^{-m(\tau_i-\tau_{i-1})}\cos(\phi_i - \phi_{i-1})$$

$\tau_{2n-1} = \tau_i^-$, $\tau_{2n} = \tau_i^+$, $\tau_0 = \tau_{2p} - \beta$, $\phi_0 = \phi_{2p}\ (\mathrm{mod}\ 2\pi)$,

$\epsilon_{2n-1} = 0$ and $\epsilon_{2n} = 2\epsilon$

For $0 \leq \epsilon \leq 1$, large separation of instanton and anti-instanton $\rightarrow$

$\det{}''\Delta \approx$ product of determinant of constituent (anti-)instantons

Complexify τ_r, ϕ_r and determine integration paths (**thimbles**) and their weight (by intersection of **dual thimbles** with the original path)

Lagrange multiplier σ to impose periodicity

$$2\pi\delta\left(\sum_i \tau_i - \beta\right) = m\int_{-\infty}^{\infty} d\sigma \exp\left[im\sigma\left(\sum_i \tau_i - \beta\right)\right]$$

$$\frac{Z_p}{Z_0} = \frac{1}{p} \int \prod_{i=1}^{2p} \left[d\tau_i \wedge d\phi_i \, \frac{2m^2}{\pi g^2} \exp\left(-\frac{m}{g^2} - V_i \right) \right]$$

$$E \; = \; E_0 - \lim_{\beta \to \infty} \frac{1}{\beta} \log \left(1 + \sum_{p=1}^{\infty} \frac{Z_p}{Z_0} \right)$$

$$E_p^{(1)} = -e^{\frac{2pm}{g^2}} \lim_{\epsilon \to 1} \lim_{\beta \to \infty} \frac{1}{\beta} \frac{\partial}{\partial \epsilon} \frac{Z_p}{Z_0} = -2m$$

$$E_p^{(2)} = -\frac{e^{\frac{2pm}{g^2}}}{2} \lim_{\epsilon \to 1} \lim_{\beta \to \infty} \frac{1}{\beta} \left[\partial_\epsilon^2 \frac{Z_p}{Z_0} - \sum_{i=1}^{p-1} \partial_\epsilon \frac{Z_{p-i}}{Z_0} \partial_\epsilon \frac{Z_i}{Z_0} \right]$$

$$= 4mp^2 \left(\gamma + \log \frac{2m}{g^2} \pm \frac{\pi i}{2} \right)$$

7 Conclusions

1. **Factorially divergent** perturbation series can be summed by using **Borel resummation**. In some cases, imaginary ambiguities arise from the Borel resummed perturbative contributions, which are cancelled by nonperturbative contributions, leading to **resurgence** : intimate relation between perturbative and nonperturbative contributions.

2. We obtained exact results for near SUSY **$\mathbb{C}\boldsymbol{P}^1$ quantum mechanics** revealing resurgence to infinitely many powers of nonperturbative exponentials.

3. We have found an infinite tower of **exact multi-bion solutions** for **finite time interval** in the complexified theory with fermions.

4. Semi-classical contributions of arbitrary numbers of bions give **nonperturbative contributions** in $\mathbb{C}\boldsymbol{P}^1$ quantum mechanics **exactly**.

5. By using dispersion relations (resurgence), we can recover the exact results completely from bion amplitudes in the case of near SUSY $\mathbb{C}\boldsymbol{P}^1$ quantum mechanics.

6. We have explicitly the evaluated the **quasi-moduli** integral and the **1-loop determinant** for multi-bion saddle points.

7. The integration path and weight for the quasi-moduli are determined by computing the **Lefschetz thimbles** and **dual thimbles**.

8. Our results can be generalized to other cases such as sine-Gordon quantum mechanics, $\mathbb{C}\boldsymbol{P}^{\boldsymbol{N-1}}$ quantum mechanics, and more general nonlinear target spaces, such as squashed $\mathbb{C}\boldsymbol{P}^{\boldsymbol{1}}$.

9. Near SUSY situation can be generalized to quasi-exactly-solvable (QES) cases, such as particular excited states of the sine-Gordon quantum mechanics.

10. Extending our analysis to quantum field theories such as 2d $\mathbb{C}\boldsymbol{P}^{\boldsymbol{N-1}}$ nonlineaer sigma models are interesting. Hopefully it will eventually lead to the understanding of nonperturbative effects in asymptotically free gauge theories in 4 dimensions.