Non-perturbative QCD inputs for axion phenomenology: status and perspectives

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Axion Theory and Lattice QCD: Advances and Challenges
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Based on a review article in preparation with C. Bonati and M. D'Elia (Pisa)

Gluon topology and the QCD θ parameter

Quantum Chromo-Dynamics (QCD) possesses non-trivial topological features.
Non-trivial topology of $\mathrm{SU}(N)$ gauge group $\rightarrow$ integer-valued **topological charge**

$$S_{\mathrm{QCD}} = -\frac{1}{2g^2}\int \mathrm{d}^4x\, \mathrm{Tr}\left[G_{\mu\nu}(x)G^{\mu\nu}(x)\right] + \int \mathrm{d}^4x \sum_{\mathrm{f}=1}^{N_{\mathrm{f}}} \overline{\psi}_{\mathrm{f}}(x)(\mathrm{i}\not{D} - m_{\mathrm{f}})\psi_{\mathrm{f}}(x)$$

$$Q = \frac{1}{32\pi^2}\int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\, \mathrm{Tr}\left[G_{\mu\nu}(x)G_{\rho\sigma}(x)\right] \in \mathbb{Z}$$

$$S_{\mathrm{QCD}} \qquad \longrightarrow \qquad S_{\mathrm{QCD}}(\theta) = S_{\mathrm{QCD}} + \theta Q$$

Via the topological charge, one can introduce a new coupling in QCD: the θ parameter.

Non-trivial θ-dependence of physical observables $\rightarrow$ theoretical feature of QCD with fundamental phenomenological implications.

The strong CP problem

One of the most important open problems of the Standard Model is a consequence of QCD topology: the **strong CP problem**

Q odd under charge conjugation (C) + parity (P):

$\theta \neq 0 \implies$ **strong CP violations**

This should lead to a non-vanishing **Neutron Electric Dipole Moment**:

$$d_{\mathrm{N}} \approx \theta\, e\, \frac{m_\pi^2}{m_{\mathrm{n}}^3} \approx 4 \cdot 10^{-3}\, \theta\, e\, \mathrm{fm}$$

(e = elementary electric charge)

However, experimental measures found no signal above zero:

$$d_{\mathrm{N}}^{(\mathrm{exp})} = (0.0 \pm 1.2) \cdot 10^{-13}\, e\, \mathrm{fm} \qquad \implies \qquad |\theta|_{\mathrm{exp}} \lesssim 10^{-9} - 10^{-10}$$

PSI experiment — PRL 124 (2020) 8, 081803 [2001.11966]

QCD θ-dependence and the axion

Absence of dynamical mechanism to set $\theta = 0$ in the Standard Model
$\Longrightarrow$ Beyond Standard Model solution.

The axion is one of the most appealing. Pseudo-Goldstone of a new axial U(1) symmetry, spontaneously broken below a very high scale f_a.

The dynamics of this particle relaxes θ to zero due to its peculiar coupling to gluons via the topological charge density:

$$\mathcal{L}_a + \mathcal{L}_{\text{QCD}} \supset \left(\frac{a(x)}{f_a} + \theta\right) q(x) \qquad q(x) = \frac{1}{32\pi^2}\varepsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[G_{\mu\nu}(x)G_{\rho\sigma}(x)]$$

- Axion shift symmetry sets $\theta = 0$: $\quad a \to a + \theta f_a \implies \theta \to 0$
- Axion couplings to Standard Model suppressed by $f_a \implies$ **Dark Matter candidate**

Ongoing experimental effort towards axion detection needs to be matched by accurate theoretical predictions $\longrightarrow$ inputs from QCD topology needed for this task

Once QCD is integrated out:

$$\mathcal{V}_{\mathrm{eff}}(T,a) = f\,(T, \theta = a/f_a)$$

Shape of axion effective potential $\to$ θ-dependence of QCD free energy

In terms of the QCD Euclidean functional integral:

$$f(T,\theta) \equiv F(T,\theta) - F(T,0) = -\frac{1}{V}\log\frac{Z(T,\theta)}{Z(T,0)}$$

$$Z(T,\theta) = \int [\mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi]\exp\left\{-\int_0^{\frac{1}{T}}\mathrm{d}\tau\int\mathrm{d}^3x\,\mathcal{L}_{\mathrm{QCD}}\right\}\exp\{\mathrm{i}\theta Q\}$$

$f(T,\theta)$ determines the cosmological evolution of the axion field
$\to$ axion abundance from *misalignement* requires knowledge of $f(T,\theta)$

$$\ddot{a} + 3H\dot{a} + \mathcal{V}'_{\mathrm{eff}} = 0 \qquad\qquad a_0 = \theta_0 f_a$$

Taylor expansion about $\theta = 0$, where CP symmetry implies $f(T, \theta) = f(T, -\theta)$:

$$f(T, \theta) = \frac{1}{2}\chi(T)\theta^2 \left(1 + b_2(T)\theta^2 + b_4(T)\theta^4 + \dots\right)$$

$$\mathcal{V}_{\rm eff}(T, a) = \frac{1}{2} m_a^2(T) a^2 \left(1 + \frac{2}{m_a^2}\frac{\lambda_a(T)}{4!} a^2 + \dots\right)$$

Coefficients related to **cumulants** of $\theta = 0$ topological charge distribution:

$$\chi = \left.\frac{\langle Q^2 \rangle}{V}\right|_{\theta=0} = \frac{m_a^2}{f_a^2}$$

(**Topological Susceptibility**)

$$b_2 = -\frac{1}{12}\left.\frac{\langle Q^4 \rangle - 3\langle Q^2 \rangle^2}{\langle Q^2 \rangle}\right|_{\theta=0} = \frac{\lambda_a}{12}\frac{f_a^2}{m_a^2}$$

(Quartic Coefficient)

- Below chiral crossover $T_c \simeq 155$ MeV $\rightarrow$ Chiral Perturbation Theory (χPT)

- For asymptotically high temperatures $T \gg \Lambda_{\rm QCD} \rightarrow$ semiclassical methods

QUESTIONS: what happens around and above T_c?
In what temperature regimes are these analytic predictions reliable?

- On general grounds $\rightarrow$ Lattice QCD is a natural framework to determine $f(T,\theta)$ from first-principle and in a fully non-perturbative fashion

STATUS: in practice, determining $f(T,\theta)$ is a serious computational challenge.
In the high-temperature phase it is still an **open problem**.
Ongoing work to progress this research field.

REST OF THE TALK: review of available analytic/numerical results, discussion of future directions

θ-dependence in Chiral Perturbation Theory

Effective theory in terms of pseudo Nambu–Goldstone bosons of

$$\mathrm{SU}(N_\mathrm{f})_\mathrm{L} \times \mathrm{SU}(N_\mathrm{f})_\mathrm{R} \to \mathrm{SU}(N_\mathrm{f})_\mathrm{V}$$

At leading order $\mathcal{O}(p^2)$ and $\mathcal{O}(m)$:

$$\mathcal{L}_\mathrm{eff}(\theta) = \frac{F_\pi^2}{4}\partial_\mu U \partial^\mu U^\dagger + \frac{\Sigma}{4}\Re\left\{U\mathcal{M}(\theta)^\dagger\right\}, \qquad U = \mathrm{e}^{\mathrm{i}\pi/F_\pi}, \qquad \Sigma = -\lim_{m\to 0}\langle\overline{\psi}\psi\rangle$$

By virtue of the chiral anomaly: trade θ-term for **phase** in quark mass matrix:

$\psi \to \mathrm{e}^{\mathrm{i}\gamma_5\alpha}\psi$	$\mathrm{U}(1)_\mathrm{A}$ rotation
$\overline{\psi}_\mathrm{L}\mathcal{M}\psi_\mathrm{R} \to \mathrm{e}^{2\mathrm{i}\alpha}\overline{\psi}_\mathrm{L}\mathcal{M}\psi_\mathrm{R}$	non invariance of mass term
$\theta \to \theta - 2\alpha N_\mathrm{f}$	chiral anomaly

Choosing $2\alpha = \theta/N_\mathrm{f}$, θ fully moved into $\mathcal{M} = \mathrm{diag}(m_\mathrm{f}) \to \mathcal{M}(\theta) = \mathrm{e}^{\mathrm{i}\theta/N_\mathrm{f}}\mathcal{M}$

If $m_u = 0$, one could rotate θ away: ruled out by Lattice QCD (more in L. Di Luzio talk)

FLAG24 PRD 113 (2026) 014508 — Alexandrou et al. PRL 125 (2020) 232001

θ-dependence in Chiral Perturbation Theory

At LO and for $N_{\mathrm{f}} = 2$, $E_{\mathrm{G}}(\theta) \equiv f(T=0,\theta)$ is just a function of F_π, m_π^2 and $z = m_u/m_d$

(Di Vecchia and Veneziano 1980 – Leutwyler and Smilga 1992 – Grilli di Cortona et al. 2016)

$$\frac{E_{\mathrm{G}}(\theta)}{F_\pi^2 m_\pi^2} = 1 - \sqrt{1 - \frac{4z}{1+z}\sin^2\left(\frac{\theta}{2}\right)} \qquad \chi = \frac{z}{(1+z)^2} m_\pi^2 F_\pi^2 \qquad b_2 = -\frac{1}{12}\frac{1+z^3}{(1+z)^3}$$

Also NLO corrections are known (Grilli di Cortona et al. JHEP 01 (2016) 034)

$\chi = [77.8(4)\ \mathrm{MeV}]^4 \quad b_2 = -0.029(2) \quad b_4 = 17(6)\times 10^{-5}$ (NLO, iso-symmetric QCD)

$\chi = [75.5(5)\ \mathrm{MeV}]^4 \quad b_2 = -0.022(1) \quad b_4 = -28(7)\times 10^{-5}$ (NLO + isospin breaking)

And also finite-T corrections (Grilli di Cortona et al. JHEP 01 (2016) 034)

$$\chi(T)/\chi(0) = 1 - \mathcal{O}(T^2) \qquad b_2(T)/b_2(0) = 1 - \mathcal{O}(T^2)$$

χPT predicts that θ-dependence is suppressed rather slowly toward $T \to T_c$

E.g., $\chi(0.9\,T_c)/\chi(0) \sim 0.65$

θ-dep. via Dilute Instanton Gas Approximation (DIGA)

When $T \gg \Lambda_{\rm QCD}$, one can assume $Z(\theta)$ to be dominated by dilute gas of weakly-interacting topological quasi-particle excitations by virtue of asymptotic freedom

Finite T is crucial: cut-off on instanton size $\to \rho \ll 1/T$

DIGA = semi-classical approximation + perturbation around 1 instanton

(Gross, Pisarski, Yaffe, 1981; Schäfer & Shuryak, 1998; Boccaletti & Nogradi, 2020)

$$f(T,\theta) = \frac{2Z_1(T)}{V}(1-\cos\theta) = \chi(T)(1-\cos\theta)$$ (Diluteness assumption)

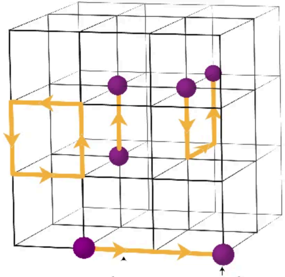
$$\implies b_2 = -\frac{1}{12} \simeq -0.0833 \sim 3.7 b_2(T=0)|_{\chi\mathrm{PT}} \qquad b_4 = \frac{1}{360} \simeq 2.78 \times 10^{-3}$$

$$\chi(T) \sim T^{4-\beta_0}\left(\frac{m}{T}\right)^{N_{\rm f}} \qquad \beta_0 = \frac{11}{3}N - \frac{2}{3}N_{\rm f}$$ (Perturbation around 1 instanton)

For 2+1 quark flavors, extremely strong T-suppression: $\chi(T) \sim T^{-8}$.
Also stronger chiral suppression compared to χPT behavior: $\chi \sim m^2$ VS $\chi \sim m$

Lattice QCD approach

$$\langle \mathcal{O} \rangle = \int [\mathcal{D}U] \, \frac{1}{Z_{\mathrm{Lat}}} e^{-S_{\mathrm{YM}}[U]} \prod_{\mathrm{f}=1}^{N_{\mathrm{f}}} \det \{ D[U] + m_{\mathrm{f}} \} \, \mathcal{O}[U] \quad \to \quad \int [\mathcal{D}U] \, P[U] \, \mathcal{O}[U]$$



- QCD formulated in **Euclidean time** $\tau = it$.
- QCD path integral regularized on a finite space-time grid.
- Gluon fields $\longrightarrow$ parallel transports $U_\mu(x) \sim e^{iaA_\mu(x)}$.
- Quark fields $\longrightarrow$ integrated away (Dirac determinant).
- Lattice time extent $aN_\tau = 1/T$ (T = temperature).
- $P[U] \sim$ probability distribution $\longrightarrow$ Monte Carlo methods.
- $\langle \mathcal{O} \rangle$ computed numerically via sample average.

Challenges

- Large lattice artifacts in the presence of a non-chiral discretized Dirac operator
- *Topological Freezing* approaching the continuum limit $a \to 0$
- Topological charge sampling in the presence of a suppressed top. susceptibility

Index theorem: $Q = n_0^{(L)} - n_0^{(R)}$ $\Longrightarrow$ Topology $\leftrightarrow$ chiral zero Dirac eigenvalues

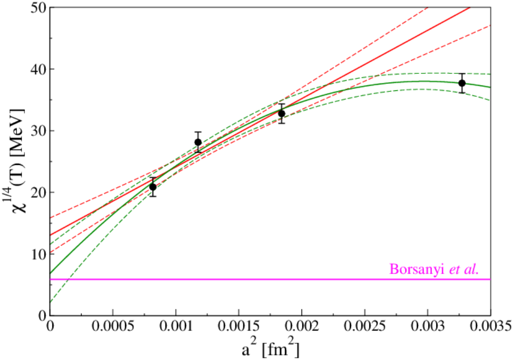
$\det\{\not{D} + m\} = \prod_{\lambda \geq 0} (\lambda^2 + m^2) \propto m^\alpha$ ensures chiral suppression $\chi \to 0$ as $m \to 0$

On the lattice, commonly employed non-chiral discretizations $\to$ no exact zero mode

Lowest modes shifted: $m \to m + \lambda_0$ $\quad$ ($\lambda_0 \to 0$ as $a \to 0$)

On coarse lattices one finds $\lambda_0 \gtrsim m \to$ spoils chiral suppression

$\Longrightarrow$ lattice χ much larger than in the continuum $\Longrightarrow$ large cut-off effects



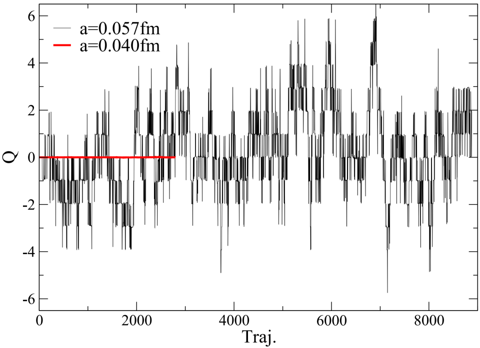
χ decreases by orders of magnitudes when approaching the continuum limit $a \to 0$

Fig. from Bonati et al. JHEP 11 (2018) 170

Reaching sufficiently fine lattice spacings → computationally rapidly unfeasible

Topological Freezing: as $a \to 0$, the simulation remains "stuck" in a corner of phase space with fixed topological charge Q for a very long Monte Carlo time

Allés et al. PLB 389 (1996) 107 — Del Debbio et al. PLB 594 (2004) 315 — ALPHA NPB 845 (2011) 93



Monte Carlo sample = Markov chain sequence $U_0 \to U_1 \to U_2 \to \ldots$

Each U_i obtained from previous one via stochastic update

Update ineffective in changing Q close to the continuum, where $Q \in \mathbb{Z}$

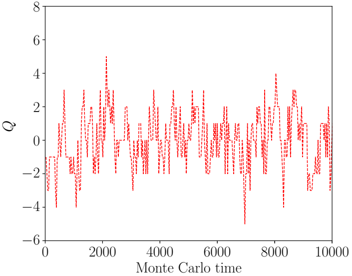
At finite T, magnitude of lattice artifact determined by $aT = 1/N_\tau \implies$ keeping $N_\tau \gtrsim \mathcal{O}(10)$ as T is increased requires **ultra fine lattice spacings**

E.g., $T \simeq 1$ GeV: $N_\tau = 6 \implies a \simeq 0.033$ fm, $N_\tau = 16 \implies a \simeq 0.012$ fm

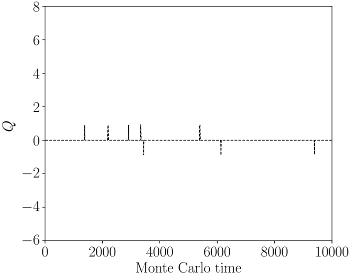
Due to the strong temperature-suppression of χ $\quad\rightarrow\quad$ $\langle Q^2 \rangle = \chi V \ll 1$

CONSEQUENCE: $P(Q = 0) \gg P(|Q| = 1) \gg P(|Q| = 2) \gg \ldots$

Sampling Q is hard: very narrow probability distribution dominated by $Q = 0$
$\rightarrow$ very rare topological jumps



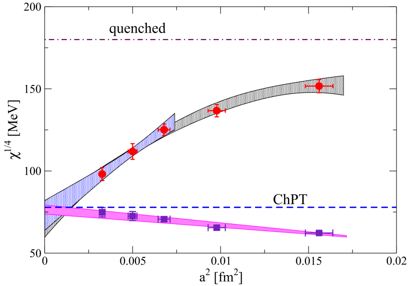
(a) 48^4 lattice, $T \simeq 0$, $a \simeq 0.0572$ fm

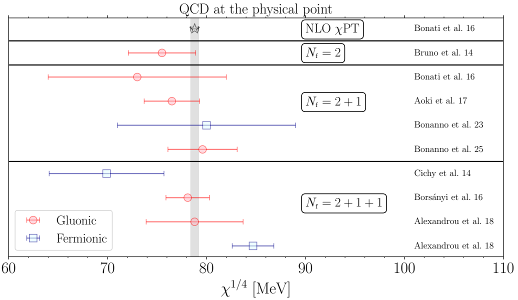


(b) $48^3 \times 8$ lattice, $T \simeq 430$ MeV, $a \simeq 0.0572$ fm

This sampling problem adds up to topological freezing, making the calculation of χ even harder than the $T = 0$ case.

Topological susceptibility at $T = 0$



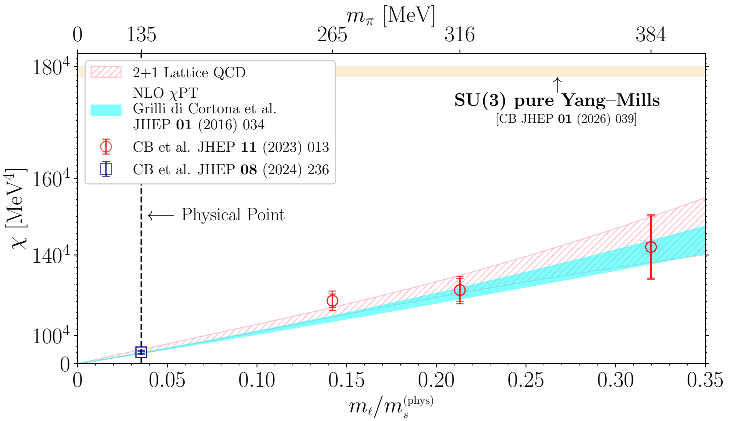


Bonati et al. JHEP 03 (2016) 155 (left figure)

Cut-off effects are large in lattice spacing range that can be explored without freezing with standard algorithms.

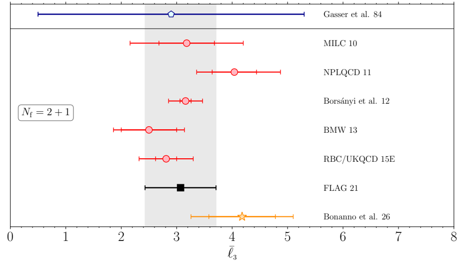
However, continuum extrapolation at $T = 0$ at physical point is under control: very good agreement among various determinations, and with NLO χPT.

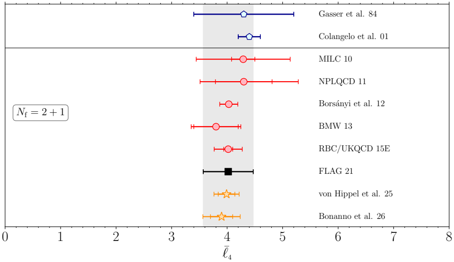
Topological susceptibility at $T = 0$

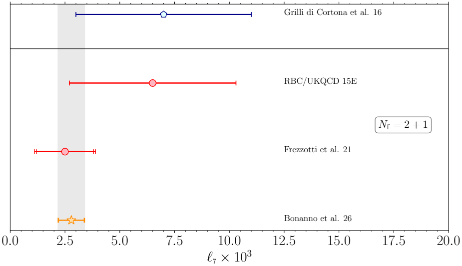


Chiral scaling agrees with χPT. Important cross-check of lattice artifacts.
Also important to determine NLO low-energy constants.

$$\chi = \frac{z}{(1+z)^2} m_\pi^2 F_\pi^2 \left\{ 1 + 2\frac{m_\pi^2}{F_\pi^2}\left[\frac{\bar{h}_1 - \bar{\ell}_4}{16\pi^2} - \bar{h}_3 + \frac{1-6z+z^2}{(1+z)^2}\,\bar{\ell}_7\right]\right\} \qquad \text{(NLO } \chi\text{PT)}$$







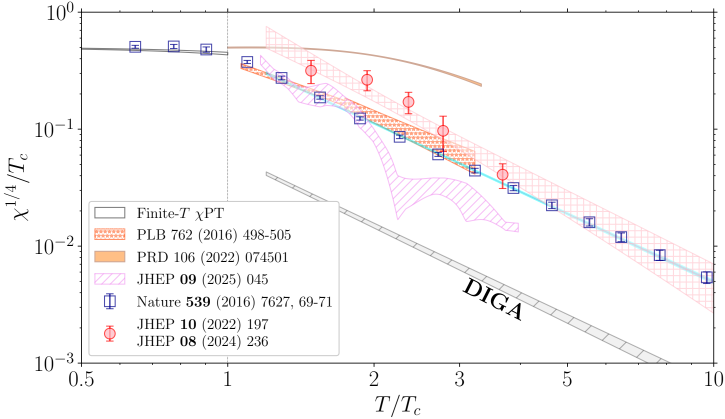
$\chi(m_\pi)$ at NLO parameterized by $(z=1)$

$$\ell \equiv h_1^{\mathrm{r}} - h_3^{\mathrm{r}} - \ell_4^{\mathrm{r}} - \ell_7^{\mathrm{r}} = \frac{\bar{h}_1 - \bar{\ell}_4}{16\pi^2} - \bar{h}_3 - \bar{\ell}_7$$

Recently: $\bar{\ell}_7$ from pion mass splitting
New data for $\bar{\ell}_3$ and $\bar{\ell}_4$ too

CB, Colangelo, D'Angelo, D'Elia, Dionisio, Frezzotti, Gagliardi, Lubicz, Martinelli, Sanfilippo, Simula
to appear in PRD [arXiv:2512.09880]

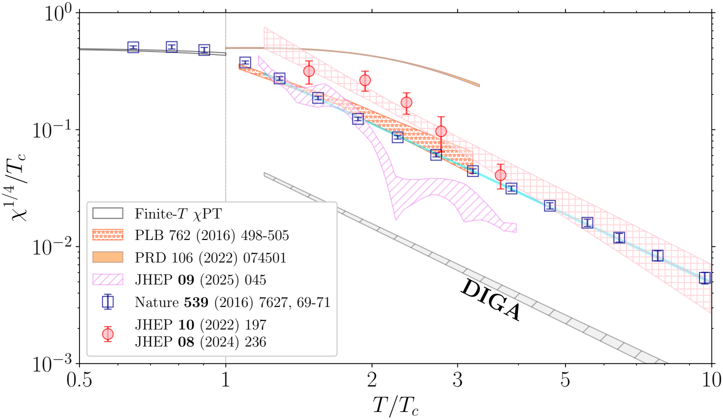
- $\bar{\ell}_7 = [2.79 \pm 0.58_{\mathrm{stat}} \pm 0.19_{\mathrm{stat}}] \times 10^{-3}$
- $\bar{\ell}_4 = 3.90 \pm 0.20_{\mathrm{stat}} \pm 0.27_{\mathrm{stat}}$
- $\ell + \bar{\ell}_7 + \frac{1}{16\pi^2}\bar{\ell}_4 = h_1^{\mathrm{r}} - h_3^{\mathrm{r}} = \frac{\bar{h}_1}{16\pi^2} - \bar{h}_3 = \left[3.50 \pm 0.55_{\mathrm{stat}} \pm 0.26_{\mathrm{syst}}\right] \times 10^{-2}$



$\chi(T) \to$ huge discrepancies among independent determinations.
Reflects inherent difficulties in determining χ, which get worse at high T

Most observe $\dfrac{\chi(T)}{T_c^4} \approx A \left(\dfrac{T}{T_c}\right)^{-c} \quad c \approx 7-9 \quad (T/T_c \gtrsim 2)$ (DIGA: T^{-8})

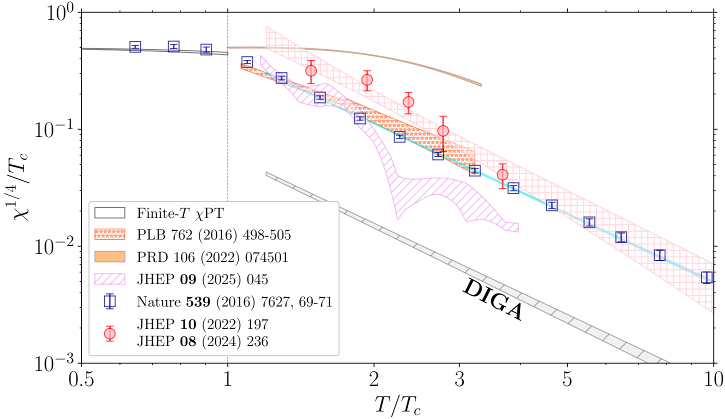
Huge uncertainties on A: reported estimates range from $\sim \mathcal{O}\left(10^{-2}\right)$ to $\sim \mathcal{O}(1)$



BW Collaboration [Nature **539** (2016)]: report smallest uncertainties.

BUT they use a series of *ad hoc* assumptions based on DIGA.

- Determinant reweighting $\longrightarrow$ DIGA expected number of Dirac zero-modes
- Assume that only $Q = 0, \pm 1$ matter: $\chi = \frac{1}{V}\langle Q^2 \rangle = \frac{1}{V}\frac{2P_1+\ldots}{P_0+P_1+\ldots} \approx \frac{2}{V}\frac{P_1}{P_0}$

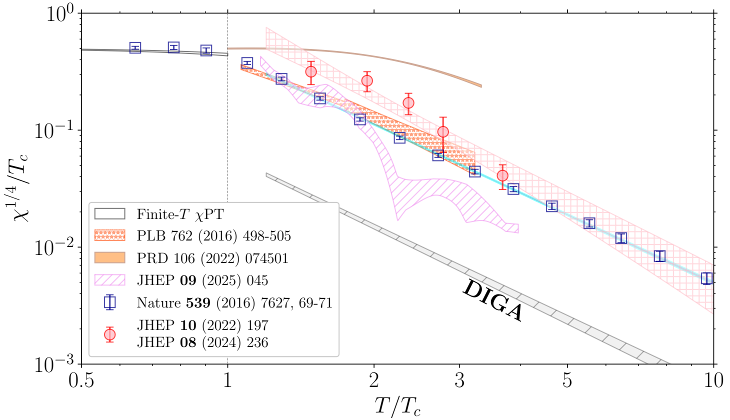


PLB **762** (2016) and JHEP **09** (2025): same method, incompatible results

Assume $U(1)_A$ restoration above T_c: $\chi \approx \frac{1}{V} m_\ell^2 \left[\langle (\overline{\psi}\psi)^2 \rangle - (\langle \overline{\psi}\psi \rangle)^2 \right]$

- $U(1)_A$ restoration $\rightarrow$ no consensus in the community

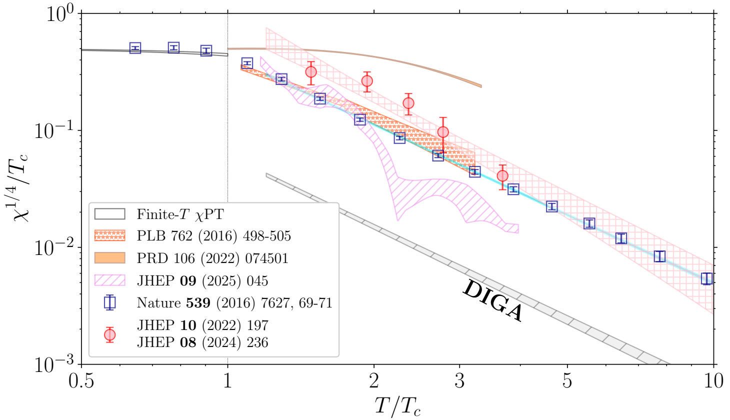
- $\chi \sim \mathcal{O}(m^2)$ and $U(1)_A$ always explicitly broken at $\mathcal{O}(m)$ (quark mass term)



PRD **106** (2022): first investigation with chiral domain wall fermions

At Lattice 2025, JLQCD reported preliminary results with exact same discretization: *completely incompatible*. Unclear origin of discrepancy.

Proceeding of talk by I. Kanamori on top. susceptibility: arXiv:2603.23022



Research program aiming at computing $\chi(T)$ without extra assumptions

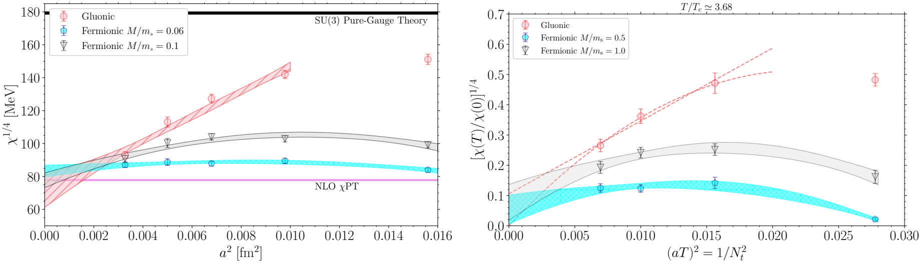
JHEP **10** (2022): study of χ via a fermionic definition (index theorem)

Athenodorou, CB, Bonati, Clemente, D'Angelo, D'Elia, Maio, Martinelli, Sanfilippo, Todaro

JHEP **08** (2024): new algorithm for topology sampling → future avenue

CB, Clemente, D'Elia, Maio, Parente

Athenodorou et al. JHEP **10** (2022) 197

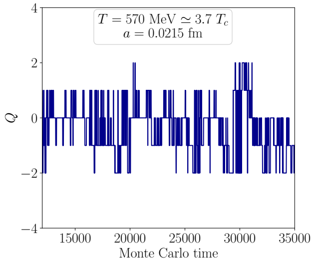


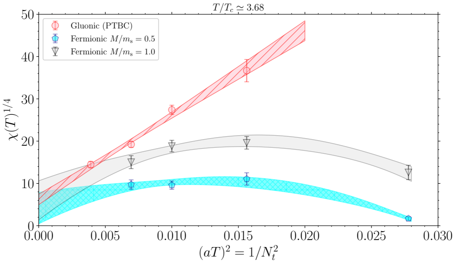
Fermionic definition of Q (index theorem): reduced cut-off effects
Non-chiral fermions: include contributions of all modes up to a certain scale M

Can be used to cross check extrapolation based on $G\widetilde{G}$ definition.

However, at high T cut-off effects are still significant.
Necessary to reach finer lattice spacings $\rightarrow$ algorithm to deal with topological freezing

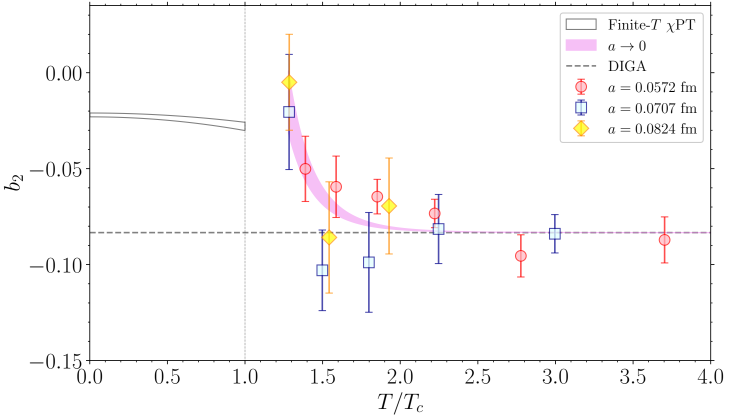
Parallel Tempering on Boundary Conditions (PTBC)





- PTBC significantly reduces severity of topological freezing by combining open and periodic boundaries simulations.
  - First implementation in $\mathrm{SU}(N)$ Yang–Mills CB et al. JHEP **03** (2021) 111
- Recently, first implementation in 2+1 QCD at physical point CB et al. JHEP **08** (2024) 236
  Significant improvement in the gluonic determination of χ at high T

Quartic coefficient b_2



b_2 has small deviations from DIGA above $T \gtrsim 1.5T_c$

No determinations in full QCD below T_c:
χ larger → distribution much wider → very noisy to detect deviations from Gaussian

Currently only known in Yang–Mills via imaginary-θ simulations: $b_2^{(\mathrm{YM})} = -0.0216(15)$

Cè et al. PRD 92 (2015) 074502 – Bonati et al. PRD 93 (2015) 025028 / PRD 94 (2016) 085017 – CB et al. JHEP 03 (2021) 111

Another mechanism for hot axion production in the primordial Universe is via out-of-equilibrium real-time topological transitions (talk by A. Notari)

Notari et al. PRL 131 (2023) 011004 [2211.03799]

Theoretical axion abundance computed from Boltzmann equation:

$$\frac{\mathrm{d} f_{\vec{p}}}{\mathrm{d}t} = (1 + f_{\vec{p}})\Gamma_a^{(+)} - f_{\vec{p}}\,\Gamma_a^{(-)}$$

($f_{\vec{p}}$ = axion distribution function)

$$\Gamma_a^{(-)} = \mathrm{e}^{E_{\vec{p}}/T}\,\Gamma_a^{(+)} = \frac{\Gamma_{\mathrm{top}}}{2E_{\vec{p}} f_a^2}$$

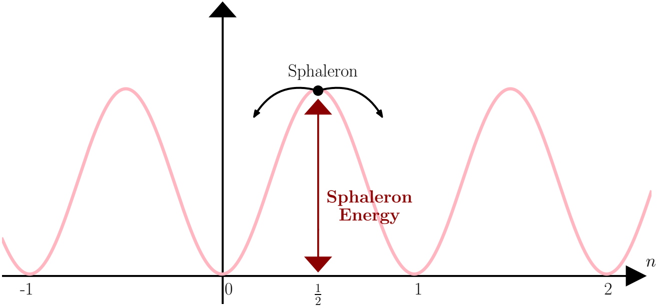
$$E_{\vec{p}}^2 = m_a^2 + |\vec{p}|^2$$

$\Gamma_a^{(\pm)}$ = Axion rate $\longleftrightarrow$ Γ_{top} = **QCD topological rate**

In the zero-momentum/zero-energy limit $\Gamma_{\mathrm{top}} \to \Gamma_{\mathrm{s}}$ = **QCD *sphaleron* rate**

The QCD sphaleron rate

Sphaleron rate = **real-time** thermal diffusion rate of topology-changing fluctuations



$$Q = \Delta n$$

$$\Gamma_s = \int dt\, d^3x\, \langle q(\vec{x}, t) q(\vec{0}, 0)\rangle_T\,, \qquad \langle \mathcal{O}\rangle_T \equiv \frac{1}{\mathrm{Tr}\left\{e^{-\mathcal{H}_{\mathrm{QCD}}/T}\right\}}\mathrm{Tr}\left\{e^{-\mathcal{H}_{\mathrm{QCD}}/T}\,\mathcal{O}\right\}$$

PROBLEM: how to compute real-time quantities from Euclidean lattice QCD?

The sphaleron rate from lattice QCD

$$\Gamma_s = 2T \lim_{\omega \to 0} \frac{\rho(\omega)}{\omega}$$

$\rho(\omega) \longrightarrow$ spectral density of thermal Euclidean time correlator of $q(\tau, \vec{x})$

$$G_E(\tau) = -\int_0^\infty \frac{d\omega}{\pi} \boxed{\rho(\omega)} \left[\frac{\cosh\left(\frac{\omega}{2T} - \omega\tau\right)}{\sinh\left(\frac{\omega}{2T}\right)} \right]$$

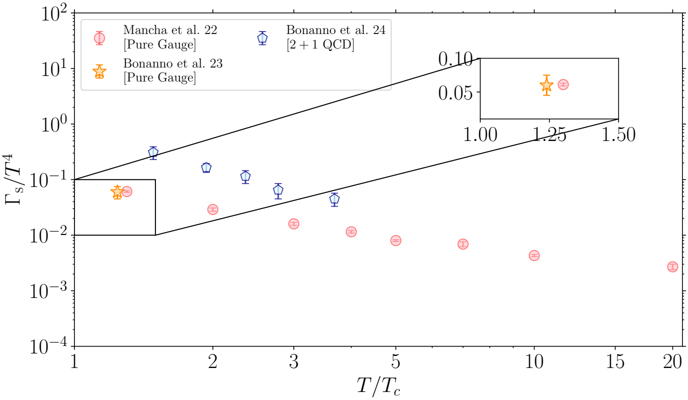
$$G_E(\tau) \equiv \int d^3x \, \underbrace{\langle q(\tau, \vec{x}) q(0, \vec{0}) \rangle_E}_{\text{Euclidean path integral}} \qquad \longleftarrow \qquad \text{amenable to be computed on the lattice}$$

Ill-posed **inverse problem**: reconstruct $\rho(\omega)$ from $G_E(\tau)$, with $G_E(\tau)$ known only for a few values of τ and with statistical errors.

▶ CB et al. PRD 108 (2023) 074515 → new method to determine Γ_s

Based on the new framework to solve inverse problems on the lattice with controlled systematic errors: **Hansen–Lupo–Tantalo (HLT)** PRD 99 (2019) 094508

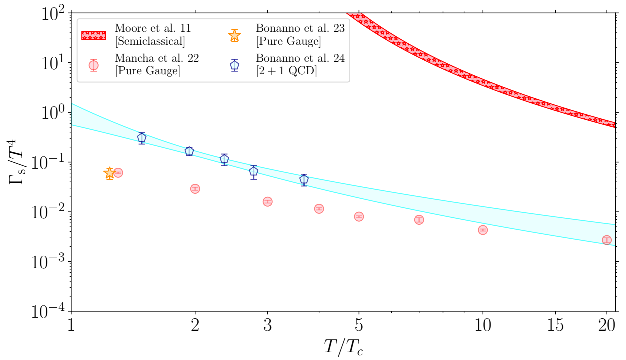
▶ CB et al. PRL 132 (2024) 051903 → First determination in 2+1 QCD



Full QCD rate larger than pure-gauge → opposite behavior compared to χ

New method in pure-gauge agrees with previous alternative avoiding inverse problem

Temperature-dependence of the sphaleron rate



Semiclassical estimate [Moore & Tassler JHEP 02 (2011) 105 – arXiv:1011.1167]

$$\frac{1}{T^4}\Gamma_s \propto \alpha_s^5 \qquad \alpha_s(T) \propto \frac{1}{\log\left(T^2/\Lambda^2_{\mathrm{QCD}}\right)} \qquad \text{(1 loop)}$$

$$\frac{\Gamma_s}{T^4} = \left[\frac{A_0}{\log(T^2/T_c^2) + A_1}\right]^C \qquad \text{(semiclassical-inspired fit ansatz)}$$

Perfectly works for $C = 5$. If C left as free parameter: 5 with 100% error

Recent claims on the absence of θ-dependence in QCD

Garbrecht–Tamarit claimed in a series of articles that QCD is independent of θ.

Ai et al. PLB 822 (2021) 136616 – Univ. 10 (2024) 189 – arXiv:2403.00747 – arXiv:2511.04216

ARGUMENT: semiclassically Q = winding number of $A_\mu(x)$ around $\mathrm{SU}(N)$ in the limit $|x| \to \infty$, when $\mathbb{R}^4$ is compactified to a 3-sphere S^3.
Thus, $Q \in \mathbb{Z}$ and well-defined only when $V \to \infty$

CLAIM: correct procedure to evaluate path integral is to take $V \to \infty$ at fixed topological charge Q and then sum over all Q.

Garbrecht-Tamarit : $$\langle O \rangle = \sum_{n=-\infty}^{+\infty} \lim_{V\to\infty} \langle O \rangle \big|_{V,Q=n}$$ with Q fixed as $V \to \infty$

Standard : $$\langle O \rangle = \lim_{V\to\infty} \sum_{n=-Q_{\max}}^{Q_{\max}} \langle O \rangle \big|_{V,Q=n}$$ with $Q_{\max} = Q_{\max}(V) \sim \sqrt{\chi V}$

In Garbrecht–Tamarit approach, $\mathrm{e}^{\mathrm{i}\theta Q}$ drops off the path integral (overall factor) and simplifies $\to$ no QCD θ-dependence, vanishing neutron dipole moment

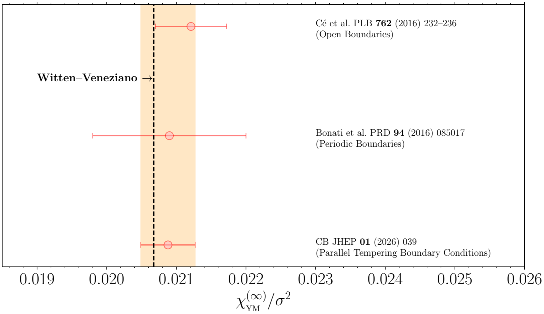
Criticisms to Garbrecht–Tamarit approach:

• Topology can be defined in finite volume (Hamiltonian formalism)

Pedagogical review: Strong CP problem, θ term and QCD topological properties – CB, Bonati, D'Elia [2510.03059]

• Q can also be non-integer. E.g., finite box with open boundaries $\longrightarrow$ Q non-integer. This will not matter when $V \longrightarrow \infty$.

E.g., periodic/open boundaries give same results for large-N limit of χ_{YM}, related to η' mass



Witten–Veneziano relation:

$$\chi_{\mathrm{YM}}^{(\infty)} = \frac{F_\pi^2}{6}\left(m_{\eta'}^2 + m_\eta^2 - 2m_K^2\right)$$

Criticisms to Garbrecht–Tamarit approach:

Well-known papers worked out QCD finite-volume corrections in a fixed topology to understand bias introduced by topological freezing

Brower et al. PLB 560 (2003) 64–74 — Aoki et al. PRD 76 (2007) 054508

$$C_\theta = \langle \mathcal{O}_1 \ldots \mathcal{O}_k \rangle_\theta \qquad C_0^{(n)} = \frac{\mathrm{d}^n}{\mathrm{d}\theta^n} C_\theta(\tau)\Big|_{\theta=0} = \langle \mathcal{O}_1 \ldots \mathcal{O}_k \, Q^n \rangle_c$$

$$C_Q = C_0 + C_0^{(2)} \frac{1}{2\chi V} + \mathcal{O}\left(\frac{1}{V^2}\right)$$ (CP-even correlator)

$$C_Q = C_0^{(1)} \frac{\mathrm{i}Q}{2\chi V} + \mathcal{O}\left(\frac{1}{V^2}\right)$$ (CP-odd correlator)

Fixed-topology approach can only compute $\theta = 0$ CP-even correlation functions.
Garbrecht–Tamarit approach trivially yields $d_{\mathrm{N}} \to 0$ when $V \to \infty$ by construction.

As $V \to \infty$, $\langle Q^2 \rangle = \chi V \to \infty$, more and more topological sector become relevant.
This is fundamental to probe θ-dependence of functional integral.

Conclusions

Summary of present status

- QCD does depend on θ, and this introduces the strong CP problem. Lattice QCD confirmed this in several way.
- Low T: general consensus. All lattice calculations of χ agree among them and with χPT. No result for b_2 in full QCD.
- High T: no consensus yet. Significant discrepancies among different lattice calculations of $\chi(T)$.
- Qualitative indication that DIGA-like behavior of $\chi(T)$ sets in for $T \gtrsim 2T_c$ (much larger pre-factor, though). Same for b_2.
- First determination of QCD sphaleron rate in $1.5T_c \lesssim T \lesssim 4T_c$ with 2+1 flavors

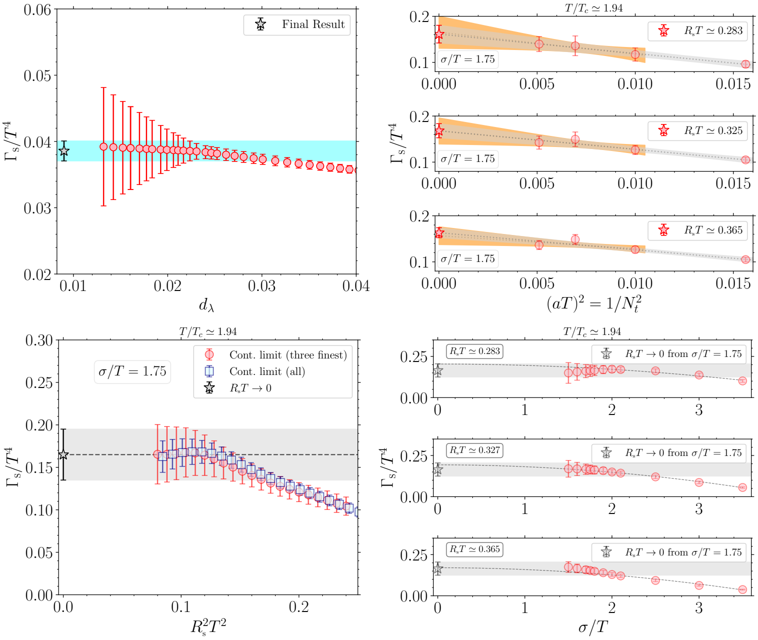
Future directions

- Topological freezing is the main obstacle hindering progress in QCD topology from the lattice. Lots of work in this field at present.
  - New algorithmic improvement (PTBC) is effective in taming topological freezing in QCD.
  - We plan to use PTBC to study $\chi(T)$ in 2+1+1 QCD up to the GeV scale
- Forthcoming study of JLQCD collaboration with chiral (domain wall) fermions
  - We are working to extend the study of the real-time topological rate.

Wanted: dynamical charm, higher temperatures, non-zero energy/momentum. First determination of $\Gamma_{\rm top}$ in quenched QCD at non-zero momentum is currently ongoing and is expected to be soon completed

Extra

In the case of the sphaleron rate, lattice artifacts are much milder compared to the χ.



Lattice QCD calculations starting to become sensitive to the neutron dipole moment.
Expected further improvements in the future.
Present results reasonably agree with χPT.