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Asymmetric soft resummation of Semi-Inclusive Deep Inelastic Scattering

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May 6, 2026

Based on [2026 2601.06245]

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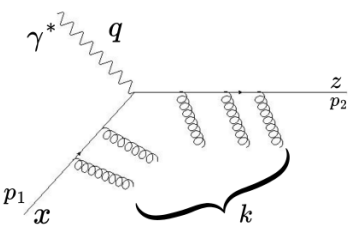
SIDIS kinematics

Factorization

$$\sigma(\xi, \zeta, Q^2) = \sum_{ij} f_i \otimes \hat{\sigma}_{ij} \otimes D_j^{h_2}$$

$$\mathcal{P}_1(p_1) + \gamma^*(q) \to \mathcal{P}_2(p_2) + \mathcal{X}(k)$$



$$\hat{\sigma}_{ij} := \hat{\sigma}_{ij}(x, z, Q^2)$$

$$x := \frac{Q^2}{2p_1 \cdot q}$$

$$z := \frac{p_1 \cdot p_2}{p_1 \cdot q}$$

$$x, z \in (0, 1)$$

Mellin space $\left(\tilde{f}(N) = \int_0^1 dx x^{N-1} f(x)\right)$:

- $\otimes \to$ simple product
  - $(x, z) \to N, M$

Key message

- Threshold limit $\Rightarrow$ Large logs to all orders $\Rightarrow$ Resummation
- Double-soft:

$$x, z \to 1 \Rightarrow \textbf{Soft} \text{ radiation}$$

- Single-soft:

$$x \to 1 \text{ at fixed } z\,, \qquad z \to 1 \text{ at fixed } x \Rightarrow \text{Purely } \textbf{Collinear} \text{ radiation}$$

- Double-soft resummation [M. Abele et al., 2021 2109.00847; M. Abele et al., 2022 2203.07928]
- Single-soft $\Rightarrow$ unlocks resummation in new kinematical regions

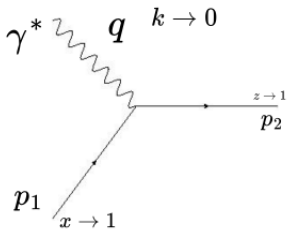
Soft limits

- $x \to 1$
  - $\Rightarrow (p_1 + q)^2 = 0 \Rightarrow (p_2 + k)^2 = k^2 + 2p_2 \cdot k \to 0$
  - $\Rightarrow$ Either $k \to 0$ or $k^2 = 0$ and $k \parallel p_2$.
- $z \to 1$
  - $\Rightarrow p_1 \cdot p_2 = p_1 \cdot q \Rightarrow p_1 \cdot (p_1 - k) = p_1 \cdot k \to 0$
  - $\Rightarrow$ Either $k \to 0$ or $k^2 = 0$ and $k \parallel p_1$.

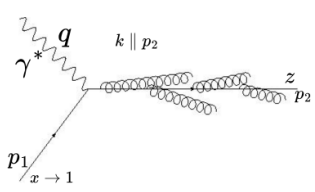
But if $k \to 0 \Rightarrow x, z \to 1$

$\Rightarrow$ Single-soft limits = fully collinear radiation

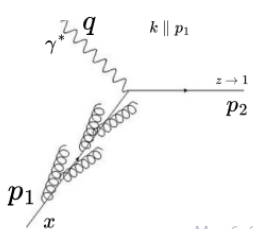
Double-soft



$x-$single-soft



$z-$single-soft



Which terms survive in the soft limits

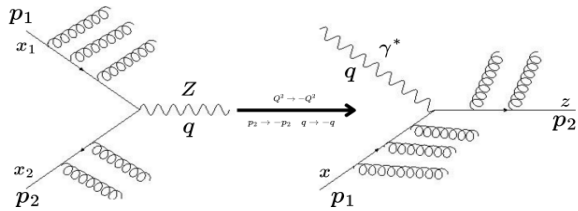
- x and/or $z \to 1$ (N and/or $M \to \infty$)
- $\Rightarrow \log^k N$ and/or $\log^k M +$ const. $+ \mathcal{O}\left(\frac{1}{N}\right)$ and/or $\mathcal{O}\left(\frac{1}{M}\right)$.
- Resummation via RG: evolution $\hat{\sigma}$: $\Lambda^2 \to Q^2$ [S. Forte et al., 2003 hep-ph/0209154]
- **Double-soft** $N, M \to \infty$:
  - Only logs in M & $N+$ const
- x-**Single-soft** resummation $N \to \infty$ M fixed:
  - resums all terms that do not vanish as $N \to \infty$, including full M dependence
- z-**Single-soft** resummation $M \to \infty$ N fixed:
  - resums all terms that do not vanish as $M \to \infty$, including full N dependence

Soft scales Λ^2

Phase space analysis $\Rightarrow$ soft-scale dependence

- Double soft: $\Lambda^2_{\text{ds}} = Q^2(1-x)(1-z)$
- Single soft: $\Lambda^2_{\text{xss}} = Q^2(1-x)$ or $\Lambda^2_{\text{zss}} = Q^2(1-z)$
- x, z dependence encoded in a single dimensionful soft-scale

Correspondence with the DY process



- Double soft: $x_1, x_2 \to 1 \Rightarrow k$ soft
  - Absolute threshold (minimal energy for final state)
  - Same limit as inclusive DY
  - Resummation formula: [S. Catani et al., 1989, *Nucl. Phys. B* 327 pp. 323–352]
- Single soft: $x_{1 \text{ or } 2} \to 1$ with the other fixed $\Rightarrow k^2 = 0$, $k \parallel p_{1 \text{ or } 2}$
  - Threshold at fixed rapidity of Z ($\Rightarrow$ fixed boost)
  - Resummation: SCET [B. Mistlberger et al., 2025 2502.17548; G. Lustermans et al., 2019 1908.00985] = dQCD (RG) [L. De Ros et al., 2026 2601.04309]
- DY same soft scales of SIDIS

$\Rightarrow$ SIDIS same RG argument of DY $\Rightarrow$ Same form of the resummation formula

Resummation formulas

$$\Lambda^2_{\rm ds} \Rightarrow \bar{\Lambda}^2_{\rm ds} = \frac{Q^2}{NM} \qquad \Lambda^2_{\rm xss} \Rightarrow \bar{\Lambda}^2_{\rm xss} = \frac{Q^2}{N} \qquad \Lambda^2_{\rm zss} \Rightarrow \bar{\Lambda}^2_{\rm zss} = \frac{Q^2}{M}$$

- $x-$Single soft

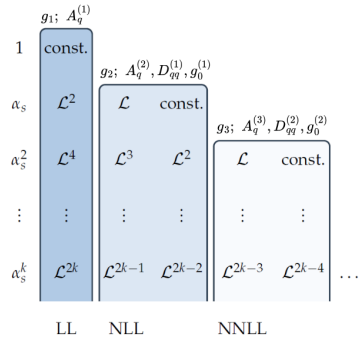
$$\hat{\sigma}^{\rm xss}_{qq}\left(N, M, \alpha_s(Q^2)\right) = C^{\rm xss}_{c,qq}\left(\alpha_s(Q^2), M\right) \exp \int_{\frac{Q^2}{N}}^{Q^2} \frac{dk^2}{k^2} \left[A_q(\alpha_s(k^2)) \ln \frac{Nk^2}{Q^2} - D^{\rm xss}_{qq}(\alpha_s(k^2), M) \right].$$

- $z-$single-soft: $N \leftrightarrow M$.
- Double-soft: resummation coefficients $\Rightarrow$ constants
- $A_q =$ **universal**: cusp $\Rightarrow$ soft–collinear logs in γ_{qq}

Resummation accuracy

$$\hat{\sigma}(N,M) = g_0(\alpha_s)\,\exp\left[\frac{1}{\alpha_s}g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \dots\right] \qquad \text{with } \lambda = \beta_0\alpha_s\mathcal{L}$$

- Same structure for double & single soft in terms of $\mathcal{L}$
- $\mathcal{L} = \ln(NM)$ (double) $\mathcal{L} = \ln N$ or $\ln M$ (single).
- Equivalent form of SS $\bar{\Lambda}^2 \to k\,\bar{\Lambda}^2$ (k finite), e.g using DS soft scale $\Rightarrow \mathcal{L} = \ln N \to \ln N + \ln M$



- A_q up to N^{3}LO [M. Abele et al., 2022 2203.07928]
- NNLL: g_0, D_{qq} obtained by matching to NNLO [S. Goyal et al., 2025 2412.19309; L. Bonino et al., 2024 2401.16281].
- Double soft up to N^{3}LL [M. Abele et al., 2021 2109.00847; M. Abele et al., 2022 2203.07928]

Fixed order result in the soft limit

Resummation formula expanded

$$\hat{\sigma}_{qq}\left(N, M, \alpha_s(Q^2)\right) = 1 + \frac{\alpha_s}{\pi}\left(\frac{1}{2}A_1\mathcal{L}^2 + (-D_1\mathcal{L}) + g_0^{(1)}\right) + \frac{\alpha_s^2}{\pi^2}\left[g_0^{(2)} + \mathcal{L}\left(-D_1 g_0^{(1)} - D_2\right)\right.$$
$$\left. + \mathcal{L}^2\left(\frac{A_1 g_0^{(1)}}{2} + \frac{A_2}{2} - \frac{b_0\pi D_1}{2} + \frac{D_1^2}{2}\right) + \mathcal{L}^3\left(\frac{A_1\pi b_0}{6} - \frac{A_1 D_1}{2}\right) + \frac{A_1^2\mathcal{L}^4}{8}\right] + \mathcal{O}(\alpha_s^3)\,.$$

Fixed order x-single-soft limit:

$$\hat{\sigma}_{qq}(N, M) = 1 + \left(\frac{\alpha_s}{\pi}\right)\left(\ln^2 N\, f_2^{(1)}(M) + \ln N\, f_1^{(1)}(M) + f_0^{(1)}(M)\right)$$
$$+ \left(\frac{\alpha_s}{\pi}\right)^2\left(\ln^4 N\, f_4^{(2)}(M) + \ln^3 N\, f_3^{(2)}(M) + \ln^2 N\, f_2^{(2)}(M) + \ln N\, f_1^{(2)}(M) + f_0^{(2)}(M)\right) + \mathcal{O}\left(\frac{1}{N}\right) + \mathcal{O}(\alpha_s^3)$$

Resummation coefficients

- Double-soft:
  - $A_q^{(i)}$ cusp anomalous dimension: soft & collinear
  - $D_q^{\mathrm{ds}(1)} = 0$ but $D_q^{\mathrm{ds}(2)} \neq 0$: soft but no collinear
  - $g_0^{(1)}$ and $g_0^{(2)} \Rightarrow$ constant terms
- $x-$single soft, scale Q^2/NM:

$$
\begin{aligned}
D_{qq}^{\mathsf{xss},(1)}(M) &= \bar{\gamma}_{qq}^{(0)}(M) \\
D_{qq}^{\mathsf{xss},(2)}(M) &= D_{qq}^{\mathsf{ds},(2)} - \pi\beta_0 \bar{F}(M) + \bar{\gamma}_{qq,\mathsf{NS}}^{(1),t}(M) \\
g_{0,qq}^{\mathsf{xss},(1)}(M) &= g_{0,qq}^{\mathsf{ds},(1)} + \bar{F}(M)
\end{aligned}
$$

- z-single soft same but $\Rightarrow N \leftrightarrow M\,, \quad \gamma^t \leftrightarrow \gamma\,, \quad F \leftrightarrow H$

$\bar{\gamma}$: subtracted anomalous dimension: no const. + logs

Interpretation of results

- Double soft: as inclusive result with $N^2 \to N\,M$ ([M. Abele et al., 2021 2109.00847] or with GR derivation [L. De Ros et al., 2026 2601.04309])
- Single soft purely collinear:
  - Double-soft=Single-soft when non-soft variable $\to \infty$
  - A_q soft-collinear
  - $D^{\rm ss}$ contains subtracted anomalous dimension $\Rightarrow$ $A_q \supset$ soft piece, $g_0 \supset$ constant
  - $x \to 1$ collinear to timelike leg, $z \to 1$ collinear to spacelike leg $\Rightarrow$ γ^t or γ
  - $D^{\rm ss,\,(2)} \supset \bar{H}, \bar{F}$ from $O(\alpha_s)$ constants $\Rightarrow$ running α_s: $Q^2 \to Q^2/N,\ Q^2/M$
- Expanding at $\mathcal{O}(1/N)$ or $\mathcal{O}(1/M)$ $\Rightarrow$ NLP corrections [M. Abele et al., 2021 2109.00847]
- $\Rightarrow$ single-soft confirms & extends [M. Abele et al., 2021 2109.00847]

Outlook

- Extend single-soft resummation to the other channels $\Rightarrow$ nontrivial because of parton mixing in single soft limit [Forte, S. and Groenendijk, E. Ridolfi, G. Stagnitto, G. Ventola F. *In preparation*]
- Matching procedure with the fixed order
- EIC phenomenology: combined fixed-order and resummed predictions

Backup

SIDIS: soft limits in terms of mandelstam invariants

Mandelstam Invariants

$$s = (p_1 + q)^2 = Q^2 \frac{1-x}{x}$$

$$t = (p_1 - p_2)^2 = -Q^2 \frac{z}{x}$$

Boundaries

$$0 \leq s < \infty$$

$$-\frac{Q^2}{x} \leq t \leq 0$$

$$0 \leq k^2 \leq Q^2 \frac{(1-x)(1-z)}{x}$$

- **Double-soft**: $k \to 0$
  $s \to 0 \Rightarrow x \to 1$
  $\Rightarrow t \to -Q^2 \Rightarrow x = z$
  $\Rightarrow x, z \to 1$
- $x-$**single-soft**:
  $\Rightarrow s \to 0$ and $t = -Q^2 z$ fixed
- $z-$**single-soft**
  $\Rightarrow t \to -\frac{Q^2}{x}$, $s = \frac{Q^2(1-x)}{x}$ fixed

DY: definition of the scaling variables

Scaling variables

$$x_1 x_2 = \frac{Q^2}{s}, \qquad \frac{x_1}{x_2} = e^{2y}$$

- Double soft: $\Rightarrow y \to 0 \Rightarrow s \to Q^2 =$ minimum energy needed to produce $Z \Rightarrow k$ soft
- Single soft: $x_1 \to 1$ and x_2 fixed (or $x_2 \to 1$ and x_1 fixed) at fixed rapidity $\Rightarrow$ $s \to s_{\min}$ at **fixed rapidity** $\Rightarrow k^2 = 0$ and $k \parallel p_1$ (or $\parallel p_2$)

Explicit resummation coefficients

Double-soft

$$A_q^{(1)} = C_F$$

$$A_q^{(2)} = \frac{1}{2} C_F \left[\left(\frac{67}{18} - \frac{\pi^2}{6} \right) C_A - \frac{5}{9} N_f \right]$$

$$D_{qq}^{\mathsf{ds}(1)} = 0$$

$$D_{qq}^{\mathsf{ds},(2)} = \frac{1}{2} C_F \left[\left(-\frac{101}{27} + \frac{7}{2}\,\zeta_3 \right) C_A + \frac{14}{27}\, N_f \right] ,$$

$$g_{0,qq}^{\mathsf{ds}(1)} = C_F \left(\frac{\pi^2}{6} - 4 \right) = C_F \left(\zeta_2 - 4 \right)$$

$$\begin{aligned} g_{0,qq}^{\mathsf{ds},(2)} = {} & C_F^2 \left[\frac{511}{64} - \frac{\pi^2}{16} - \frac{\pi^4}{60} - \frac{15}{4}\,\zeta_3 \right] \\ & + C_F C_A \left[-\frac{1535}{192} - \frac{5\pi^2}{16} + \frac{7\pi^4}{720} + \frac{151}{36}\,\zeta_3 \right] \\ & + C_F N_f \left[\frac{127}{96} + \frac{\pi^2}{24} + \frac{\zeta_3}{18} \right] \end{aligned}$$

Explicit resummation coefficients

x-single soft limit: scale Q^2/N

$$D_{qq}^{\mathsf{xss},(1)}(M) = \tilde{\gamma}_{qq}^{(0)}(M)$$
$$D_{qq}^{\mathsf{xss},(2)}(M) = D_{qq}^{\mathsf{ds},(2)} - \pi\beta_0 F(M) + \tilde{\gamma}_{qq,\mathsf{NS}}^{(1),t}(M)$$
$$g_{0,qq}^{\mathsf{xss},(1)}(M) = g_{0,qq}^{\mathsf{ds},(1)} + F(M) = f_0^{(1)}(M)$$
$$g_{0,qq}^{\mathsf{xss},(2)}(M) = f_0^{(2)}(M)$$

$\tilde{\gamma}_{qq}$ timelike anomalous dimension no const. term

z-single soft limit: scale Q^2/M

$$D_{qq}^{\mathsf{zss},(1)}(N) = \tilde{\gamma}_{qq}^{(0)}(N)$$
$$D_{qq}^{\mathsf{zss},(2)}(N) = D_{qq}^{\mathsf{ds},(2)} - \pi\beta_0 H(N) + \tilde{\gamma}_{qq,\mathrm{NS}}^{(1)}(N)$$
$$g_{0,qq}^{\mathsf{zss},(1)}(N) = g_{0,qq}^{\mathsf{ds},(1)} + H(N) = h_0^{(1)}(N)$$
$$g_{0,qq}^{\mathsf{zss},(2)}(N) = h_0^{(2)}(N)$$

$\tilde{\gamma}_{qq}$spacelike anomalous dimension no const. term
$h_j^{(i)}(N)$ *as* $f_j^{(i)}$ *but in the* $z-$*soft*

Explicit resummation coefficients

x-single soft limit: scale Q^2/NM

$$\bar{D}_{qq}^{\text{xss},(1)}(M) = \bar{\gamma}_{qq}^{(0)}(M)$$
$$\bar{D}_{qq}^{\text{xss},(2)}(M) = D_{qq}^{\text{ds},(2)} - \pi\beta_0 \bar{F}(M) + \bar{\gamma}_{qq,\text{NS}}^{(1),t}(M)$$
$$\bar{g}_{0,qq}^{\text{xss},(1)}(M) = g_{0,qq}^{\text{ds},(1)} + \bar{F}(M)$$
$$= f_2^{(1)}(M)\ln^2 M - f_1^{(1)}(M)\ln M + f_0^{(1)}(M)$$
$$\bar{g}_{0,qq}^{\text{xss},(2)}(M) = f_4^{(2)}(M)\ln^4 M - f_3^{(2)}(M)\ln^3 M + f_2^{(2)}(M)\ln^2 M$$
$$- f_1^{(2)}(M)\ln M + f_0^{(2)}(M)$$

z-single soft limit: scale Q^2/NM

$$\bar{D}_{qq}^{\text{zss},(1)}(N) = \bar{\gamma}_{qq}^{(0)}(N)$$
$$\bar{D}_{qq}^{\text{zss},(2)}(N) = D_{qq}^{\text{ds},(2)} - \pi\beta_0 \bar{H}(N) + \bar{\gamma}_{qq,\text{NS}}^{(1)}(N)$$
$$\bar{g}_{0,qq}^{\text{zss},(1)}(N) = g_{0,qq}^{\text{ds},(1)} + \bar{H}(N)$$
$$= h_2^{(1)}(N)\ln^2 N - h_1^{(1)}(N)\ln N + h_0^{(1)}(N)$$
$$g_{0,qq}^{\text{zss},(2)}(M) = h_4^{(2)}(N)\ln^4 N - h_3^{(2)}(N)\ln^3 N + h_2^{(2)}(N)\ln^2 N$$
$$- h_1^{(2)}(N)\ln N + h_0^{(2)}(N)$$

$\bar{\gamma}_{qq} \Rightarrow$ anomalous dimension without const+logs

$\bar{F}, \bar{H} \Rightarrow H, F$ without logs

Fixed order result in the soft limit

$$f_2^{(1)}(M) = \frac{C_F}{2}$$

$$f_1^{(1)}(M) = C_F\left(S_1(M) - \frac{2}{4M^2+4M}\right)$$

$$f_0^{(1)}(M) = \frac{C_F\left(2M^2-M-1\right)}{2M^2(M+1)^2} + \frac{1}{2}C_F S_1(M)^2 - \frac{C_F S_1(M)}{2M(M+1)} + \frac{3}{2}C_F S_2(M) - \frac{C_F\zeta(2)}{2} - 4C_F$$

$$f_4^{(2)}(M) = \frac{1}{8}C_F^2,$$

$$f_3^{(2)}(M) = C_F^2\left(\frac{S_1(M)}{2} - \frac{1}{4M(M+1)}\right) + \frac{11C_F C_A}{72} - \frac{C_F N_f}{36},$$

$$f_2^{(2)}(M) = C_F^2\left(\frac{4M^2-2M-1}{8M^2(M+1)^2} + \frac{3}{4}S_1(M)^2 - \frac{3S_1(M)}{4M(M+1)} + \frac{3S_2(M)}{4} - \frac{\zeta(2)}{4} - 2\right) + C_F C_A\left(\frac{11S_1(M)}{24} - \frac{11}{48M(M+1)} - \frac{\zeta(2)}{4} + \frac{67}{72}\right) + C_F N_f\left(-\frac{1}{12}S_1(M) + \frac{1}{24M(M+1)} - \frac{5}{36}\right),$$

$$f_1^{(2)}(M) = C_F^2\left(-\frac{\left(3M^2+3M+5\right)S_2(M)}{4M(M+1)} + \frac{3\left(M^2+M+1\right)\zeta(2)}{4M(M+1)} - \frac{\left(16M^4+32M^3+12M^2+6M+3\right)S_1(M)}{4M^2(M+1)^2} + \frac{8M^4+19M^3+14M^2+8M+4}{4M^3(M+1)^3} - \frac{3}{2}\zeta(2)S_1(M) + \frac{1}{2}S_1(M)^3 - \frac{3S_1(M)^2}{4M(M+1)} + \frac{5}{2}S_2(M)S_1(M) - S_3(M) + \zeta(3)\right) + C_F N_f\left(-\frac{\left(10M^2+10M-3\right)S_1(M)}{36M(M+1)} - \frac{1}{12}S_1(M)^2 - \frac{S_2(M)}{12} + \frac{5M+8}{36M(M+1)^2} + \frac{\zeta(2)}{12} - \frac{7}{27}\right) + C_F C_A\left(\frac{\left(134M^2+134M-33\right)S_1(M)}{72M(M+1)} - \frac{85M^4+203M^3+154M^2+36M+18}{72M^3(M+1)^3} - \frac{1}{2}\zeta(2)S_1(M) + \frac{11}{24}S_1(M)^2 + \frac{11S_2(M)}{24} + \frac{S_3(M)}{2} + \frac{\zeta(2)}{4M(M+1)} - \frac{11\zeta(2)}{24} - \frac{9\zeta(3)}{4} + \frac{101}{54}\right)$$

$$S_{a_1,\ldots,a_k}(N) = \sum_{N\geq i_1\geq i_2\geq\cdots\geq i_k\geq 1} \frac{\text{sign}(a_1)^{i_1}}{i_1^{|a_1|}} \cdots \frac{\text{sign}(a_k)^{i_k}}{i_k^{|a_k|}}$$

Plus distributions in z space $\leftrightarrow$ Harmonic sums in M Mellin space

$$C_F = \frac{4}{3} \quad C_A = 3 \quad N_f = \text{active flavours}$$

$$f_0^{(2)}(M) = C_F^2\left(\frac{1}{8}S_1(M)^4 - \frac{S_1(M)^2}{2M(M+1)} - \frac{5}{4}\zeta(2)S_1(M)^2 + \frac{(-M^2+13M^2+41M^2+13M^2+2)S_1(M)}{8M^2(M+1)^3} - \frac{5\zeta(2)S_1(M)}{4M^2+4M} - \frac{1}{2}\zeta(3)S_1(M) + \frac{S_1(M)^3\left(7M^2(M+1)^2S_1(M)-2\left(4M^4+8M^3+3M^2+3M+1\right)\right)}{4M^3(M+1)^3} + \frac{\left(-7M(M+1)S_1(M)\right)S_1(M)}{4M^2(M+1)^2} - \frac{3}{2}\left(5S_2(M)-2S_{2,1}(M)\right)S_1(M) - \frac{35\zeta(2)^2}{80} - \frac{-318M^7-561M^6+294M^5+813M^4+653M^3+483M+135}{8M^3(M+1)^3} + \frac{\left(63M^5-24M-1\right)\zeta(2)}{8M^2(M+1)^2} + \frac{\left(189M^4+284M^3+123M^2+M+1\right)\zeta(2)}{4M^2(M+1)^3} + \frac{13\zeta(2)}{8} - \frac{\zeta(3)}{4M(M+1)} + \frac{15\zeta(3)}{4} + \frac{3\left(7\zeta(3)M^3+7\zeta(3)M+2\zeta(3)\right)}{8M(M+1)} - \frac{\left(27M^4+54M^3+18M^2+7M+5\right)S_2(M)}{8M^2(M+1)^2} + \frac{5}{4}\zeta(2)S_2(M) - \frac{\left(94M^3+9M+10\right)S_2(M)}{8M(M+1)} - \frac{335S_2(M)}{8} - \frac{\left(13M^3+3M-2\right)S_{2,1}(M)}{4M(M+1)} + \frac{13}{4}S_{2,1}(M) - 2S_{3,1}(M) + \frac{5}{2}S_{3,2}(M) + \frac{S_4(M)}{2} + \frac{521}{96} + \frac{1}{36}\left(11C_A - 2N_f\right)\zeta(3)C_F + N_f C_F\left(-\frac{1}{36}S_1(M)^3 - \frac{\left(101M^2+101M-3\right)S_1(M)^2}{72M(M+1)} + \frac{\left(28M^3+56M^2+13M-24\right)S_1(M)}{108M^2(M+1)^2} + \frac{1}{12}\zeta(2)S_1(M) - \frac{1}{12}S_2(M)S_1(M) - \frac{11M^4-74M^3-109M^2-6M+9}{216M^3(M+1)^3} + \frac{\left(5M^2+5M-3\right)\zeta(2)}{36M(M+1)} + \frac{\zeta(2)}{24M(M+1)} + \frac{7\zeta(2)}{18} + \frac{\zeta(3)}{12} + \frac{\left(20M^2+20M-3\right)S_1(M)}{72M(M+1)} + \frac{S_2(M)}{36} + \frac{127}{96}\right) + C_F C_A\left(\frac{11}{72}S_1(M)^3 + \frac{\left(484M^3+1239M^2+1038M^2-783M^2-273M+54\right)S_1(M)}{216M^3(M+1)^3} - \frac{11}{24}\zeta(2)S_1(M) - \frac{13}{4}\zeta(3)S_1(M) - \frac{S_1(M)^2\left(-134M^2+36(M+1)S_1(M)M-134M+33\right)}{144M(M+1)} + \frac{-6S_1(M)\left(11M^2+11M+6\right)S_2(M)}{144M(M+1)} + \frac{1}{4}\left(S_2(M)-2S_{2,1}(M)\right)S_1(M) + \frac{21\zeta(2)^2}{20} + \frac{1245M^7-1044M^6-3679M^5-2396M^4-1002M^3-765M-270}{432M^3(M+1)^3} - \frac{11\zeta(2)}{8M(M+1)} - \frac{103\zeta(2)}{36} - \frac{29\zeta(2)M^4+58\zeta(2)M^3+17\zeta(2)M^2-30\zeta(2)M-9\zeta(2)}{36M^2(M+1)^2} + \frac{\left(11M^2+11M+39\right)\zeta(3)}{24M(M+1)} + \frac{43\zeta(3)}{12} + \frac{\left(250M^3+250M^2-69M-36\right)S_2(M)}{144M^2(M+1)} - \frac{5}{2}\zeta(2)S_2(M) - \frac{\left(11M^3+11M-18\right)S_2(M)}{72M(M+1)} + S_3(M) - \frac{S_{2,1}(M)}{2M(M+1)} + S_3(M) + S_{3,1}(M) - \frac{5}{2}S_{3,1,1}(M) - \frac{1535}{1920}\right)$$