# The four-loop non-singlet splitting functions

**Vasily Sotnikov**

University of Zurich

*Based on*

Thomas Gehrmann, Andreas von Manteuffel, VS, Tong-Zhi Yang [arXiv:2604.09534]

**DIS 2026**

**Hotel Carlton, Bologna (Italy)**

7th May 2026

Universität Zürich[UZH]

# Introduction

# Non-singlet splitting functions

Splitting functions are DGLAP evolution kernels

$$\frac{\mathrm{d}f_i(x,\mu)}{\mathrm{d}\ln\mu} = 2\sum_j \left[P_{ij}\otimes f_j(\mu)\right](x),$$

$$i,j \in \{q_1, \bar{q}_1, \ldots, q_{n_f}, \bar{q}_{n_f}, g\}$$

Non-singlet evolution decouples:

$$q^{\pm}_{\mathrm{ns},ij} = q_i \pm \bar{q}_i - (q_j \pm \bar{q}_j),$$
$$q^{V}_{\mathrm{ns}} = \sum_i (q_i - \bar{q}_i)$$

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\begin{pmatrix}\Sigma\\ g\\ \hline q^{+}_{\mathrm{ns}}\\ q^{-}_{\mathrm{ns}}\\ q^{V}_{\mathrm{ns}}\end{pmatrix} = \left(\begin{array}{cc|ccc} P_{qq} & P_{qg} & 0 & 0 & 0\\ P_{gq} & P_{gg} & 0 & 0 & 0\\ \hline 0 & 0 & P^{+}_{\mathrm{ns}} & 0 & 0\\ 0 & 0 & 0 & P^{-}_{\mathrm{ns}} & 0\\ 0 & 0 & 0 & 0 & P^{V}_{\mathrm{ns}} \end{array}\right) \otimes \begin{pmatrix}\Sigma\\ g\\ \hline q^{+}_{\mathrm{ns}}\\ q^{-}_{\mathrm{ns}}\\ q^{V}_{\mathrm{ns}}\end{pmatrix}$$

N$^3$LO accuracy requires four-loop splitting functions, only known partially.
This talk: complete non-singlet sector @ N$^3$LO

# Four-loop non-singlet splitting functions: partial results

- $n_f^2$ [Davies, Vogt, Ruijl, Ueda, Vermaseren '16] [Gehrmann, von Manteuffel, VS, Yang '23]
- Leading color [Moch, Ruijl, Ueda, Vermaseren, Vogt '17]
- Fixed Mellin moments $n \leq 16$ [Moch, Ruijl, Ueda, Vermaseren, Vogt '17]
- $n_f C_F^3$ [Gehrmann, von Manteuffel, VS, Yang '24]
- $n_f$ in $P_{\rm ns}^{(3)\pm}$ [Kniehl, Moch, Velizhanin, Vogt '25]

- Small $x$ limit: $\ln^k x,\ k = \{6, 5, 4\}$ [Davies, Kom, Moch, Vogt '22]
- Large $x$ limit: [Dokshitzer, Marchesini, Salam '05] [Grozin '18] [Henn, Korchemsky, Mistlberger '19] [von Manteuffel, Panzer, Schabinger '20] [Das, Moch, Vogt '19]

- aN$^3$LO PDFs use approximations based on these partial results (see e.g. [Cooper-Sarkar, Cridge, Giuli, Harland-Lang, Hekhorn, Huston, Magni, Moch, Thorne '24]).

# Computation

# Off-shell operator matrix elements method

- Mellin transform, convolution $\to$ product: $\gamma_{\rm ns}(n) = -\int_0^1 dx\, x^{n-1} P_{\rm ns}(x).$
- $\gamma_{\rm ns}$ is anomalous dimension of twist-two operator $O_{\rm ns}(n) \sim \bar\psi\, \Delta\cdot\gamma\, (\Delta\cdot D)^{n-1}\, \frac{\lambda}{2}\psi, \quad \Delta^2 = 0$
- Simple multiplicative renormalization (non-singlet):

$$O^{\rm R}_{\rm ns}(\mu, n) = Z_{\rm ns}(\mu, n) O^{\rm B}_{\rm ns}(n)\,, \qquad \frac{dZ_{\rm ns}(\mu, n)}{d\ln\mu} = -2\gamma_{\rm ns}(\mu, n)\, Z_{\rm ns}(\mu, n)\,.$$

# Off-shell operator matrix elements method

- Mellin transform, convolution $\to$ product: $\gamma_{\rm ns}(n) = -\int_0^1 dx\, x^{n-1} P_{\rm ns}(x).$
- $\gamma_{\rm ns}$ is anomalous dimension of twist-two operator $O_{\rm ns}(n) \sim \bar\psi\, \Delta\cdot\gamma\, (\Delta\cdot D)^{n-1}\, \frac{\lambda}{2}\psi, \quad \Delta^2 = 0$
- Simple multiplicative renormalization (non-singlet):

$$O^{\rm R}_{\rm ns}(\mu, n) = Z_{\rm ns}(\mu, n) O^{\rm B}_{\rm ns}(n)\,, \qquad \frac{dZ_{\rm ns}(\mu, n)}{d\ln\mu} = -2\gamma_{\rm ns}(\mu, n)\, Z_{\rm ns}(\mu, n)\,.$$

**Extract** $\gamma_{\rm ns}^{(l-1)}$

1. Compute off-shell OME $A_{\rm ns}(n) = \langle q(p)|\, O_{\rm ns}(n)\, |q(p)\rangle$ with $p^2 < 0$ through $l$ loops.
2. Renormalized OME finite $\implies$ $Z_{\rm ns}$ through $\alpha_s^l$.
3. Read off $\gamma_{\rm ns}^{(l-1)}$ from $1/\epsilon$ at order $\alpha_s^l$.

# Tracing parameter

$$A_{\rm ns}(n) = \langle q(p)| \, \bar{\psi} \, \Delta \cdot \gamma \, (\Delta \cdot D)^{n-1} \, \frac{\lambda}{2} \psi \, |q(p)\rangle$$

- Infinite number of OME computations at fixed $n$.
- Introduce tracing parameter to compute all simultaneously
  [Ablinger, Blümlein, Hasselhuhn, Schneider, Wissbrock '12]

$$(\Delta \cdot k)^{n-1} \to \sum_{n=1}^{\infty} t^n (\Delta \cdot k)^{n-1} = \frac{t}{1 - t\Delta \cdot k},$$

| | | |
|---|---|---|
| Rank $n-1$ numerator tensors | $\Longleftrightarrow$ | Additional denominators |
| Fixed $n$ moments | | Coefficients of Taylor series in $t$ |

# Tracing parameter

$$A_{\rm ns}(n) = \langle q(p)| \, \bar{\psi} \, \Delta \cdot \gamma \, (\Delta \cdot D)^{n-1} \, \frac{\lambda}{2} \psi \, |q(p)\rangle$$

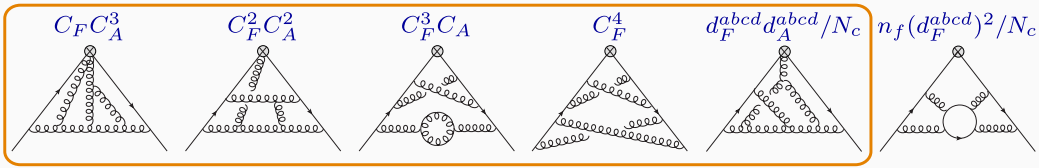
- Infinite number of OME computations at fixed $n$.
- Introduce tracing parameter to compute all simultaneously
  [Ablinger, Blümlein, Hasselhuhn, Schneider, Wissbrock '12]

$$(\Delta \cdot k)^{n-1} \to \sum_{n=1}^{\infty} t^n (\Delta \cdot k)^{n-1} = \frac{t}{1 - t\Delta \cdot k},$$

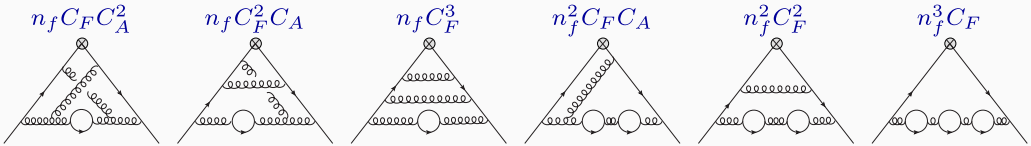
| | | |
|---|---|---|
| Rank $n-1$ numerator tensors | $\Longleftrightarrow$ | Additional denominators |
| Fixed $n$ moments | | Coefficients of Taylor series in $t$ |

- Diagrams with up to $4$ additional linear denominators, standard IBP relations.
- Expand in $t$ to sufficiently high powers to recover all-$n$ form of $\gamma_{\rm ns}^{(3)}(n)$.

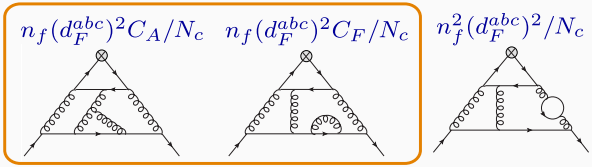
# Feynman diagrams




$\gamma_{\rm ns}^{(3)\pm}$




$\gamma_{\rm ns}^{(3)V}$



# Integral reduction

⚠ Main technical challenge: integration-by-parts (IBP) reduction of around $3 \cdot 10^6$ integrals:

$$\int \mathrm{d}^d\ell_1 \cdots \mathrm{d}^d\ell_4 \, \frac{\partial}{\partial \ell_i^\mu} \left( v^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right) = 0, \qquad D_j = \{q_j^2,\ 1 - t(\Delta \cdot q_j)\}\,.$$

- Custom private reduction code.
- Finite-field sampling and analytic reconstruction [von Manteuffel, Schabinger '14] [Peraro '15].
- Aggressively optimized identity generation and filtering
  [Driesse, Jakobsen, Mogul, Plefka, Sauer, Usovitsch '24][Guan, Liu, Ma, Wu '24]
  [Bern, Herrmann, Roiban, Ruf, Smirnov, Smirnov, Zeng '24] [von Hippel, Wilhelm '25] [Song, Yang, Cat, Leo, Zhu '25] [Lange, Usovitsch, Wu '25]
- Spanning cuts and sectors [Larsen, Zhang '15] [Guan, Liu, Ma, Wu '24].
- Avoiding mixed $\epsilon$ and $t$ dependence in denominators through improved basis choice
  [Smirnov, Smirnov '20] [Usovitsch '20].

# Differential equations

- Reduce OME to basis of around 6k master integrals $A = \sum_i c_i(\epsilon, t)\, I_i(\epsilon, t)\,.$
- Derive differential equations in tracing parameter $\partial_t\, I \;=\; M(\epsilon, t)\, I\,.$
- Use DE to derive recursion relations [Blümlein, Kauers, Klein, Schneider '09] [Lee, Smirnov, Smirnov '17] for Laurent series in $\epsilon$ and Taylor series in $t$ coefficients of $I_i(\epsilon, t)$.
- Boundary condition: at $t \to 0$ four-loop self-energy integrals [Baikov, Chetyrkin '10] [Lee, Smirnov, Smirnov '11].

# Surprise: elliptic geometry

- Homogeneous DE with $\epsilon \to 0$ related to class of special functions.
- OME @ three loops: rational and (rationalizeable) square roots
  $\implies$ solutions through multiple polylogarithms in $t$.
- OME @ four loops: more square roots, elliptic curves.

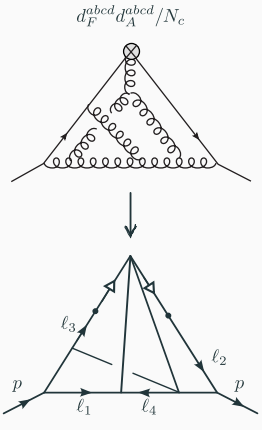
# Surprise: elliptic geometry

- Homogeneous DE with $\epsilon \to 0$ related to class of special functions.
- OME @ three loops: rational and (rationalizeable) square roots
  $\Longrightarrow$ solutions through multiple polylogarithms in $t$.
- OME @ four loops: more square roots, elliptic curves.

**Example**

Nonplanar graph with $9$ standard (massless) propagators
and $2$ denominators $1 - t(\Delta \cdot \ell_i)$.

Found coupled $2 \times 2$ DE block, solutions are periods of elliptic curves.



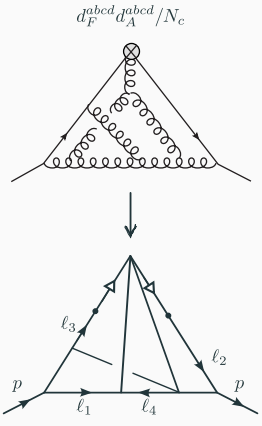
# Surprise: elliptic geometry

- Homogeneous DE with $\epsilon \to 0$ related to class of special functions.
- OME @ three loops: rational and (rationalizeable) square roots
  $\Longrightarrow$ solutions through multiple polylogarithms in $t$.
- OME @ four loops: more square roots, elliptic curves.

**Example**

Nonplanar graph with $9$ standard (massless) propagators
and $2$ denominators $1 - t(\Delta \cdot \ell_i)$.

Found coupled $2 \times 2$ DE block, solutions are periods of elliptic curves.




- No special treatment in our approach.
- Elliptic contributions must drop out in OME, at least in $1/\epsilon$ poles.

# Reconstruction of all-$n$ form

Series in $t \implies$ samples of fixed moments: $\gamma_{\rm ns}^{(3)}(n_i) = s_i^{(7)} + \sum_{j=3}^{5} s_i^{(7-j)} \zeta_j$

Match $s_i^{(w)}$ samples to generic ansatz of monomials of harmonic sums [Vermaseren '99] and denominators

$$\mathcal{A}^{(w)}(n) = \sum_{j=0}^{w} \sum_{k+|\vec{i}|=j} \sum_{a} c^{j}_{k,\vec{i},a} \, \frac{1}{(n+a)^k} \, S_{\vec{i}}(n),$$

$$S_{i_1,\ldots,i_d}(n) = \sum_{j=1}^{n} \frac{(\operatorname{sgn} i_1)^j}{j^{|i_1|}} \, S_{i_2,\ldots,i_d}(j),$$

$$\left|\vec{i}\right| = \sum_j |i_j|$$

# Reconstruction of all-$n$ form

Series in $t \implies$ samples of fixed moments: $\gamma_{\rm ns}^{(3)}(n_i) = s_i^{(7)} + \sum_{j=3}^{5} s_i^{(7-j)} \zeta_j$

Match $s_i^{(w)}$ samples to generic ansatz of monomials of harmonic sums [Vermaseren '99] and denominators

$$\mathcal{A}^{(w)}(n) = \sum_{j=0}^{w} \sum_{k+|\vec{i}|=j} \sum_{a} c^{j}_{k,\vec{i},a} \frac{1}{(n+a)^k} S_{\vec{i}}(n),$$

$$S_{i_1,\ldots,i_d}(n) = \sum_{j=1}^{n} \frac{(\operatorname{sgn} i_1)^j}{j^{|i_1|}} S_{i_2,\ldots,i_d}(j), \qquad \left|\vec{i}\right| = \sum_j |i_j|$$

- Poles in $n$: $\{0,-1\}$ for $\gamma_{\rm ns}^{(3)\pm}$, $\{0,\pm1,\pm2\}$ for $\gamma_{\rm ns}^{(3)s} = \gamma_{\rm ns}^{(3)V} - \gamma_{\rm ns}^{(3)-}$.
- Not using constraints from limits or reciprocity relations [Moch, Ruijl, Ueda, Vermaseren, Vogt '17] [Kniehl, Moch, Velizhanin, Vogt '25].
- No "elliptic" contributions.

# Results

# Exact four-loop non-singlet splitting functions

**$n$ space**

Around 4k fixed moments to successfully fit the ansatz to obtain all-$n$ form, confirming expected analytic structure.

**Inverse Mellin transform to $x$ space [Ablinger '09]**

$P_{\rm ns}^{(3)\pm}$

- HPLs weight 6, $\zeta$ values weight 7.
- Denominators $1 \pm x$.
- Distribution contributions $\delta(1-x), [1/(1-x)]_+$.

$P_{\rm ns}^{(3)s}$

- HPLs weight 6, $\zeta$ values weight 6.
- Denominators $\{1 \pm x, x\}$.

✓ All earlier partial results independently confirmed*

# Large $x$ limit

Universal all-order structure in $x \to 1$ limit [Dokshitzer, Marchesini, Salam '05]:

$$P_{\text{ns}}^{(3)\pm,V} = A_4 \left[\frac{1}{1-x}\right]_+ + B_4\,\delta(1-x) + C_4 \log(1-x) + D_4 - A_4 + \mathcal{O}(1-x)$$

# Large $x$ limit

Universal all-order structure in $x \to 1$ limit [Dokshitzer, Marchesini, Salam '05]:

$$P_{\text{ns}}^{(3)\pm,V} = A_4 \left[\frac{1}{1-x}\right]_+ + B_4\,\delta(1-x) + C_4 \log(1-x) + D_4 - A_4 + \mathcal{O}(1-x)$$

- Confirmed conjectured [Dokshitzer, Marchesini, Salam '05][Basso, Korchemsky '06] relations $C = A^2$ and $D = A \cdot (B - \frac{\beta}{\alpha_s})$ through four loops.

# Large $x$ limit

Universal all-order structure in $x \to 1$ limit [Dokshitzer, Marchesini, Salam '05]:

$$P_{\text{ns}}^{(3)\pm,V} = A_4 \left[\frac{1}{1-x}\right]_+ + B_4\,\delta(1-x) + C_4 \log(1-x) + D_4 - A_4 + \mathcal{O}(1-x)$$

- Confirmed conjectured [Dokshitzer, Marchesini, Salam '05][Basso, Korchemsky '06] relations $C = A^2$ and $D = A \cdot (B - \frac{\beta}{\alpha_s})$ through four loops.
- Cusp anomalous dimension $A_4$ reproduced [Grozin '18] [Henn, Korchemsky, Mistlberger '19] [von Manteuffel, Panzer, Schabinger '20].

# Large $x$ limit

Universal all-order structure in $x \to 1$ limit [Dokshitzer, Marchesini, Salam '05]:

$$P_{\text{ns}}^{(3)\pm,V} = A_4 \left[\frac{1}{1-x}\right]_+ + B_4\,\delta(1-x) + C_4 \log(1-x) + D_4 - A_4 + \mathcal{O}(1-x)$$

- Confirmed conjectured [Dokshitzer, Marchesini, Salam '05][Basso, Korchemsky '06] relations $C = A^2$ and $D = A \cdot (B - \frac{\beta}{\alpha_s})$ through four loops.
- Cusp anomalous dimension $A_4$ reproduced [Grozin '18] [Henn, Korchemsky, Mistlberger '19] [von Manteuffel, Panzer, Schabinger '20].
- Analytic result for virtual anomalous dimension $B_4$ new, consistent with numerical [Das, Moch, Vogt '19].

# Virtual and rapidity anomalous dimensions

$$
\begin{aligned}
B_4 = {} & C_A^3 C_F \left(295.7 \pm 0.5\right) \\
& + C_A^2 C_F^2 \left(1219.5 \pm 2.0\right) \\
& + C_A C_F^3 \left(-687.5 \pm 1.5\right) \\
& + C_F^4 \left(196.5 \pm 1.0\right) \\
& + \frac{d_F^{abcd} d_A^{abcd}}{N_c} \left(-998.0 \pm 0.2\right) \\
& + C_A^2 C_F n_f \left(-274.466 \pm 0.01\right) \\
& + C_A C_F^2 n_f \left(-455.247 \pm 0.005\right) \\
& + C_F^3 n_f \left(80.780 \pm 0.005\right) \\
& + \frac{(d_F^{abcd})^2 n_f}{N_c} \left(-143.6 \pm 0.2\right) \\
& + C_A C_F n_f^2 \left(-\frac{88}{9}\zeta_5 + \frac{80}{3}\zeta_3\zeta_2 - \frac{80}{9}\zeta_4 - \frac{320}{9}\zeta_3 + \frac{3170}{81}\zeta_2 - \frac{193}{54}\right) \\
& + C_F^2 n_f^2 \left(\frac{368}{9}\zeta_5 - \frac{160}{9}\zeta_3\zeta_2 - \frac{2104}{27}\zeta_4 + \frac{56}{27}\zeta_3 + \frac{1244}{27}\zeta_2 - \frac{188}{27}\right) \\
& + C_F n_f^3 \left(-\frac{32}{27}\zeta_4 + \frac{304}{81}\zeta_3 + \frac{32}{81}\zeta_2 - \frac{131}{81}\right)
\end{aligned}
$$

[Das, Moch, Vogt '19]

# Virtual and rapidity anomalous dimensions

$$
\begin{aligned}
B_4 = {} & C_A^3 C_F \left(-\frac{8960}{3}\zeta_7 + \frac{1472}{3}\zeta_5\zeta_2 + \frac{32}{3}\zeta_4\zeta_3 + \frac{73333}{108}\zeta_6 + \frac{1672}{3}\zeta_3^2 + \frac{11522}{9}\zeta_5 + \frac{584}{3}\zeta_3\zeta_2 - \frac{11206}{27}\zeta_4 - \frac{152284}{81}\zeta_3 + \frac{13864}{9}\zeta_2 - \frac{373793}{648}\right) \\
& + C_A^2 C_F^2 \left(8610\zeta_7 - 2104\zeta_5\zeta_2 - 32\zeta_4\zeta_3 - \frac{5497}{2}\zeta_6 - \frac{7102}{3}\zeta_3^2 + \frac{5354}{9}\zeta_5 + \frac{2096}{9}\zeta_3\zeta_2 - \frac{60850}{27}\zeta_4 + \frac{129662}{27}\zeta_3 - \frac{46771}{27}\zeta_2 + \frac{29639}{36}\right) \\
& + C_A C_F^3 \left(-10920\zeta_7 + 2064\zeta_5\zeta_2 + 128\zeta_4\zeta_3 + \frac{79297}{18}\zeta_6 + 3220\zeta_3^2 - 976\zeta_5 - \frac{1988}{3}\zeta_3\zeta_2 + 2167\zeta_4 - 3260\zeta_3 + 1167\zeta_2 - \frac{2085}{4}\right) \\
& + C_F^4 \left(5880\zeta_7 - 384\zeta_5\zeta_2 + 64\zeta_4\zeta_3 - 2111\zeta_6 - 1152\zeta_3^2 - 2520\zeta_5 - 120\zeta_3\zeta_2 - 342\zeta_4 + 2004\zeta_3 - 450\zeta_2 + \frac{4873}{24}\right) \\
& + \frac{d_F^{abcd} d_A^{abcd}}{N_c} \left(2800\zeta_7 + 320\zeta_5\zeta_2 - 64\zeta_3\zeta_4 - \frac{1562}{9}\zeta_6 - 704\zeta_3^2 - 400\zeta_5 - 896\zeta_3\zeta_2 + \frac{32}{3}\zeta_4 - \frac{1232}{3}\zeta_3 - \frac{944}{3}\zeta_2 + 96\right) \\
& + C_A^2 C_F n_f \left(-\frac{3913}{27}\zeta_6 + \frac{416}{3}\zeta_3^2 + \frac{1130}{9}\zeta_5 - \frac{580}{3}\zeta_3\zeta_2 + \frac{1234}{9}\zeta_4 + \frac{9554}{27}\zeta_3 - \frac{41092}{81}\zeta_2 + \frac{20027}{108}\right) \\
& + C_A C_F^2 n_f \left(351\zeta_6 - \frac{1232}{3}\zeta_3^2 - \frac{7432}{9}\zeta_5 + \frac{2672}{9}\zeta_3\zeta_2 + \frac{27854}{27}\zeta_4 - \frac{15400}{27}\zeta_3 - \frac{3892}{27}\zeta_2 - \frac{7751}{54}\right) \\
& + C_F^3 n_f \left(-\frac{6434}{9}\zeta_6 + 224\zeta_3^2 + 912\zeta_5 - \frac{256}{3}\zeta_3\zeta_2 - 204\zeta_4 - 308\zeta_3 + 162\zeta_2 + 32\right) \\
& + \frac{(d_F^{abcd})^2 n_f}{N_c} \left(\frac{1792}{9}\zeta_6 + 256\zeta_3^2 - 1120\zeta_5 + 64\zeta_3\zeta_2 - \frac{352}{3}\zeta_4 - \frac{992}{3}\zeta_3 + \frac{1888}{3}\zeta_2 - 192\right) \\
& + C_A C_F n_f^2 \left(-\frac{88}{9}\zeta_5 + \frac{80}{3}\zeta_3\zeta_2 - \frac{80}{9}\zeta_4 - \frac{320}{9}\zeta_3 + \frac{3170}{81}\zeta_2 - \frac{193}{54}\right) \\
& + C_F^2 n_f^2 \left(\frac{368}{9}\zeta_5 - \frac{160}{9}\zeta_3\zeta_2 - \frac{2104}{27}\zeta_4 + \frac{56}{27}\zeta_3 + \frac{1244}{27}\zeta_2 - \frac{188}{27}\right) \\
& + C_F n_f^3 \left(-\frac{32}{27}\zeta_4 + \frac{304}{81}\zeta_3 + \frac{32}{81}\zeta_2 - \frac{131}{81}\right)
\end{aligned}
$$

[Gehrmann, von Manteuffel, VS, Yang '26]

# Virtual and rapidity anomalous dimensions

$$B_4 = C_A^3 C_F \left(-\frac{8960}{3}\zeta_7 + \frac{1472}{3}\zeta_5\zeta_2 + \frac{32}{3}\zeta_4\zeta_3 + \frac{73333}{108}\zeta_6 + \frac{1672}{3}\zeta_3^2 + \frac{11522}{9}\zeta_5 + \frac{584}{3}\zeta_3\zeta_2 - \frac{11206}{27}\zeta_4 - \frac{152284}{81}\zeta_3 + \frac{13864}{9}\zeta_2 - \frac{373793}{648}\right)$$
$$+ C_A^2 C_F^2 \left(8610\zeta_7 - 2104\zeta_5\zeta_2 - 32\zeta_4\zeta_3 - \frac{5497}{2}\zeta_6 - \frac{7102}{3}\zeta_3^2 + \frac{5354}{9}\zeta_5 + \frac{2096}{9}\zeta_3\zeta_2 - \frac{60850}{27}\zeta_4 + \frac{129662}{27}\zeta_3 - \frac{46771}{27}\zeta_2 + \frac{29639}{36}\right)$$
$$+ C_A C_F^3 \left(-10920\zeta_7 + 2064\zeta_5\zeta_2 + 128\zeta_4\zeta_3 + \frac{79297}{18}\zeta_6 + 3220\zeta_3^2 - 976\zeta_5 - \frac{1988}{3}\zeta_3\zeta_2 + 2167\zeta_4 - 3260\zeta_3 + 1167\zeta_2 - \frac{2085}{4}\right)$$
$$+ C_F^4 \left(5880\zeta_7 - 384\zeta_5\zeta_2 + 64\zeta_4\zeta_3 - 2111\zeta_6 - 1152\zeta_3^2 - 2520\zeta_5 - 120\zeta_3\zeta_2 - 342\zeta_4 + 2004\zeta_3 - 450\zeta_2 + \frac{4873}{24}\right)$$
$$+ \frac{d_F^{abcd} d_A^{abcd}}{N_c} \left(2800\zeta_7 + 320\zeta_5\zeta_2 - 64\zeta_3\zeta_4 - \frac{1562}{9}\zeta_6 - 704\zeta_3^2 - 400\zeta_5 - 896\zeta_3\zeta_2 + \frac{32}{3}\zeta_4 - \frac{1232}{3}\zeta_3 - \frac{944}{3}\zeta_2 + 96\right)$$
$$+ \left(-\frac{2912}{9}\zeta_5 - \frac{416}{9}\zeta_3\zeta_2 - \frac{1120}{27}\zeta_4 + \frac{580}{27}\zeta_3 + \frac{1324}{27}\zeta_2 - \frac{9554}{27}\right)$$
$$+ C_F n_f \left(\frac{1280}{9}\zeta_5 - \frac{64}{9}\zeta_3\zeta_2 - \frac{1120}{27}\zeta_4 + \frac{2752}{27}\zeta_3 + \frac{1324}{27}\zeta_2 - \frac{41092}{27}\right)$$
$$+ C_F n_f^2 \left(-\frac{32}{27}\zeta_4 + \frac{304}{81}\zeta_3 + \frac{32}{81}\zeta_2 - \frac{131}{81}\right)$$

[Gehrmann, von Manteuffel, VS, Yang '26]

Last missing ingredient to get analytic result for four-loop rapidity anomalous dimension

$$\gamma_r^{(3)q}\Big|_{n_f=4} = 0.61362 \pm 5\cdot 10^{-5} \qquad \gamma_r^{(3)g}\Big|_{n_f=4} = 1.58840 \pm 5\cdot 10^{-5}$$

[Moult, Zhu, Zhu '21]
[Duhr, Mistlberger, Vita '21]

(and other anomalous dimensions [Moch, Vogt '26]).

# Small $x$ limit

Expand exact results to obtain all logarithmic contributions, for $N_c = 3$:

$$P_{\rm ns}^{(3)+} = \frac{256}{729}\ln^6 x + \left(\frac{3200}{243} - \frac{256}{243}n_f\right)\ln^5 x + \left(-\frac{14240}{243}\zeta_2 + \frac{89008}{243} - \frac{7616}{243}n_f + \frac{64}{81}n_f^2\right)\ln^4 x$$
$$+ \left\{\frac{1024}{243}\zeta_3 - \frac{85504}{81}\zeta_2 + \frac{1062692}{243} + \left(\frac{2368}{27}\zeta_2 - \frac{43208}{81}\right)n_f + \frac{4016}{243}n_f^2 - \frac{32}{243}n_f^3\right\}\ln^3 x$$
$$+ \quad \cdots$$

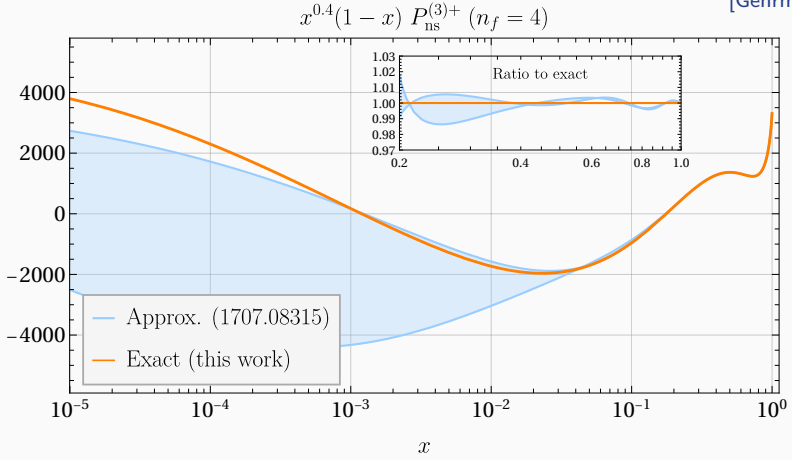
- Subleading power in $x$ compared to singlet (no $1/x$ pole).
- $\ln^3 x$ and beyond not previously known.

✓ LL and NLL agree with [Davies, Kom, Moch, Vogt '22].

⚠ NNLL: deviations proportional to $\zeta_2$ for all color structures, conspire to agree at leading color.

# Comparison to earlier approximations I

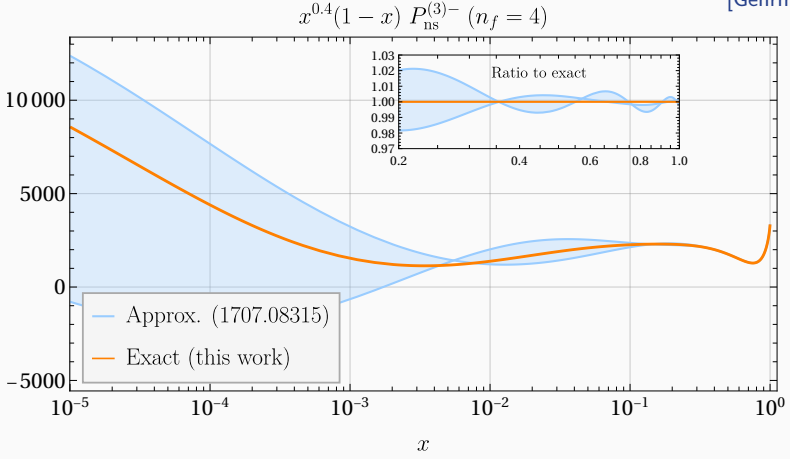
[Gehrmann, von Manteuffel, VS, Yang '26]




Approximation from [Moch, Ruijl, Ueda, Vermaseren, Vogt '17]

# Comparison to earlier approximations II

[Gehrmann, von Manteuffel, VS, Yang '26]




Approximation from [Moch, Ruijl, Ueda, Vermaseren, Vogt '17]

# Conclusions and outlook

# Conclusions and outlook

## Conclusions

- Non-singlet four-loop splitting functions complete, also timelike.
- Analytical result for virtual and rapidity anomalous dimensions.
- Precise fit for PDF evolution provided.

## Outlook

- Towards complete four-loop splitting functions.
- Renormalization in singlet sector.
- Gaps in understating of small-$x$ resummation (hint at "super-leading" logarithms)?

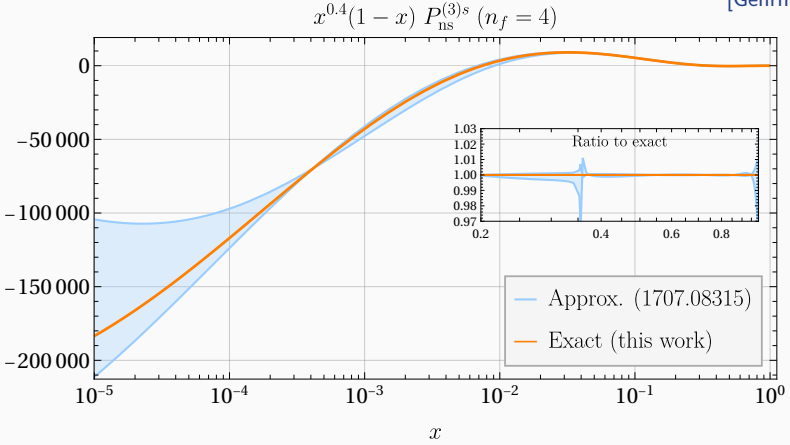
# Funding statement

This work has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme grant agreement 101019620 (ERC Advanced Grant TOPUP).

# Backup

# Comparison to earlier approximations: $P_{\rm ns}^{(3)s}$

[Gehrmann, von Manteuffel, VS, Yang '26]




Approximation from [Moch, Ruijl, Ueda, Vermaseren, Vogt '17]

## Stats for experts

- Complete sets of standard and linear propagators ("integral families") : $52$.
- Top-level sectors with $13$ different denominators: $549$.
- Unreduced integrals: $\sim 3 \cdot 10^6$.
- Symmetry relations alone leave $220k$, up to rank 5 numerators, up to 4 dots.
- "Standard" IBP relations with $\sim 9k$ spanning sectors: reduce to $120k$ integrals.
- Symmetries and "anomalous" relations leave $6k$ master integrals.
- Computed $300$ differential equations (DEs).
- DEs contain up to 2k master integrals, commonly 12 per sector.

# Non-singlet renormalization

$$
\begin{aligned}
Z_{\mathsf{ns}} =& 1 + a_s \frac{\gamma_{\mathsf{ns}}^{(0)}}{\epsilon} + a_s^2 \left( \frac{\gamma_{\mathsf{ns}}^{(1)}}{2\epsilon} + \frac{1}{2\epsilon^2} \left[ -\beta_0 \gamma_{\mathsf{ns}}^{(0)} + \left(\gamma_{\mathsf{ns}}^{(0)}\right)^2 \right] \right) \\
&+ a_s^3 \left( \frac{1}{3\epsilon} \gamma_{\mathsf{ns}}^{(2)} + \frac{1}{6\epsilon^2} \left[ -2\beta_1 \gamma_{\mathsf{ns}}^{(0)} - 2\beta_0 \gamma_{\mathsf{ns}}^{(1)} + 3\gamma_{\mathsf{ns}}^{(0)} \gamma_{\mathsf{ns}}^{(1)} \right] + \frac{1}{6\epsilon^3} \left[ 2\beta_0^2 \gamma_{\mathsf{ns}}^{(0)} - 3\beta_0 \left(\gamma_{\mathsf{ns}}^{(0)}\right)^2 + \left(\gamma_{\mathsf{ns}}^{(0)}\right)^3 \right] \right) \\
&+ \frac{a_s^4}{24} \left( \frac{1}{\epsilon^4} \left[ -6\beta_0^3 \gamma_{\mathsf{ns}}^{(0)} - 6\beta_0 (\gamma_{\mathsf{ns}}^{(0)})^3 + 11\beta_0^2 (\gamma_{\mathsf{ns}}^{(0)})^2 + (\gamma_{\mathsf{ns}}^{(0)})^4 \right] \right. \\
&\quad + \frac{1}{\epsilon^3} \left[ 6\beta_0^2 \gamma_{\mathsf{ns}}^{(1)} - 14\beta_0 \gamma_{\mathsf{ns}}^{(0)} \gamma_{\mathsf{ns}}^{(1)} + 12\beta_0 \beta_1 \gamma_{\mathsf{ns}}^{(0)} + 6\gamma_{\mathsf{ns}}^{(1)} (\gamma_{\mathsf{ns}}^{(0)})^2 - 8\beta_1 (\gamma_{\mathsf{ns}}^{(0)})^2 \right] \\
&\quad \left. + \frac{1}{\epsilon^2} \left[ -6\beta_0 \gamma_{\mathsf{ns}}^{(2)} - 6\beta_1 \gamma_{\mathsf{ns}}^{(1)} - 6\beta_2 \gamma_{\mathsf{ns}}^{(0)} + 8\gamma_{\mathsf{ns}}^{(0)} \gamma_{\mathsf{ns}}^{(2)} + 3(\gamma_{\mathsf{ns}}^{(1)})^2 \right] + \frac{6\gamma_{\mathsf{ns}}^{(3)}}{\epsilon} \right) + \mathcal{O}(a_s^5)
\end{aligned}
$$