Heavy-Quark Production in Deep-Inelastic Scattering

Mellin Moments of Structure Functions

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DIS 2026

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CERN-TH-2026-014, DESY-26-013

Heavy-quark production in deep-inelastic scattering
– Mellin moments of structure functions –

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Abstract

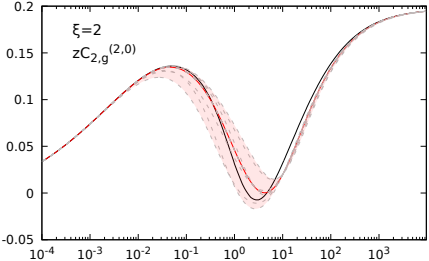
We compute Mellin moments of the heavy-quark structure functions in deep-inelastic scattering at next-to-leading order in quantum chromodynamics, retaining their full dependence on the heavy-quark mass. Using the optical theorem and the operator

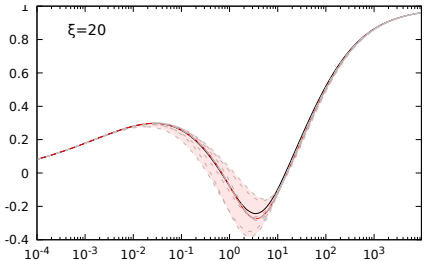
Motivation

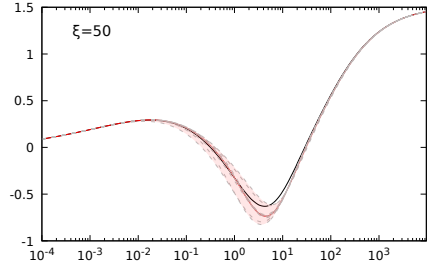
- At NNLO all light-quark coefficient functions are analytically known [Moch, Vermaseren, Vogt '04].
- Heavy-quark coefficient functions are known analytically at NLO in the pure-singlet [Blümlein et al. '19] and non-singlet [Buza et al. '96; Blümlein, Falcioni, De Freitas '16] channels.
- The gluon channel remains analytically unknown at NLO; only asymptotic results are available via for example operator matrix elements [Bierenbaum, Blümlein, Klein '08].
- All three-loop operator matrix elements are available at NNLO [Blümlein et al. '10-'25], yet the full coefficient functions remain unknown, perhaps for a while.
- Mellin moments constrain approximations to the full result, each additional moment significantly tightens the allowed region for the coefficient function.

Motivation

An approximation of the NLO gluon coefficient function [Laurenti '21]







$$\xi = -\frac{q^2}{m^2}$$

$$\eta = \frac{\xi}{4}\frac{1-z}{z} - 1$$

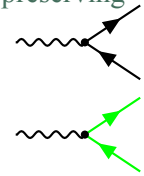
Hadronic Tensor

unpolarised hadronic state

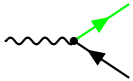
electroweak current

$J_\mu = \bar{\psi}\left(Q^v + Q^a\gamma_5\right)\gamma_\mu\psi$

mass preserving



mass changing



$$W_{\mu\nu} = \frac{1}{4\pi}\int \mathrm{d}^4 z\; e^{iq\cdot z}\langle \mathrm{P}\,|\,J_\mu^\dagger(z)\,J_\nu(0)\,|\,\mathrm{P}\rangle$$

tensor decomposition

$$W_{\mu\nu} = \frac{1}{2x}\,e_{\mu\nu}\,F_L + \frac{1}{2x}\,d_{\mu\nu}\,F_2 + i\,\epsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{p\cdot q}F_3$$

symmetric tensors

structure functions F_L, F_2 and F_3

Operator Product Expansion

hadronic operator matrix elements

$$\langle \mathrm{P} | \mathcal{O}_k^{\{\lambda_1 \cdots \lambda_N\}} | \mathrm{P} \rangle = p^{\lambda_1} \cdots p^{\lambda_N} A^N_{\mathrm{P},k}(p)$$

time-ordered product

$$T_{\mu\nu} = i \int \mathrm{d}^4 z \, e^{iq\cdot z} \langle \mathrm{P} | \mathscr{T} \{ J^\dagger_\mu(z) \, J_\nu(0) \} | \mathrm{P} \rangle$$

operator product expansion
on the light-cone $z^2 = 0$ for $x \to \infty$

$$T_{\mu\nu} = \sum_{N \in \mathbb{N}} \sum_{k \in \{a,\mathrm{q},\mathrm{g}\}} \left(\frac{1}{2x} \right)^N A^N_{\mathrm{P},k}(p) \left[e_{\mu\nu} \, C^N_{L,k}(q) + d_{\mu\nu} \, C^N_{2,k}(q) + i \, \epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p \cdot q} \, C^N_{3,k}(q) \right] + \mathcal{O}(p^2)$$

leading twist approximation

tensor decomposition

$$T_{\mu\nu} = e_{\mu\nu} \, T_L + d_{\mu\nu} \, T_2 + i \, \epsilon_{\mu\nu\alpha\beta} \frac{p^\alpha q^\beta}{p \cdot q} \, T_3$$

hadronic invariants T_L, T_2 and T_3

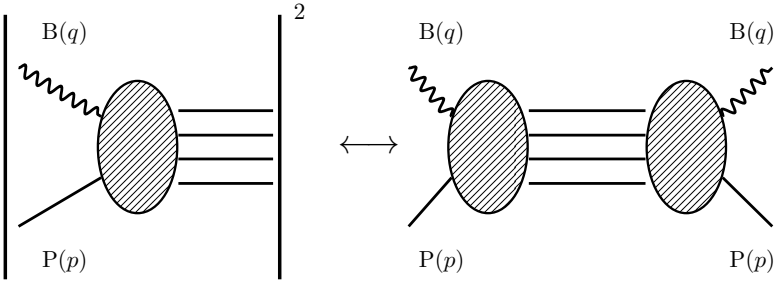
Forward Scattering

optical theorem

$$W_{\mu\nu} = \frac{1}{2\pi} \operatorname{Im} T_{\mu\nu}$$

projectors

$$T_i = \mathscr{P}^{\mu\nu} T_{\mu\nu}$$



$$\frac{1}{2\pi} \operatorname{Im} T_i = \frac{1}{2x} F_i$$

moments of structure functions

$$F_i^N \equiv \int_0^1 x^{N-2} F_i = \sum_{k \in \{a,\mathrm{q},\mathrm{q}\}} \frac{A_{\mathrm{P},k}^N(p)\, C_{i,k}^N(q)}{2^{N+1}}$$

Harmonic Projection

- Symmetric, traceless tensors are well-studied in classical physics (e. g. multipole expansions); when normalised appropriately, they are termed harmonic tensors,

$$q_{\mu_i} H_N^{\mu_1 \cdots \mu_i \cdots \mu_n} = q^2 \, H_{N-1}^{\mu_1 \cdots \widehat{\mu_i} \cdots \mu_n}$$

- The harmonic projection operator [Larin et al. '96] annihilates polynomials, by taking derivatives and nullifying the partonic momentum.

$$\mathscr{P}_M(\,\cdot\,) = \frac{H_q^{\mu_1 \cdots \mu_M}}{M\,!} \frac{\partial^M}{\partial p^{\mu_1} \cdots \partial p^{\mu_M}} (\,\cdot\,) \Big|_{p=0}$$

moments of hadronic invariants

$$\mathscr{P}_M(T_i) = (-1)^M \sum_{k \in \{a,\mathrm{q},\mathrm{g}\}} A_{\mathrm{P},k}^N(p) \, C_{i,k}^N(q) \Big|_{p=0} \equiv T_i^N(q)$$

no derivatives of OMEs

$$A(p) = A(p^2) \implies \mathrm{d}A \sim p \, \partial A$$

Renormalisation

- Whilst hadronic operator matrix elements are non-perturbative but the coefficients functions are universal

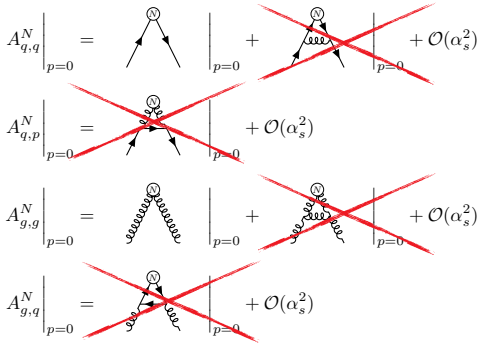
hadronic states ⟶ partonic states

- Harmonic projection alone does not allow one to extract coefficient functions without complete knowledge of the operator matrix elements unless the are evaluated in a momentum subtraction scheme…

mass factorisation

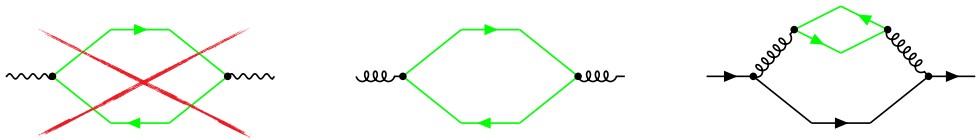
$$T_{i,\mathrm{p}}^{N,\mathrm{ren}} = (-1)^N \sum_{j\in\{a,\mathrm{q},\mathrm{g}\}} C_{N,j}^{i}\, Z_{\mathrm{jp}}\, A_{\mathrm{p},j}^{N,(0),\mathrm{ren}}(p)\Big|_{p=0}$$

may be calculated using Feynman graphs



Renormalisation

- The moments of the forward amplitudes are calculated in dimensional regularisation at and require charge, wave-function, and mass renormalisation
- The coupling is renormalised with $n_l + n_h$ active flavors, but since mass factorisation uses only n_l, light flavours the discrepancy must be corrected via decoupling relations [Chetyrkin, Kniehl, Steinhauser '99]
- In contrast to the massless case, wave-function renormalisation, does not fully cancel; heavy-quark insertions must be included [Chen, Li, Niggetiedt '25],



- Likewise, masses must be renormalised, which is achieved through the insertion of appropriate counterterms.

Setup

- Graphs contributing to the forward-scattering amplitude are generated with the built-in graph generator in FORM 5 [Davies et al. '26],

$$T_{i,\mathrm{q}} = T_i^{\mathrm{qq}} \qquad T_{i,\mathrm{g}} = T_i^{\mathrm{gg}} + 2T_i^{\mathrm{cc}}$$

- External gluons are treated in an unphysical gauge, which necessitates the introduction of external ghosts, while internal gluons are for small moments in a covariant gauge.
- The workflow is similar to the massless calculation: assign Feynman rules, split and calculate colour [Ritbergen, Schellekens, Vermaseren '99] and flavour factors, apply the harmonic projection, etc.

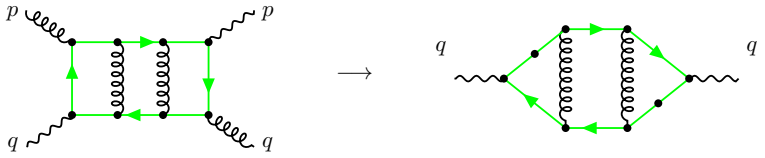
no Forcer [Ruijl, Ueda, Vermaseren '17] available

integral reductions — master integrals

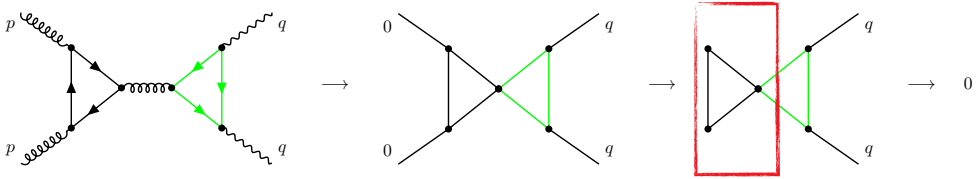
Kira 3 [Lange, Usovitsch, Wu '25] — differential equations

Setup

- Graphically, the harmonic projection amputates external partonic edges while preserving the vertex structure, reducing four-point functions to two-point functions.



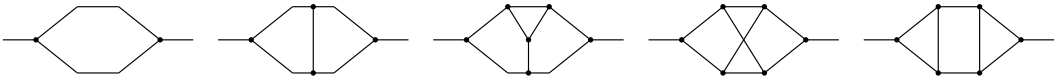
- Scaleless graphs contain articulation points whose removal, after harmonic projection and contraction, splits the graph into disjoint components with no masses or external momenta.



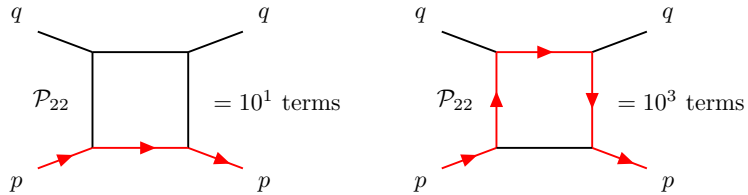
Setup

- Laporta families generally require auxiliary propagators. A specific choice allows up to three loops to express all internal momenta with the same momentum routing.

$$\mathscr{R}_3 = \{k_1, k_1+q, k_2, k_2+q, k_3, k_3+q, k_1-k_2, k_3-k_2, k_1-k_2+k_3\}$$



- As a derivative operator, the harmonic projection generates the more terms the more edges depend on the partonic momentum; optimal momentum routing minimises them.

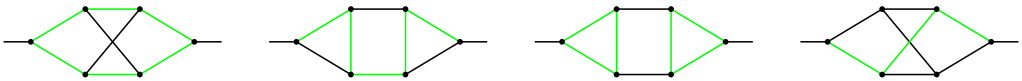


Integral Families

- Adding arbitrary auxiliary propagators can generate inequivalent integral families, even if the integrals are identical. The goal is to minimise the number of required IBP reductions.
- Incomplete families can be assigned an equivalence class by considering all non-degenerate momentum shifts that map to the momentum routing.
- Working with equivalence classes allows to directly translate the problem into set covering problem, allowing to find the exact minimum number of integral families
- The independent equivalence classes generate the universe, while reference families, that is the momentum routings with masses taken modulo symmetries, define the covering sets.
- Integral families are determined separately for mass preserving and changing currents, yielding $2 + 1$, $2 + 2$, and $6 + 6$ families at one, two, and three loops.

Master Integrals

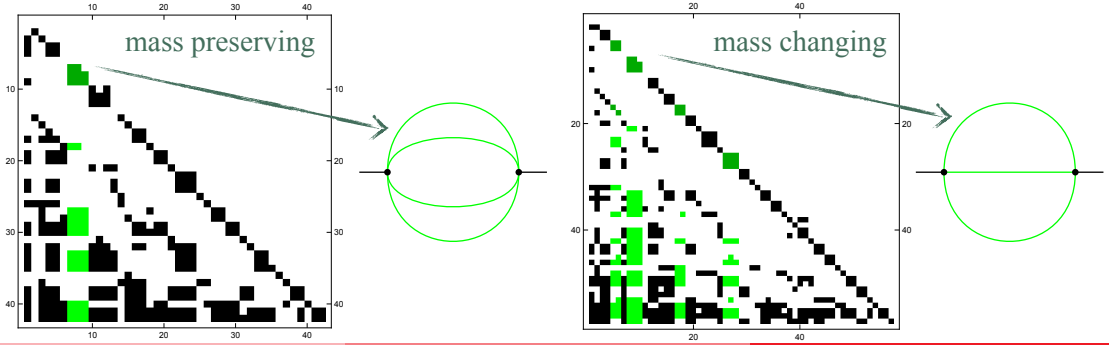
- For mass preserving currents at three loops and , there are roughly 15 million integrals to be reduced with Kira. Integrals have many dots and scalar products.
- At three loops, there are 65 + 57 unique master integrals for the mass-preserving and mass-changing currents, respectively



- All QED-type integrals have been computed previously, [Duhr et al. '24, Former, Tancredi, Nega '24], while the remaining integrals are structurally very similar.
- Using a dot basis makes sector symmetries manifest and enables a straightforward computation of boundary conditions via expansion by subgraphs [Smirnov '89].
- Resulting constants are expressed in terms of polylogarithms at the sixth roots of unity [Henn, Smirnov, Smirnov '16; Kniehl, Pikelner, Veretin '17] up to weight five.

Master Integrals

- The master integrals were computed using the method of differential equations [Kotikov '91, Remiddi '96].
- Polylogarithmic systems are cast into canonical form using Canonica [Henn '13, Meyer '17], whilst elliptic systems transformed by splitting the Wronskian [Görges et al. '23].
- The factorised differential equation contains 26 + 16 integral kernels, all but two coincide exactly with the QED-type integral kernels.



Results

- Moments were calculated independently using differential equations for the unexpanded forward-scattering amplitude, yielding full agreement.
- The results are rather lengthy. At two loops, they involve harmonic polylogarithms $\mathrm{H}_{\mathbf{a}}(\lambda)$, zeta values ζ_n and numerator polynomials $P_{i,j,k}$. In the gluon channel the moment, reads

$$C_{2,g}^2 = a_{s,\mathrm{ren}} \left[T_F \langle h_{\{V,V\}} \rangle^g n_h \left(\frac{4\mathrm{H}_0\, \mathrm{P}_{g,2,02}}{3(\lambda-1)(\lambda+1)^3} + \frac{\mathrm{P}_{g,2,01}}{(\lambda+1)^2} \right) - T_F \langle l_{\{V,V\}} \rangle^g n_l \right]$$
$$+ a_{s,\mathrm{ren}}^2 \left[C_A T_F \langle h_{\{V,V\}} \rangle^g n_h \left(\frac{4\mathrm{H}_{-1,0}\, \mathrm{P}_{g,2,13}}{45(\lambda-1)\lambda^3(\lambda+1)^3} + \cdots \right) \right.$$
$$\left. + \left(\frac{115}{162} - 4\,\zeta_3 \right) C_A T_F \langle l_{\{V,V\}} \rangle^g n_l + \left(\frac{32\,\zeta_3}{5} - \frac{4799}{405} \right) C_F T_F \langle l_{\{V,V\}} \rangle^g n_l \right]$$

$\xi = \frac{(1-\lambda)^2}{\lambda}$

flavour factors

- At leading order, the analytic moments are exactly reproduced; at NLO, the massless limit $n_h \to 0$ is recovered [Witten '76, Larin et al.'96]

Results

- In the asymptotic small mass limit the moments factorise correctly in the known operator matrix elements and massless coefficient functions [Kawamura et al. '12].
- For a consistent numerical comparison with [Riemersma et al. '95] at NLO, the heavy-quark form factor must be subtracted twice

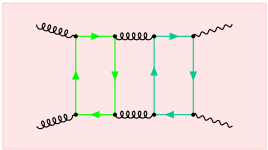
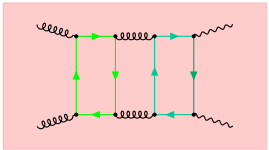
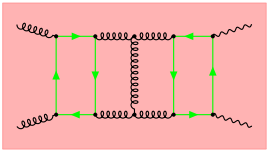
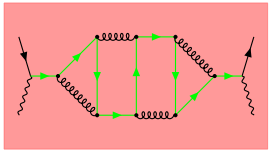
$$C_{2,\mathrm{q}}^{N,\mathrm{ns}}\Big|_{\mathrm{RSvN}} = C_{2,\mathrm{q}}^{N,\mathrm{ns}}\Big|_{\mathrm{KMS}} - 2\,F_{\mathrm{h}}^{(2)}(\kappa)$$

- Numerical agreement with semi-analytical result. Moreover, this allows an estimate of their accuracy. For the NLO contribution to C , consider $Q = 7, m = 2$ and $\mu_r = \mu_f = Q$:

N	KMS [this work]	RSvN [Riemersma et al. (1995)]	LeProHQ [Hekhorn (2019)]
2	$1.6065 \times 10^{+1}$	$1.6\ 121 \times 10^{+1}$	$1.60\ 47 \times 10^{+1}$
4	$3.6805 \times 10^{+0}$	$3.68\ 72 \times 10^{+0}$	$3.6\ 743 \times 10^{+0}$
⋮	⋮	⋮	⋮
22	7.9061×10^{-5}	$7.9\ 104 \times 10^{-5}$	$7.\ 8587 \times 10^{-5}$

Outlook

- At NNLO, for the moments $N = 2,4,6$ the calculation is nearly complete for both vector and axial currents; IBP reductions and assembly remaining
- Coefficient functions for polarised partons can be computed using the same master integrals. As in the massless case, the formalism is versatile and allows treatment of scalar and tensor currents.
- Moments involving more than one mass are relevant due to the closeness of the charm and bottom quark masses, $m_c \approx m_b$, [Ablinger et al. '25] or even a third mass.

Thank you for your attention.