# THE DIS DIPOLE PICTURE CROSS SECTION IN EXACT KINEMATICS

33RD INTERNATIONAL WORKSHOP ON DEEP INELASTIC SCATTERING AND RELATED SUBJECTS (DIS2026) - BOLOGNA

**Magnus Bertilsson**

University of Jyväskylä

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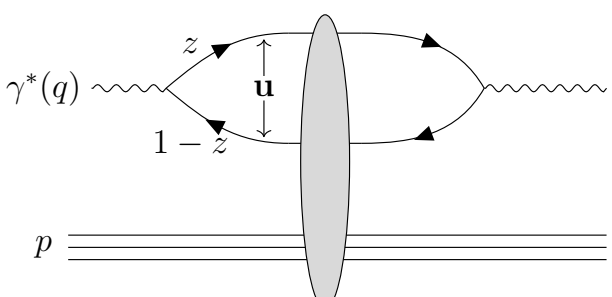
Based on arXiv:2601.07302 with T. Lappi, H. Mäntysaari, X.B. Tong

# THE DIPOLE PICTURE AND DIS

- Requirement : High precision theory calculations at small $x$ for EIC.

$$F_L(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{\text{em}}}\sigma_L^{\gamma^* p}, \qquad F_2(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{\text{em}}}\left(\sigma_L^{\gamma^* p} + \sigma_T^{\gamma^* p}\right).$$

- The total cross section is usually calculated with the Optical Theorem (OT).




$$\Leftrightarrow \quad \sigma_{L,T}^{\gamma^* p,\text{OT}} = \sigma_0 \int_{z,\mathbf{u}} N(\mathbf{u}) \left|\Psi_{L,T}^{\gamma^* q\bar{q}}(z,|\mathbf{u}|)\right|^2$$

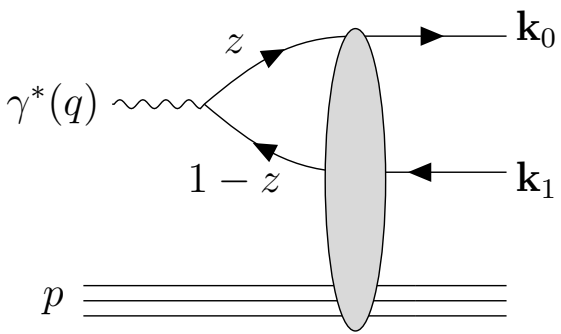
- Problem : Implicitly assumes asymptotically large $\gamma^* p$ center-of-mass energy, $W$.

# THE FINITE ENERGY CONSTRAINT

- Colliders operate at finite energy $\Rightarrow M^2_{q\bar{q}} \leq W^2$.

$$M^2_{q\bar{q}} = (k_0 + k_1)^2 = \frac{\mathbf{P}^2 + m_q^2}{z(1-z)}$$

$$\mathbf{P} = (1-z)\mathbf{k}_0 - z\mathbf{k}_1$$




- Finite energy constrains final state phase space.
- Purpose : Impose $M^2_{q\bar{q}} \leq W^2$ . Allows us to
  1. derive a finite $W$ constrained $\sigma_{\text{tot}}^{\gamma^* p}$ in the dipole picture at leading order.
  2. numerically quantify importance of constraint for light and heavy quarks.

# Part I

## FINITE-ENERGY CONSTRAINED CROSS SECTION

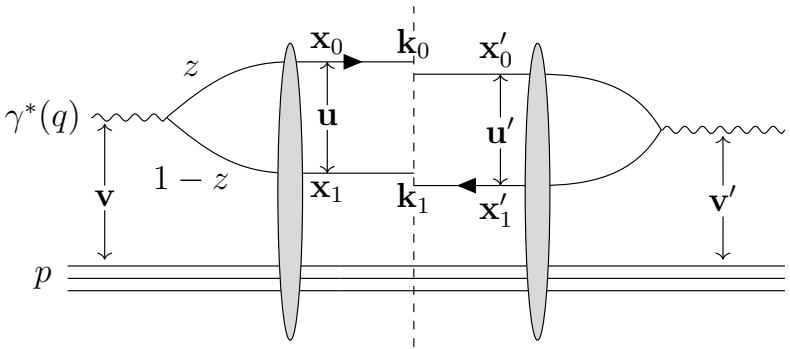
# DI-JET DIS CROSS SECTION IN THE DIPOLE PICTURE

Dipole size and center of mass

$$\mathbf{u} = \mathbf{x}_0 - \mathbf{x}_1, \quad \mathbf{v} = z\mathbf{x}_0 + (1-z)\mathbf{x}_1,$$
$$\mathbf{u}' = \mathbf{x}'_0 - \mathbf{x}'_1, \quad \mathbf{v}' = z\mathbf{x}'_0 + (1-z)\mathbf{x}'_1$$

Final state momenta

$$\mathbf{P} = (1-z)\mathbf{k}_0 - z\mathbf{k}_1, \quad \mathbf{K} = \mathbf{k}_0 + \mathbf{k}_1$$




The di-jet dipole picture cross section

$$\frac{\mathrm{d}\sigma^{\gamma^* p}_{L,T}}{\mathrm{d}^2\mathbf{K}\mathrm{d}^2\mathbf{P}} = \int_{z,\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'} e^{i\mathbf{K}\cdot(\mathbf{v}'-\mathbf{v})} e^{i\mathbf{P}\cdot(\mathbf{u}'-\mathbf{u})} \underbrace{\mathcal{N}(z,\,\mathbf{u},\,\mathbf{v},\,\mathbf{u}',\,\mathbf{v}')}_{\text{Target interaction}} \underbrace{\Psi^{\gamma^* q\bar{q}}_{L,T}(z,\,|\mathbf{u}|)\Psi^{\gamma^* q\bar{q}}_{L,T}(z,\,|\mathbf{u}'|)^\dagger}_{\text{Lightcone wavefunction squared}}$$

Target interaction $\Leftrightarrow$ correlators of Wilson lines in fundamental representation

$$\left.\begin{aligned}\mathcal{N}(z,\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}') = \mathbb{1} &- \frac{1}{N_c}\underbrace{\left\langle \mathrm{Tr}\left[U(\mathbf{x}_0)U^\dagger(\mathbf{x}_1)\right]\right\rangle}_{\text{dipole}} - \frac{1}{N_c}\left\langle \mathrm{Tr}\left[U(\mathbf{x}'_1)U^\dagger(\mathbf{x}'_0)\right]\right\rangle \\ &+ \underbrace{\frac{1}{N_c}\left\langle \mathrm{Tr}\left[U(\mathbf{x}_0)U^\dagger(\mathbf{x}_1)U(\mathbf{x}'_1)U^\dagger(\mathbf{x}'_0)\right]\right\rangle}_{\text{quadrupole}}\end{aligned}\right\}, \quad \begin{aligned}&\to 0 \quad \text{as } |\mathbf{u}| \text{ and } |\mathbf{u}'| \to 0 \\ &\to 1 \quad \text{as } |\mathbf{u}| \text{ and } |\mathbf{u}'| \to \infty\end{aligned}$$

# RECOVERING THE OPTICAL THEOREM

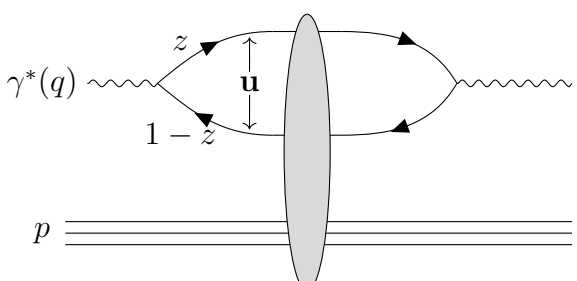
Integrating the di-jet cross section assuming translational invariance $\int_{\mathbf{v}} \to \sigma_0/2$ of the target

$$\sigma_{L,T}^{\gamma^* p} = \int_{z,\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'} \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2}\frac{\mathrm{d}^2\mathbf{P}}{(2\pi)^2} e^{i\mathbf{K}\cdot(\mathbf{v}'-\mathbf{v})} e^{i\mathbf{P}\cdot(\mathbf{u}'-\mathbf{u})} \mathcal{N}(z,\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}') \Psi_{L,T}^{\gamma^* q\bar{q}}(z,|\mathbf{u}|) \left(\Psi_{L,T}^{\gamma^* q\bar{q}}(z,|\mathbf{u}'|)\right)^\dagger$$

Total DIS cross section of DIS

$$\sigma_{L,T}^{\gamma^* p,\mathrm{OT}} = \sigma_0 \int_{z,\mathbf{u}} N(\mathbf{u}) \left|\Psi_{L,T}^{\gamma^* q\bar{q}}(z,|\mathbf{u}|)\right|^2$$

$$N(\mathbf{u}) = \mathbb{1} - \frac{1}{N_c}\left\langle \mathrm{Tr}\left[U((1-z)\mathbf{u})U^\dagger(-z\mathbf{u})\right]\right\rangle$$




Optical theorem structure functions

$$F_L^{\mathrm{OT}}(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{\mathrm{em}}}\sigma_L^{\gamma^* p,\mathrm{OT}}, \quad F_2^{\mathrm{OT}}(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{\mathrm{em}}}\left(\sigma_L^{\gamma^* p,\mathrm{OT}} + \sigma_T^{\gamma^* p,\mathrm{OT}}\right)$$

Note : At LO no dependence on CM ENERGY $W$ of $\gamma^* p$-system (No $x$ evolution)

# THE FINITE ENERGY CONSTRAINT AND TOTAL CROSS SECTION

Change variables from $\mathbf{P}$ to $M^2_{q\bar{q}}$ with delta function

$$\frac{\mathrm{d}\sigma^{\gamma^* p}_{L,T}}{\mathrm{d}M^2_{q\bar{q}}} = \int_{z,\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}'} \frac{\mathrm{d}^2\mathbf{K}}{(2\pi)^2}\frac{\mathrm{d}^2\mathbf{P}}{(2\pi)^2} e^{i\mathbf{K}\cdot(\mathbf{v}'-\mathbf{v})} e^{i\mathbf{P}\cdot(\mathbf{u}'-\mathbf{u})} \mathcal{N}(z,\mathbf{u},\mathbf{v},\mathbf{u}',\mathbf{v}') \Psi^{\gamma^* q\bar{q}}_{L,T}(z,|\mathbf{u}|) \left(\Psi^{\gamma^* q\bar{q}}_{L,T}(z,|\mathbf{u}'|)\right)^\dagger \times \delta\left(M^2_{q\bar{q}} - \frac{\mathbf{P}^2 + m_q^2}{z(1-z)}\right)$$

Total cross section at finite energy (i.e. imposing $M^2_{q\bar{q}} \leq W^2$)

$$\sigma^{\gamma^* p}_{L,T} = \int_{M^2_{\min}}^{M^2_{\max}} \mathrm{d}M^2_{q\bar{q}} \frac{d\sigma^{\gamma^* p}_{L,T}}{\mathrm{d}M^2_{q\bar{q}}} \quad , \qquad \begin{cases} M^2_{\max} = W^2 \\ M^2_{\min} = \dfrac{m_q^2}{z(1-z)} \end{cases}$$

# PURPOSE 1: THE TOTAL CROSS SECTION IN EXACT KINEMATICS

Integrating over $M^2_{q\bar{q}}$ assuming translational invariance $\int_{\mathbf{v}} \to \sigma_0/2$

✓ $$\boxed{\sigma^{\gamma^* p,\mathrm{FE}}_{L,T} = \sigma_0 \int_{z,\mathbf{u},\mathbf{u}'} \Psi^{\gamma^* q\bar{q}}_{L,T}(z,|\mathbf{u}|)\left(\Psi^{\gamma^* q\bar{q}}_{L,T}(z,|\mathbf{u}'|)\right)^\dagger \widetilde{\mathcal{N}}(z,\mathbf{u},\mathbf{u}')\frac{\zeta(M^2_{\max})J_1\left(\zeta(M^2_{\max})\right)}{2\pi|\mathbf{u}'-\mathbf{u}|^2}}$$

Compared to "OT": $W$-dependence and a multipole correlator structure in the target interaction

$$\zeta(M^2_{\max}) = |\mathbf{u}'-\mathbf{u}|\sqrt{M^2_{\max}\, z(1-z) - m_q^2}$$

$$\widetilde{\mathcal{N}}(z,\mathbf{u},\mathbf{u}') = \mathbb{1} - \frac{1}{N_c}\left\langle \mathrm{Tr}\left[U((1-z)\mathbf{u})U^\dagger(-z\mathbf{u})\right]\right\rangle - \frac{1}{N_c}\left\langle \mathrm{Tr}\left[U(-z\mathbf{u}')U^\dagger((1-z)\mathbf{u}')\right]\right\rangle$$
$$+ \frac{1}{N_c}\left\langle \mathrm{Tr}\left[U((1-z)\mathbf{u})U^\dagger(-z\mathbf{u})U(-z\mathbf{u}')U^\dagger((1-z)\mathbf{u}')\right]\right\rangle$$

Finite energy constrained structure functions

$$F_L^{\mathrm{FE}}(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{\mathrm{em}}}\sigma_L^{\gamma^* p,\mathrm{FE}}, \quad F_2^{\mathrm{FE}}(x,Q^2) = \frac{Q^2}{4\pi^2\alpha_{\mathrm{em}}}\left(\sigma_L^{\gamma^* p,\mathrm{FE}} + \sigma_T^{\gamma^* p,\mathrm{FE}}\right)$$

Expect to recover OT as $M^2_{\max} \to \infty$.

# Part II

# QUANTIFYING IMPORTANCE OF FINITE-ENERGY CONSTRAINT

# EXPRESSIONS FOR EVALUATING CORRELATORS

Dipole correlator in MV model ($Q_{s0}^2 = 0.104\,\mathrm{GeV}^2$, $\gamma = 1$)

$$\frac{1}{N_c}\left\langle \mathrm{Tr}\left[U((1-z)\mathbf{u})U^\dagger(-z\mathbf{u})\right]\right\rangle = \exp\left\{-\frac{1}{4}\left(|\mathbf{u}|^2 Q_{s0}^2\right)^\gamma \log\left(1/(|\mathbf{u}|\Lambda_{\mathrm{QCD}}) + e\right)\right\} = \exp\left\{\frac{-C_F}{2}\Gamma(|\mathbf{u}|)\right\}$$

Gaussian approximation is used to evaluate quadrupole[1]. Exact at LO[2].

$$\begin{aligned}\frac{1}{N_c}\left\langle \mathrm{TR}\left[U((1-z)\mathbf{u})U^\dagger(-z\mathbf{u})U(-z\mathbf{u}')U^\dagger((1-z)\mathbf{u}')\right]\right\rangle &= \exp\left\{\frac{-C_F}{2}\left(\Gamma(\mathbf{u}')-\Gamma(\mathbf{u})\right)\right\}\\ &\times\left[\left(\frac{\sqrt{\Delta}+G_1}{2\sqrt{\Delta}}-\frac{G_2}{\sqrt{\Delta}}\right)\exp\left\{\frac{N_c}{4}\mu^2\sqrt{\Delta}\right\}+\left(\frac{\sqrt{\Delta}-G_1}{2\sqrt{\Delta}}+\frac{G_2}{\sqrt{\Delta}}\right)\exp\left\{\frac{-N_c}{4}\mu^2\sqrt{\Delta}\right\}\right]\\ &\times\exp\left\{\mu^2\left(\frac{-N_c}{4}G_1+\frac{1}{2N_c}G_2\right)\right\}\end{aligned}$$

$G_1$, $G_2$ and $\Delta$ depend non-trivially on $\Gamma(|\mathbf{u}|)$.

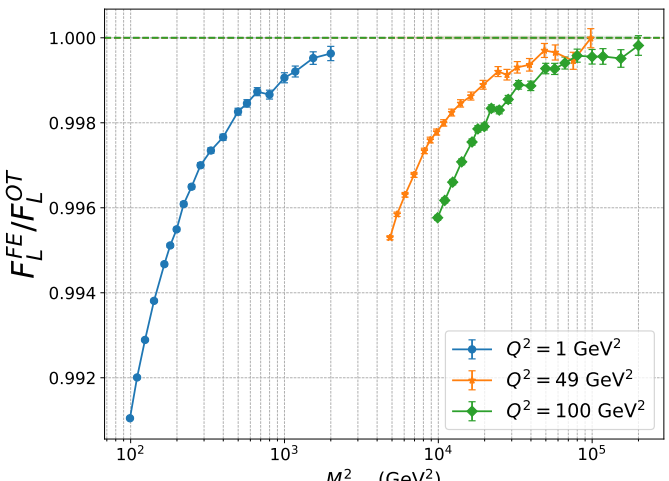
[1] See e.g. Phys. Rev. D 83, 105005, Dominguez, Marquet, Xiao and Yuan for derivation.

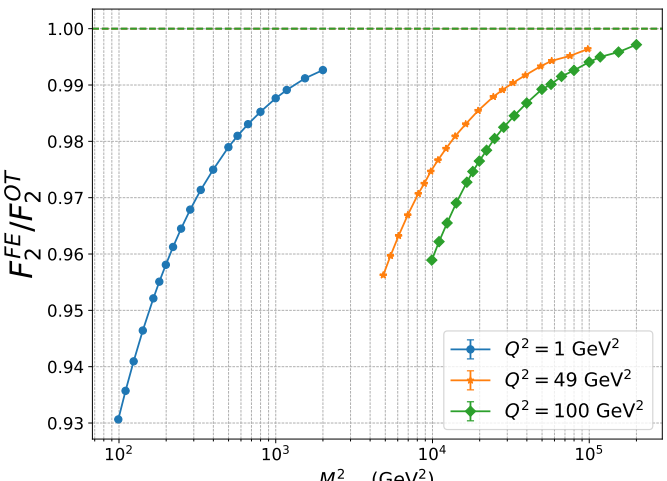
[2] Has been shown in arXiv:1108.4764, A. Dumitru, J. Jalilian-Marian, T. Lappi, B. Schenke and R. Venugopalan.

# PURPOSE 2: IMPORTANCE OF FINITE-$W$ CONSTRAINT: LIGHT QUARKS

$m_q = 0.14$ GEV

The ratio of the structure functions, $F_{L,2}^{\mathrm{FE}}/F_{L,2}^{\mathrm{OT}}$, as a function of $M_{\mathrm{max}}^2$.




$$\sigma_{L,T}^{\gamma^* p,\mathrm{OT}} = \sigma_0 \int_{z,\mathbf{u}} N(\mathbf{u}) \left|\Psi_{L,T}^{\gamma^* q\bar{q}}(z,|\mathbf{u}|)\right|^2, \quad \sigma_{L,T}^{\gamma^* p,\mathrm{FE}} = \sigma_0 \int_{z,\mathbf{u},\mathbf{u}'} \Psi_{L,T}^{\gamma^* q\bar{q}}(z,|\mathbf{u}|) \left(\Psi_{L,T}^{\gamma^* q\bar{q}}(z,|\mathbf{u}'|)\right)^\dagger \tilde{\mathcal{N}}(z,\mathbf{u},\mathbf{u}') \frac{\zeta(M_{\mathrm{max}}^2) J_1(\zeta(M_{\mathrm{max}}^2))}{2\pi|\mathbf{u}'-\mathbf{u}|^2}$$

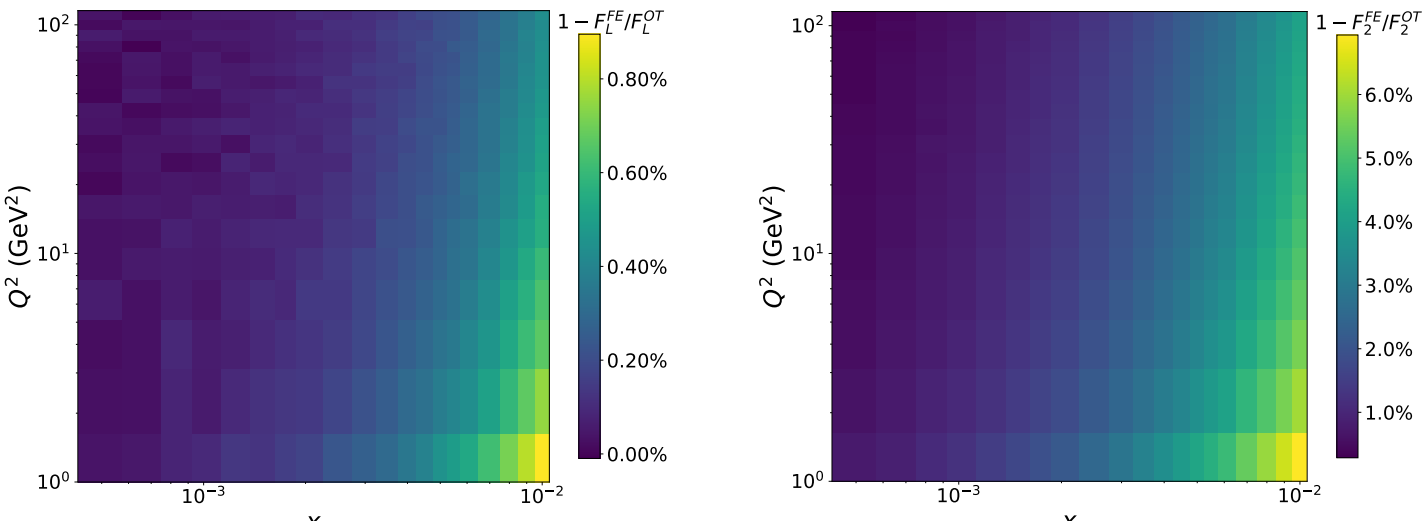
NOTE :

- Recovers OT as $M_{\mathrm{max}}^2 \to \infty$.
- Effect larger for $F_2$ than $F_L$ due to limits $z \to 0,\ 1 \Rightarrow M_{q\bar{q}}^2$ becomes large (aligned jet limit).

# PURPOSE 2: IMPORTANCE OF FINITE-$W$ CONSTRAINT: LIGHT QUARKS

$m_q = 0.14$ GEV

For physical cross sections: $M^2_{\text{max}} = W^2 \approx \frac{Q^2}{x}$. The rel. difference $1 - F^{\text{FE}}_{\text{L,2}}/F^{\text{OT}}_{\text{L,2}}$.

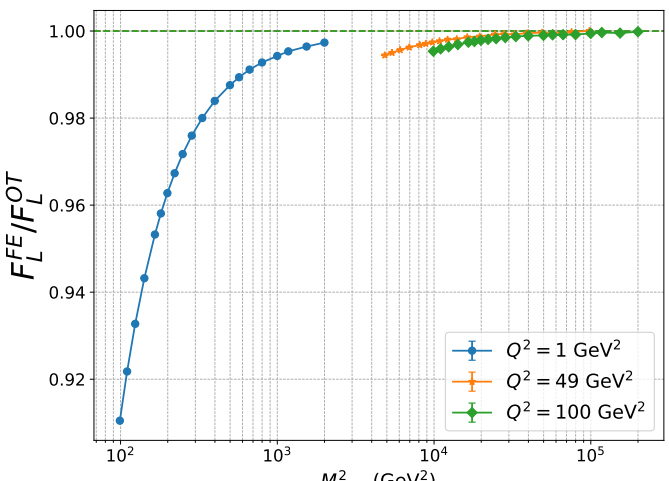


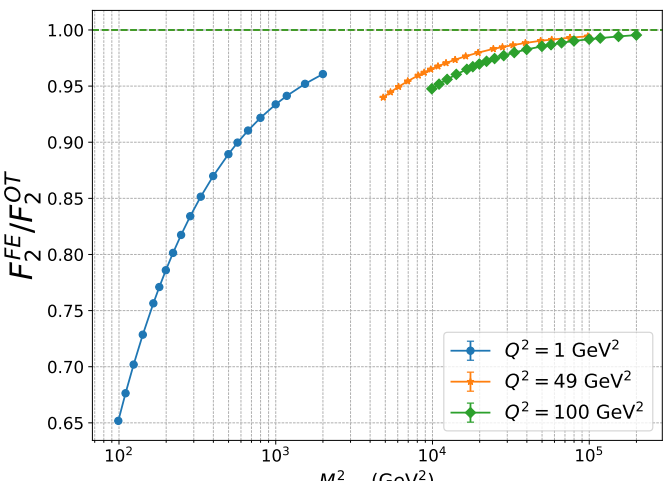

- Effect of the finite energy constraint largest at relatively small $Q^2$ and large $x$.
- Max effect: $F_L \sim 0.9\%$ and $F_2 \sim 7\%$.

# PURPOSE 2: IMPORTANCE OF FINITE-$W$ CONSTRAINT: HEAVY QUARKS

$m_q = 1.4$ GEV

The ratio of the structure functions, $F^{\mathrm{FE}}/F^{\mathrm{OT}}$, as a function of $M^2_{\mathrm{max}}$.



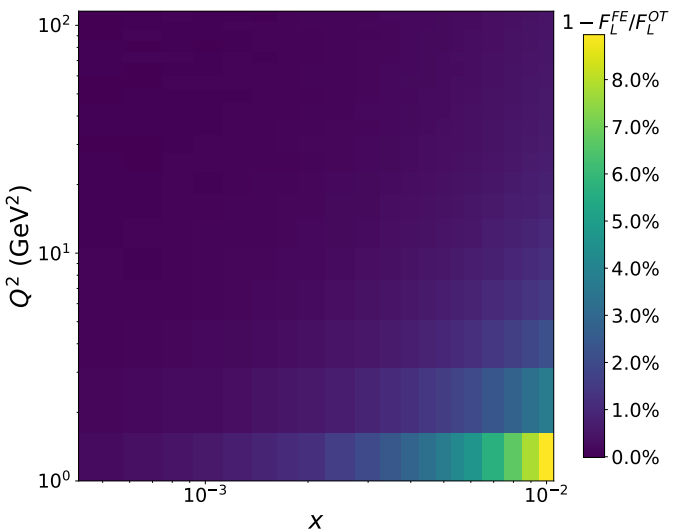

NOTE :

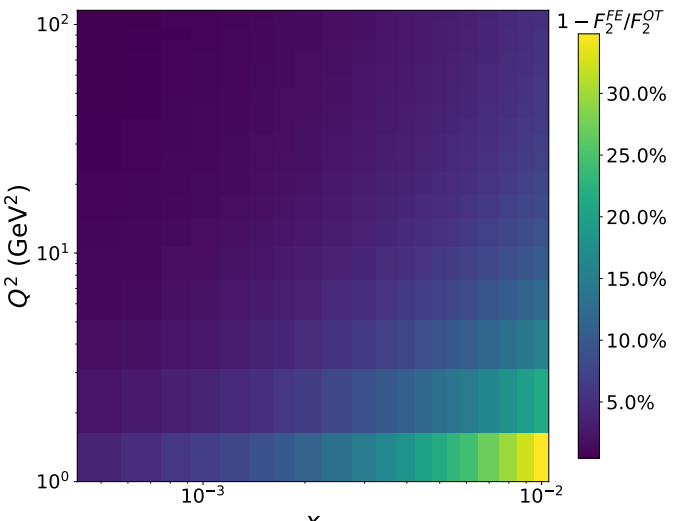
- Again recovers OT as $M^2_{\mathrm{max}} \to \infty$.
- Effect larger for $F_2$ than $F_L$ due to aligned jet limit.
- Much larger effect for heavy quarks!!

# PURPOSE 2: IMPORTANCE OF FINITE-$W$ CONSTRAINT: HEAVY QUARKS

$m_q = 1.4$ GEV

For physical cross sections: $M^2_{\text{max}} = W^2 \approx \frac{Q^2}{x}$. The rel. difference $1 - F^{\text{FE}}_{\text{L,2}}/F^{\text{OT}}_{\text{L,2}}$.




NOTE :

- Effect of finite energy constraint largest at relatively small $Q^2$ and large $x$.
- Max effect: $F_L \sim 9\%$ and $F_2 \sim 35\%$.
- 10 times larger for $F_L$ and 5 times larger for $F_2$. Quark mass surpresses aligned jet limit

# Part III

# OUTLOOK: FINITE ENERGY CONSTRAINT AND THE BK INITIAL CONDITION

# FINITE ENERGY CONSTRAINT FOR BK INITIAL CONDITION

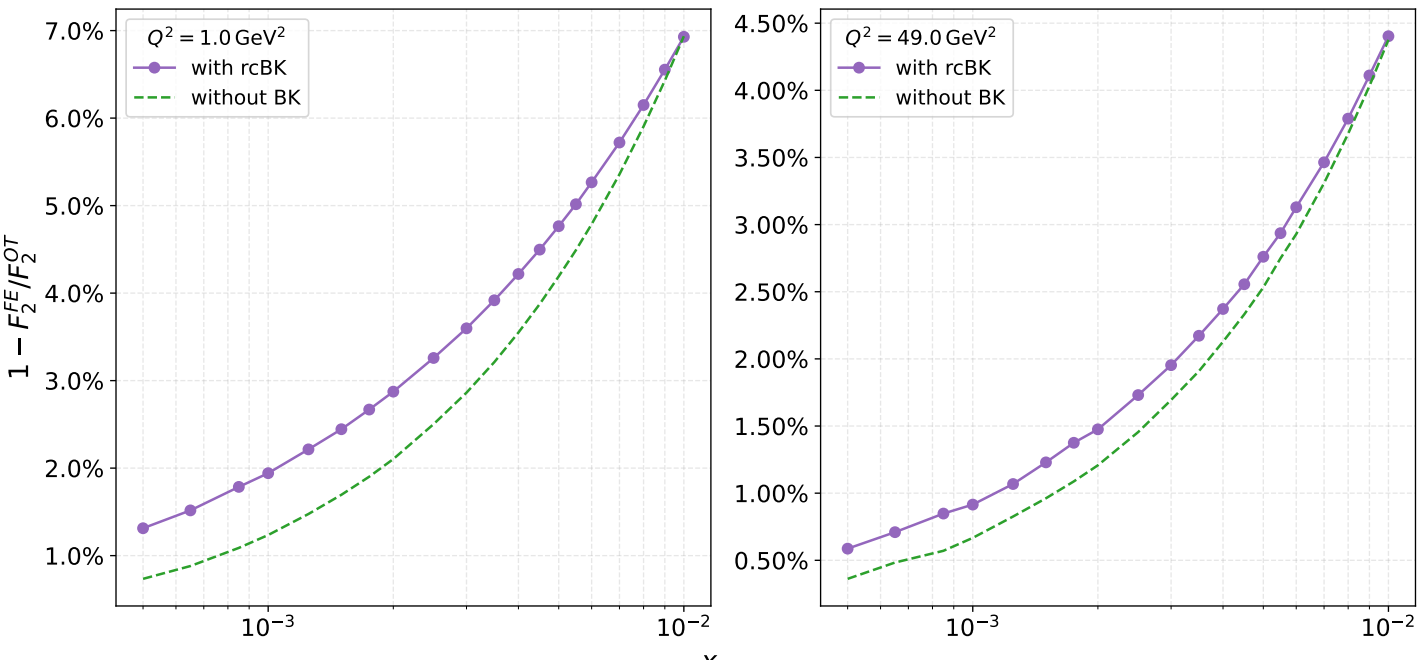
- So far, LO DIS dipole picture cross section
- Additional $x$-dependence from the evolution of the correlators e.g. BK or JIMWLK.
- Work in progress together with Casuga and Mäntysaari.
- Initial condition for BK given by MV model

$$\frac{1}{N_c}\left\langle \mathrm{Tr}\left[U((1-z)\mathbf{u})U^{\dagger}(-z\mathbf{u})\right]\right\rangle = e^{-\frac{1}{4}\left(|\mathbf{u}|^2 Q_{s0}^2\right)^{\gamma}\log\left(1/(|\mathbf{u}|\Lambda_{\mathrm{QCD}})+e\cdot e_c\right)}$$

- Previous fits of extracting non-perturbative parameters from data with $F_{L,2}^{\mathrm{OT}} \Rightarrow$ Replace with $F_{L,2}^{FE}$.
- For a proper uncertainty analysis $\rightarrow$ Bayesian inference

# INITIAL RESULTS USING BK EVOLVED CORRELATOR

- $m_q = 0.14$ GeV




Using BK dipole fit[3] $\rightarrow$ effect small for light quarks

[3] Phys. Rev. D 88, 114020, Lappi and Mäntysaari

# SUMMARY AND CONCLUSIONS

- Introduced $W$-dependence into the DIS cross section at LO. Description in exact kinematics.
- Purpose 1 : Improved cross section for DIS dipole picture calculations

$$\sigma_{L,T}^{\gamma^* p} = \sigma_0 \int_{z,\mathbf{u},\mathbf{u}'} \Psi_{L,T}^{\gamma^* q\bar{q}}(z, |\mathbf{u}|) \left(\Psi_{L,T}^{\gamma^* q\bar{q}}(z, |\mathbf{u}'|)\right)^\dagger \widetilde{\mathcal{N}}(z, \mathbf{u}, \mathbf{u}') \frac{\zeta(M_{\text{max}}^2) J_1\left(\zeta(M_{\text{max}}^2)\right)}{2\pi|\mathbf{u}' - \mathbf{u}|^2}$$

- Recover Optical Theorem cross section as $M_{\text{max}}^2 \to \infty$.
- Purpose 2 : Finite energy constraint has impact at LO. Larger effect for $F_2$ than $F_L$.
- At $Q^2 \sim 1$ GeV$^2$ and $x_B \sim 0.01$, substantial effect for $F_2$ ($\sim 35\%$) for heavy quarks.
- A finite energy constrained fit of the BK IC and the impact on $F_L$ and $F_2$ is in the works

# Part IV

## BACKUP SLIDES

# LIGHTCONE WAVEFUNCTIONS AT LO

Longitudinal wavefunciton squared

$$\Psi_L^{\gamma^* q\bar{q}}(z,|\mathbf{u}|)\left(\Psi_L^{\gamma^* q\bar{q}}(z,|\mathbf{u}'|)\right)^\dagger = N_c \sum_f \frac{\alpha_{em} Q_f^2}{4\pi^2} 8Q^2 z^2 (1-z)^2 K_0(\varepsilon_f|\mathbf{u}|) K_0^*(\varepsilon_f|\mathbf{u}'|)$$

Transverse wavefunction squared

$$\Psi_T^{\gamma^* q\bar{q}}(z,|\mathbf{u}|)\left(\Psi_T^{\gamma^* q\bar{q}}(z,|\mathbf{u}'|)\right)^\dagger = 2N_c \sum_f \frac{\alpha_{em} Q_f^2}{4\pi^2} \big[(z^2+(1-z)^2)\varepsilon_f^2 \cos(\alpha) K_1(\varepsilon_f|\mathbf{u}|) K_1^*(\varepsilon_f|\mathbf{u}'|) + m_f^2 K_0(\varepsilon_f|\mathbf{u}|) K_0^*(\varepsilon_f|\mathbf{u}'|)\big].$$

$$\varepsilon_f = Q^2 z(1-z) + m_f^2$$