Precision QCD calculations

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ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Theory and Phenomenology
of Fundamental Interactions

UNIVERSITY AND INFN · BOLOGNA

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Disclaimer

“Precision QCD” covers a very **rich and diverse** range of results.

Giving a complete summary in 25 minutes is impossible.

I will overview a **selection of topics and recent results**, but there are many more interesting subjects I will not be able to cover.

Why precision?

Precision physics is a **gateway** to **new physics**

- **Standard Model**: accurate but incomplete description of nature
- **Goal:** %-level uncertainties

Higher-order perturbative predictions

- motivates deep understanding of **physical** and **mathematical** structures in QFT
- this understanding improves our ability of making theoretical predictions

Theoretical predictions @ colliders

QCD factorization

$$\sigma = \sum_{ij} \int dx_1 dx_2 \; f_i(x_1, \mu_F) f_j(x_2, \mu_F) \; \hat{\sigma}_{ij \to X}(x_1, x_2, \mu_F) + \mathcal{O}\left(\left(\Lambda_{QCD}/Q \right)^k \right)$$

PDFs

partonic hard process

power corrections

$$\hat{\sigma}_{ij \to X} = \frac{1}{\Phi} \int |\mathscr{A}_{ij \to X}|^2 \, d\Pi_X$$

phase-space of final states

scattering amplitude

Many ingredients:

Parton Distribution Functions (PDFs), amplitudes, phase-space integration, IR subtraction, parton shower, resummation, hadronization effects, etc… [many talks about these at this conference!]

Amplitudes in perturbation theory

Perturbative expansions in the **coupling constants** (α_s in QCD)

$$\mathcal{A} = \alpha_s^k \left(\mathcal{A}_{LO} + \alpha_s \mathcal{A}_{NLO} + \alpha_s^2 \mathcal{A}_{NNLO} + \cdots \right)$$

Typical **theoretical uncertainties**:

- LO (~100%)
- NLO (10-20%)
- NNLO (a few %)
- N3LO (~1%)

LOOPS

Higher orders involve more and more **integrations** over unresolved **momenta** of **virtual particles**

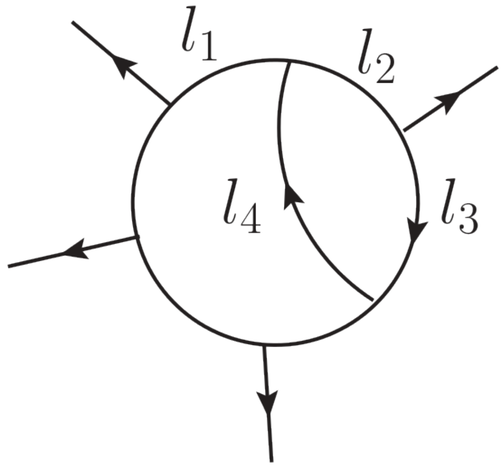
$\Rightarrow$ **loop integrals**

QCD vs EW (electroweak):

Naive expectation $\alpha_s^2 \sim \alpha_{EW} \sim \mathcal{O}(10^{-2})$, but EW enhancements are common (process dependent)

Loop amplitudes

- At the core of perturbative predictions
- Often a major algebraic and analytic **bottleneck**
- In dimensional regularization, a combination of



Inverse propagators

$$D_j = l_j^2 - m_j^2$$

$$\mathscr{A} \sim \int_{-\infty}^{\infty} \left(\prod_{i=1}^{\ell} d^d k_i \right) \frac{\mathscr{N}(k_i)}{D_1 D_2 D_3 \cdots}$$

Loop momenta k_j = unresolved momenta of virtual particles (not fixed by mom. conservation)

l_j = linear combination of:

- external (initial/final states) momenta p_j
- loop momenta k_j

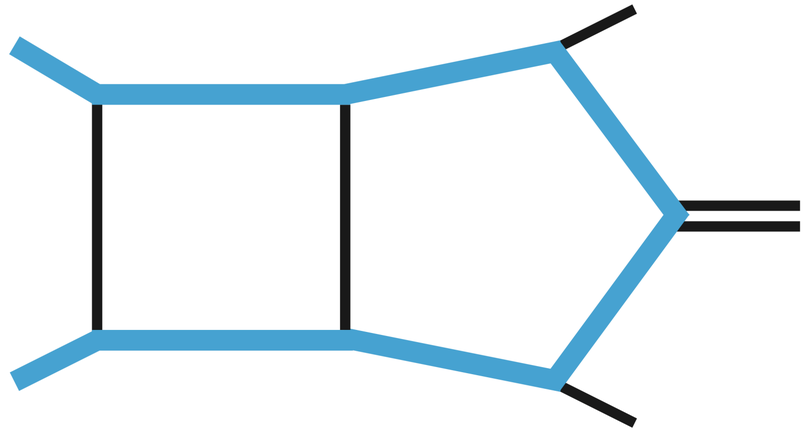
One-loop amplitudes and NLO predictions

- Calculation of amplitudes up to **one loop** is **fully automated** in **public programs**
  FormCalc, MadGraph+MadLoop, GoSam, BlackHat, NJet, OpenLoops,...
- We have **tools** for
  - systematic **reduction to known scalar integrals**
    CutTools, Samurai, Ninja, Golem, Collier,...
  - evaluation of such integrals
    OneLoop, QCDLoop, LoopTools,...
- Full **automation of NLO predictions*** when combined with **MonteCarlo** programs
  * Limited to LO for one-loop-induced processes
- Current focus: performance, numerical stability, extensions to more models, etc…

Amplitudes at higher loops

Beyond one loop, amplitude calculations can still be extremely **challenging**

- More diagrams, but generally not a bottleneck
- Higher **algebraic complexity**
  - more integration variables
  - huge intermediate expressions
- Higher **analytic complexity**
  - functional form of the integrals not well understood in the general case



Very active research on development of **new tools** and **new calculation methods**

Computing loop amplitudes: step 1/3

- Write **amplitudes** as a linear combination of **loop integrals**

$$\mathcal{A} = \sum_j a_j \; I_j$$

Rational coefficients

Integrals in a "nice/standard" form

- **Standard form** is usually

$$I = \int_{-\infty}^{\infty} \left(\prod_{i=1}^{\ell} d^d k_i \right) \frac{1}{D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}}, \qquad \alpha_j \lesseqgtr 0$$

INTEGRAL FAMILY
common set of
denominators

- **Modern methods**: tensor decomposition, integrand reduction

Computing loop amplitudes: step 2/3

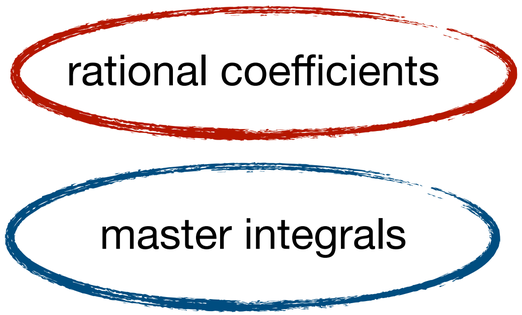
- Loop integrals obey linear relations, e.g. **Integration By Parts** (**IBPs**)
  Chetyrkin, Tkachov (1981), Laporta (2000)

$$\int \left(\prod_j d^d k_j\right) \frac{\partial}{\partial k_j^\mu} v^\mu \frac{1}{D_1^{\alpha_1} D_2^{\alpha_2} \ldots} = 0, \qquad v^\mu \in \{p_i^\mu, k_i^\mu\}$$

- Very **large** and sparse linear system ⇒ often a **huge bottleneck**

- Solution of linear system ⇔ **reduction** into a **basis** of linearly independent **master integrals (MIs)** $\{G_j\} \subset \{I_j\}$

$$I_j = \sum c_{jk} G_k$$



Computing loop amplitudes: step 2/3

IBP reduction is an **essential** step and a very **active** field of research

Public tools for generating and solving the identities: Reduze **[von Manteuffel, Studerus]**, Fire **[Smirnov, Zeng]**, Kira **[Lange, Usovitsch, Wu]**, LiteRed **[Lee]**, FiniteFlow **[T.P.]**

New important tools for building **simpler** systems of identities
(often combined with Kira or FiniteFlow)

- NeatIBP **[Z. Wu, J. Boehm, R. Ma, H. Xu, Y. Zhang (2023)]**
  polynomial constraints (**syzygy** equations) in Baikov representation

- Blade **[Guan,Liu,Ma,Wu (2024)]** (see also "improved seeding" in Kira3)
  simpler **block triangular** structures in momentum representation

- CALICO **[Bertolini, Fontana, T.P. (2025)]**
  via differential operators (**parametric annihilators**) in several parametric representations

Computing loop amplitudes: step 3/3

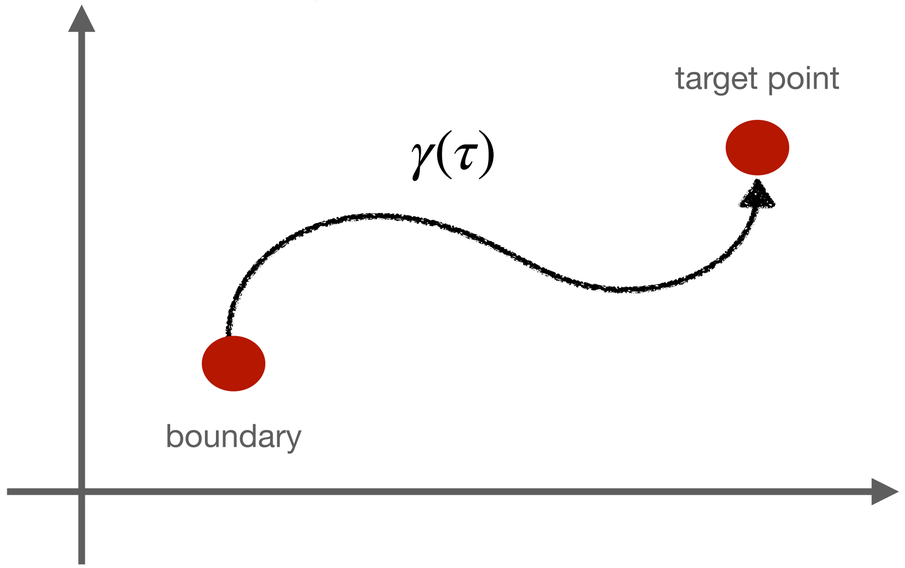
- Computing the **master integrals** (**MIs**)
- Most efficient method is **differential equations** (**DEs**)
  Kotikov (1991), Gehrmann, Remiddi (2000)
- MIs obey systems of **DEs** w.r.t. **invariants** x

$$\partial_x G_j = \sum_k A_{jk}^{(x)} G_k$$

  - take **derivates** of MIs and **reduce them to MIs**
  - both **analytic** and **numerical** methods

On a path $\gamma(\tau)$

$$\partial_\tau G_j = \sum_{x,j} \frac{\partial x}{\partial \tau} A_{jk}^{(x)} G_k$$



Computing loop amplitudes: step 3/3

Solving DEs requires **boundary conditions**

- Most straightforward method: **evaluations** with a **slower** but **more general** algorithm
- Popular choice: **auxiliary mass flow** with the `AMFlow` program **[X. Liu, Y.Q. Ma (2022)]**
  - DEs w.r.t. an **auxiliary ~~mass~~ width**

$$\frac{1}{l_j^2 - m_j^2 + i\,0^+} \to \frac{1}{l_j^2 - m_j^2 + i\eta}$$

more complex DEs but easy boundary values at $\eta \to \infty$, then integrate to $\eta \to 0^+$

- Alternative: **sector decomposition [T. Binoth, G. Heinrich]** with `PySecDec` **[G. Heinrich, S. P. Jones, M. Kerner, V. Magerya, A. Olsson, J. Schlenk, et al.]**
  - it “disentangles” divergencies and numerically integrates in a parametric representation

Finite fields and rational reconstruction

- A successful idea for dealing with **algebraic complexity**
  [Kant (2014), von Manteuffel, Schabinger (2014), T.P. (2016)]
- Reconstruct **analytic results** from **numerical evaluations**
  - sidesteps appearance of large intermediate expressions
  - massively **parallelizable**
- Evaluations over **finite fields** $\mathcal{F}_p$ (computing modulo a **prime** p) $\quad \mathcal{F}_p = \{0,1,\ldots,p-1\}$
  - using machine-size integers $p < 2^{64}$ (**fast** and **exact**)
- Examples of public codes using it: `FiniteFlow`, `FireFly+Kira`, `Fire`, `Caravel`
- Useful for both numerical and analytic calculations

From amplitudes to observables: IR divergencies

Loop and **phase-space** integrals have **infrared divergences**

They cancel in **infrared-safe** observables (~ stable under soft/collinear emissions)

- we need to sum over **final states with different multiplicity**, e.g. at NLO

$$\sigma_{NLO} = \int_{n+1} d\sigma_R + \int_n d\sigma_V$$

Similar idea but more complex structure beyond NLO

Real emission (R):
Includes additional final-state radiation
IR divergencies from phase-space

Virtual (V):
IR divergencies from loops

Explicit integration over phase-space

- **reverse unitarity** **[Anastasiou, Melnikov (2002)]** turns phase-space integrals into loop-like integrals (recycle most of the technology)
- generally viable only for simple kinematics and more inclusive observables

Infrared subtraction

The idea of infrared subtraction

Reproduces infrared behaviour of $d\sigma_R$

$$\sigma_{NLO} = \int_{n+1} \left(d\sigma_R - d\sigma_A \right) + \int_n \left(d\sigma_V + \int_1 d\sigma_A \right)$$

Finite integration:
it can be done numerically (in $d = 4$), also with phase-space cuts etc…

Add $d\sigma_A$ back, integrated over additional radiation and explicitly cancel virtual divergencies.

At **NLO** it is a **solved problem**, enabling automated NLO predictions!
[Catani, Seymour (1996), Frixione, Kunszt, Signer (1995)]

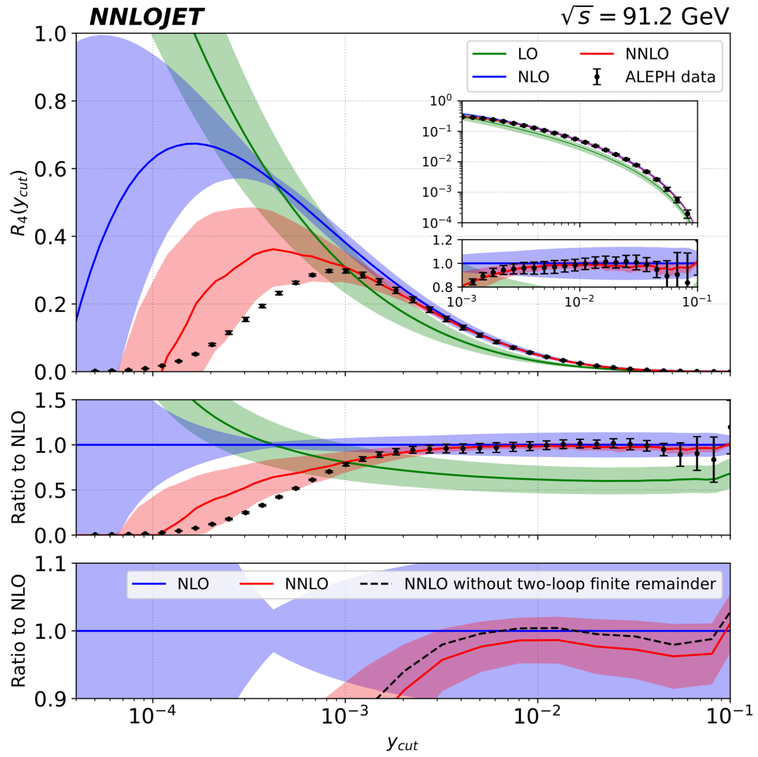
At **NNLO and beyond** a very **active field of research** with many proposals:
Antenna **[Gehrmann-De Ridder,Gehrmann,Glover]**, Nested Soft-Collinear **[Caola,Melnikov,Röntsch]**,
Local analytic sector **[Magnea,Maina,Pelliccioli,Signorile-Signorile,Torrielli, Uccirati]**,
CoLoRFul **[Del Duca, Duhr,Somogyi,Tramontano, Trocsanyi]**,
projection-to-Born **[Cacciari,Dreyer,Karlberg,Salam,Zanderighi]**
N–jettiness **[Gaunt,Stahlhofen,Tackmann,Walsh,Boughezal,Focke,Liu,Petriello]**,
q_T slicing **[Catani,Grazzini]**, Sector-improved residue **[Czakon]**

Some recent results

$e^-e^+ \to Z/\gamma^* \to 4$ jets @ NNLO

X.Chen, D.Chicherin, E.Fox, N.Glover, M.Marcoli, V.Sotnikov, H.Sun, H.Zhang, S.Zoia (2026)

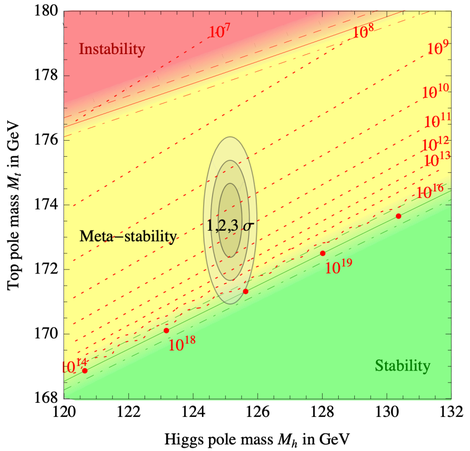
- **Antenna subtraction** (NNLOJET framework)
- Challenge: high-multiplicity of final state with massless particles
- Amplitudes: effectively a 5-point ($1 \to 4$) process
  - 2-loop uses leading colour approximation
  - crossing of $V + 2$ jets
    [Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov (2021), De Laurentis, Ita, Page, Sotnikov (2025)]
  - calculation of MIs/special functions in crossed channel ("pentagon functions" as in **[Gehrmann,Henn,Lo Presti (2015)]**)
- k_T jet algorithm



Top pair + X production @ NNLO

Recent effort in predictions for **top-pair production** plus **a jet or a massive boson**

- sensitivity to **top mass** (correlated to stability/metastability of the universe)
- the top and Higgs sectors still have **significant uncertainty**



Picture from Buttazzo, Degrassi, et al. (2013)

Unique combination of **algebraic** and **analytic complexity**

- exploring new strategies/algorithms, motivated by interesting phenomenology
- poor knowledge of space of functions and computational difficulty
  => resorting to **numerical solution of differential equations** for master integrals

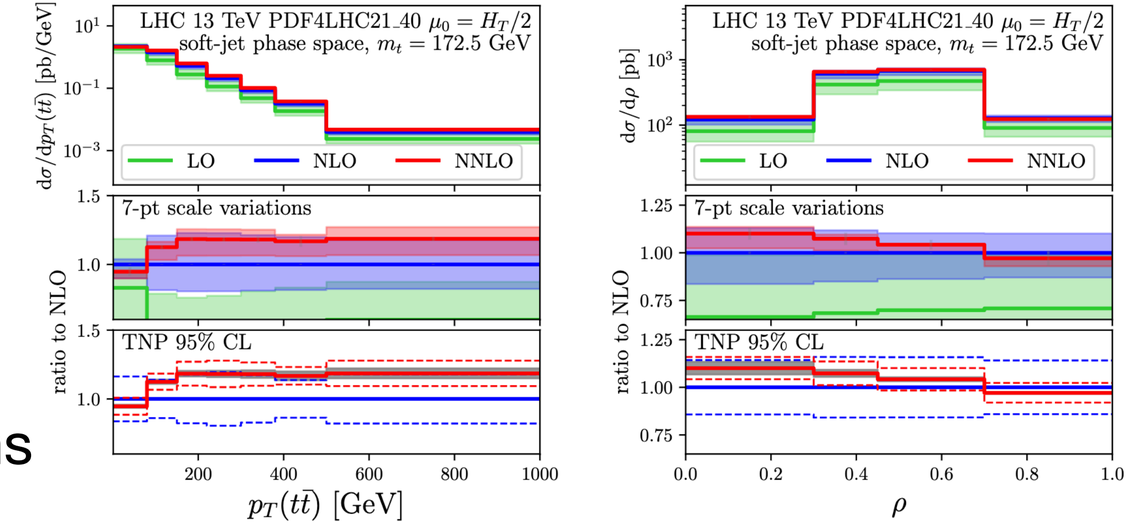
Handling highly complex rational coefficients

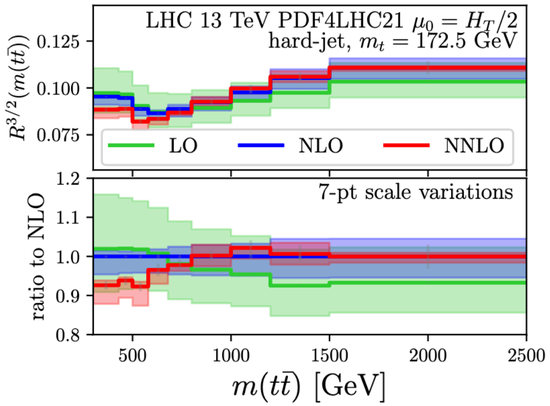
- combining **finite fields** and **rational reconstructions** with algebraic simplifications
- alternative approach: applying these techniques for **numerical evaluation**

Top pair production plus a jet @ NNLO

S. Badger, C. Brancaccio, M. Becchetti, M.Czakon, H.B.Hartanto, R.Poncelet, S.Zoia (2025)

- IBP reduction via syzygy and finite fields techniques (NeatIBP+FiniteFlow)
- **Analytic reconstruction** of amplitude coefficients (FiniteFlow)
- **Numerical master integrals** (or special functions) via solution of differential equations with custom C++ code (using Boost)
- Numerical boundary values via AMFlow+FiniteFlow+LiteRed
- **Residue-subtraction** scheme (Stripper framework)





Top pair production plus a W boson @ NNLO

M.Becchetti, D.Canko, X.Chen, V.Chestnov, M.Delto, S.Ditsch, M.Grazzini, S.Kallweit, M.Pozzoli, C.Savoini, L.Tancredi, S.Zoia, T.P. (to appear)

- Even higher algebraic+analytic complexity in DEs and amplitudes (leading colour)
- IBP reduction via syzygy and finite fields techniques (NeatIBP+FiniteFlow)
- **Numerical reconstruction** of amplitude coefficients on a grid of ~200k phase-space points (FiniteFlow)
- **Numerical evaluation** of master integrals (special functions) via iterated series expansions (custom DE solver based on the one of AMFlow)
- Numerical boundary values via AMFlow+FiniteFlow+LiteRed
- q_T **sbtraction** scheme (Matrix framework)

Top pair production plus a W boson @ NNLO

Preliminary results

$$\sigma_{NNLO,QCD+NLO,EW} = 766.66^{6.78\%}_{7.47\%} \pm 2.41\ \%\ \text{fb}$$

$$\sigma_{CMS(2023)} = 868 \pm 40 \pm 51\ \text{fb},\ \sigma_{CMS(2025)} = 938 \pm 45 \pm 52\ \text{fb}$$

$$\sigma_{ATLAS} = 880 \pm 50 \pm 70\ \text{fb}$$

Still some tension with experimental values

- Two-loop contributions ~ 10%
- Good agreement with prior results obtained with kinematic approximations of two-loop terms **[Buonocore,Devoto,Grazzini,Kallweit,Mazzitelli,Rottoli,Savoini (2023)]**

see M.Pozzoli's talk for more details!

NNLO: status and modern challenges

- 5-point processes (e.g. $2 \to 3$) with some massive states
- Focus on taming **complexity** of **internal/external masses** and **higher multiplicity**, e.g.
  - recent progress on 6-point 2-loop integrals
    [Henn,Matijašić,Miczajka,Xu,Zhang,T.P. (2022-25), Abreu,Monni,Page,Usovitsch (2024), Liu,Matijašić,Xu,Yang,Zhang,T.P. (2026)]
  - partial computation of integrals and amplitudes for top pair plus Higgs production
    [Cordero, Figueiredo, Kraus, Page, Reina (2024), Agarwal,Heinrich,Jones,Matthias, Kerner,Klein,Lang,Magerya,Olsson (2024)]
- Exploring semi-numerical approaches relying on analytic/algebraic insight
- Tremendous **progress in infrared subtraction**
- The main **bottlenecks** are often **scattering amplitudes**

N3LO: status and modern challenges

- First $2 \to 2$ amplitudes with one off-shell/massive particle
  - a recent example: H+jet **[X.Chen, X.Guan, B.Mistlberger (2025)]**
    (note: V+jet also available **[Gehrmann,Jakubčík,Mella,Syrrakos,Tancredi (2023)]**)
- First phenomenological results for $2 \to 2$ massless processes
  - $e^- e^+ \to 2$ jets, via (partial) N3LO extension of antenna subtraction scheme **[Chen,Jakubčík,Marcoli,Stagnitto (2025)]**
  - $pp \to \gamma\gamma$, via q_T subtraction scheme **[Czakon,Eschment,Generet,Poncelet (2026)]**
- **Infrared subtraction** is still a **major bottleneck**

Summary

Interesting recent **phenomenological applications**

- NNLO: $2 \to 3$ scattering with massive states
- N3LO: $2 \to 2$ scattering with massless states

The combination of **physical insights** and modern **mathematical techniques** significantly improves our ability to make important **theoretical predictions**

Modern challenges

- Handling complex **high-multiplicity**, **massive states**, and **higher-loop** combinatorics
- **Infrared subtraction**, particularly at N3LO