Spin-spin entanglement at high energy

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[M.F., Hatta, PRD 113 (2026) 3]

[M.F., Hatta, Xiao, 2604.11697 (2026)]

Deep Inelastic Scattering 2026, 4-8 May, Bologna

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THE ULAM PROGRAMME

Quantum entanglement

Einstein-Podolsky-Rosen paradox

- Quantum mechanics is not complete; cannot satisfy **local realism**

Quantum Entanglement

- The quintessential phenomenon of QM; introduced by Schrödinger (in response to the EPR paper)

Quantum Indeterminacy

- **Realism:** Hidden variables
- **Copenhagen:** Indeterminacy is intrinsic
- **Agnosticism:** Only the predictive power of the theory matters

Bell Inequality: Observable difference for Realism vs Copenhagen

Aspect, Clauser, and Zeilinger closed the debate [Nobel Prize (2022)]

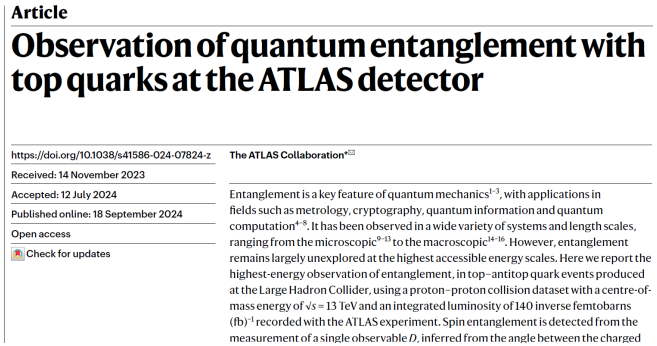
Einstein-Podolsky-Rosen

"Spooky action at a distance"

Schrödinger, Bohr and Einstein

Quantum entanglement at colliders

- **Entanglement** in $t\bar{t}$-production at LHC [Atlas (2024)] [CMS (2024)]



Article

Observation of quantum entanglement with top quarks at the ATLAS detector

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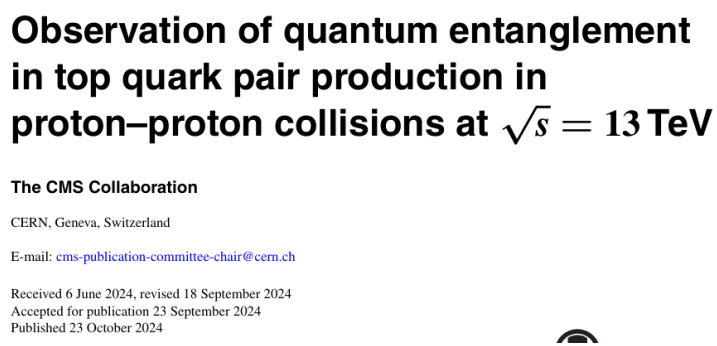
The ATLAS Collaboration*

Entanglement is a key feature of quantum mechanics[1–3], with applications in fields such as metrology, cryptography, quantum information and quantum computation[4–8]. It has been observed in a wide variety of systems and length scales, ranging from the microscopic[9–13] to the macroscopic[14–16]. However, entanglement remains largely unexplored at the highest accessible energy scales. Here we report the highest-energy observation of entanglement, in top–antitop quark events produced at the Large Hadron Collider, using a proton–proton collision dataset with a centre-of-mass energy of $\sqrt{s} = 13$ TeV and an integrated luminosity of 140 inverse femtobarns $(fb)^{-1}$ recorded with the ATLAS experiment. Spin entanglement is detected from the measurement of a single observable D, inferred from the angle between the charged



Observation of quantum entanglement in top quark pair production in proton–proton collisions at $\sqrt{s} = 13$ TeV

The CMS Collaboration

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- Set of **new observables** to test the Standard Model
- Possibility to investigate features of quantum mechanics at the highest accessible energy scales (e.g. **Spin Entanglement** between particles)
- Many works devoted to LHC physics

  see e.g. **[Tuesday's talk by E. Re]**
- Quantum entanglement at the **Electron-Ion Collider**?

  [Qi, Guo, Xiao (2025)][Cheng, Han, Trifinopoulos (2025)]
  [M.F., Hatta (2025)] [M.F., Hatta, Xiao (2026)]
- Results also usable for Ultra-Pheripheral Collisions (UPCs) at LHC

Density matrix and entanglement

- General two-qubit density matrix:

$$\rho = \frac{1}{4}\left(\mathbb{I}\otimes\mathbb{I} + B_a\,\sigma^a\otimes\mathbb{I} + \bar{B}_b\,\mathbb{I}\otimes\sigma^b + C_{ab}\,\sigma^a\otimes\sigma^b\right)$$

- Consider $B_a = \bar{B}_b = 0$ (all statements below assume this)
- **Product state** (no correlations): $\rho = \rho_A \otimes \rho_B$
- **Separable state** (classical correlations): $\rho = \sum_i p_i\ \rho_A^{(i)} \otimes \rho_B^{(i)}$
- **Entangled state** (non-classical correlations): $\rho \neq \sum_i p_i\ \rho_A^{(i)} \otimes \rho_B^{(i)}$
- Separable or Entangled $\Longrightarrow$ **Peres-Horodecki criterion**

$$\Delta_1 = \sqrt{(C_{rr} - C_{kk})^2 + (C_{rk} + C_{kr})^2} - 1 + C_{nn},$$

$$\Delta_2 = \sqrt{(C_{rr} + C_{kk})^2 + (C_{rk} - C_{kr})^2} - 1 - C_{nn}$$

$$\textbf{Entangled} \iff \Delta_1 \geq 0 \ \text{ or } \ \Delta_2 \geq 0$$

- Quantification via **Concurrence**:

$$\mathcal{C}[\rho] = \max\left\{\frac{\Delta_1}{2}, \frac{\Delta_2}{2}, 0\right\} \Rightarrow 0\ \text{(separable)} \quad \text{to} \quad 1\ \text{(maximally entangled)}$$

Bell-CHSH inequality

- Even stronger correlations $\longrightarrow$ **Bell non-locality**
- In a classical theory

$$\textbf{Bell-CHSH inequality}: \qquad S = \langle AB + A'B + AB' - A'B' \rangle, \quad |S| \leq 2$$

[Bell (1964)][Clauser, Horne, Shimony, Holt (1969)]

Its violation ruled out local hidden variable theories $\longrightarrow$ 2022 Nobel Prize

- Quantum mechanics allows violation: $|\langle \cdots \rangle| \leq 2\sqrt{2}$
- Saturation of the bound: $2\sqrt{2}$

$$\textbf{Bell state}: \qquad |\Phi^+\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right)$$

- Maximal violation depends on the maximal eigenvalues of $C^{\mathcal{T}}_{ab} C_{ab}$:

$$\textbf{Bell non-locality} \iff 1 < \mu_1 + \mu_2 \leq 2$$

- Other intriguing QI related quantity: e.g. **magic**
- Quantified via stabilizer Rényi entropy

$$S_{\mathrm{R}}^{\mathrm{stab}} = -\ln \left(\frac{1 + C_{nn}^4 + C_{rr}^4 + C_{kk}^4 + C_{kr}^4 + C_{rk}^4}{1 + C_{nn}^2 + C_{rr}^2 + C_{kk}^2 + C_{kr}^2 + C_{rk}^2} \right)$$

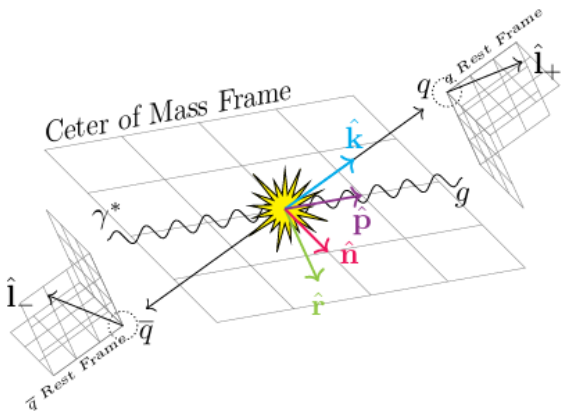
From cross section to spin density matrix

Strategy to compute the spin density matrix

- Start from the **center-of-mass frame** of the $q\bar{q}$-pair

$$S_{\alpha\alpha'\beta\beta'} = \bar{v}_{\beta'}(k')\Gamma u_{\beta}(k)\bar{u}_{\alpha}(k)\Gamma^{\dagger}v_{\alpha'}(k')$$

- Do not sum over the helicity indices
- Move to a new **rotated frame** $\{\hat{n}, \hat{k}, \hat{r}\}$ (rotating the gamma matrices)



- In this frame

$$u_{\alpha}(k) = \begin{pmatrix} \sqrt{k\cdot\sigma}\xi_{\alpha} \\ \sqrt{k\cdot\bar{\sigma}}\xi_{\alpha} \end{pmatrix}, \qquad v_{\alpha'}(k') = \begin{pmatrix} \sqrt{k\cdot\bar{\sigma}}\tilde{\eta}_{-\alpha'} \\ -\sqrt{k\cdot\sigma}\tilde{\eta}_{-\alpha'} \end{pmatrix}$$

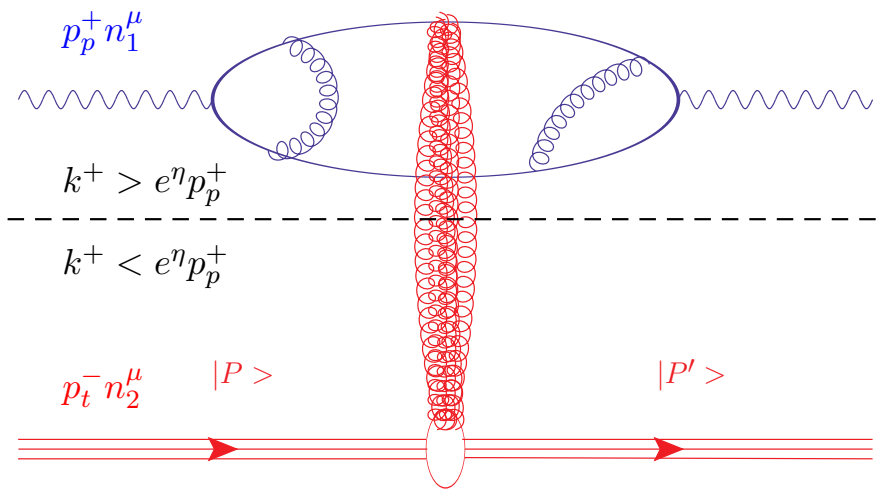
- The spin density matrix can be written

$$S_{\alpha\alpha'\beta\beta'} = \frac{A}{4}\left(\delta_{\alpha\beta}\delta_{\alpha'\beta'} + C_{ab}\xi^{\dagger}_{\alpha}\sigma^{a}\xi_{\beta}\eta^{\dagger}_{\alpha'}\sigma^{b}\eta_{\beta'}\right)$$

Diffractive heavy-quark pair production

Shockwave approach to high-energy QCD

- High-energy approximation $s = (p_p + p_t)^2 \gg \{Q^2\}$



$$p_p = p_p^+ n_1 - \frac{Q^2}{2p_p^+} n_2$$

$$p_t = \frac{m_t^2}{2p_t^-} n_1 + p_t^- n_2$$

$$p_p^+ \sim p_t^- \sim \sqrt{\frac{s}{2}}$$

$$e^\eta \ll 1$$

- Separation of the gluon field into "fast" (quantum) part and "slow" (classical) part through a rapidity parameter $\eta < 0$

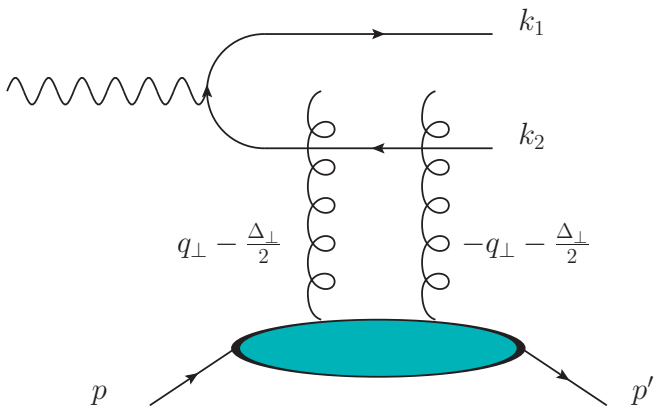
[McLerran, Venugopalan (1994)] [Balitsky (1996-2001)]

$$\mathcal{A}^\mu(k^+, k^-, \vec{k}) = A^\mu(k^+ > e^\eta p_p^+, k^-, \vec{k}) + b^\mu(k^+ < e^\eta p_p^+, k^-, \vec{k})$$

- **Eikonal** interaction between projectile and target
- High occupation number (small-x and/or large nuclei)

$\Longrightarrow$ Semi-classical description of the target

Diffractive $Q\bar{Q}$-pair production [M.F., Hatta (2025)]



- We consider the **Regge limit**
- Back-to-back configuration $\Delta_\perp = 0$

$$\beta = \sqrt{1 - \frac{4m^2}{M^2}} \qquad \cos\theta = \frac{k^3}{|\vec{k}|}$$

- **Longitudinally polarized case** $\Longrightarrow$ Simple spin density matrix

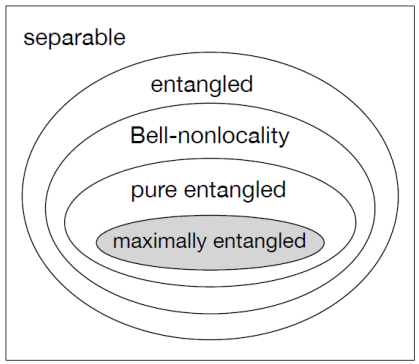
$$C^L_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1-(2-\beta^2)\cos^2\theta}{1-\beta^2\cos^2\theta} & -\frac{\sqrt{1-\beta^2}\sin 2\theta}{1-\beta^2\cos^2\theta} \\ 0 & -\frac{\sqrt{1-\beta^2}\sin 2\theta}{1-\beta^2\cos^2\theta} & \frac{1-(2-\beta^2)\cos^2\theta}{1-\beta^2\cos^2\theta} \end{pmatrix}$$

- $(C^L)^{\mathcal{T}} C^L = \mathrm{diag}(1,1,1)$, $\mu_1 + \mu_2 = 2$ $\Longrightarrow$ **Maximal entangled state**
- $\beta = 1$ or $\cos\theta = 0$ $\Longrightarrow$ $\rho^L = |\Phi^+\rangle\langle\Phi^+|$

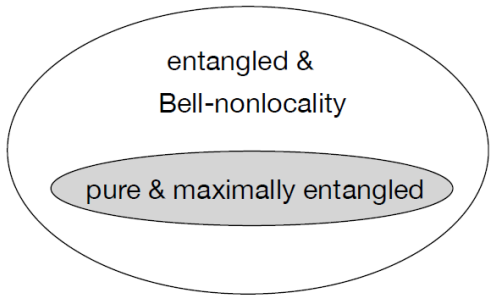
Diffractive heavy-quark pair production

- **Transversally polarized case** $\Longrightarrow$ Very complicated density matrix but:

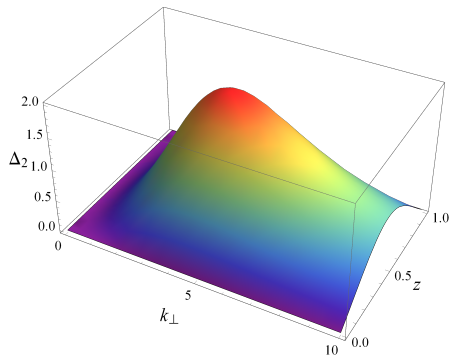
$$\Delta_2 = -\Delta_1 = -2C_{nn}^T \geq 0$$



reduces to



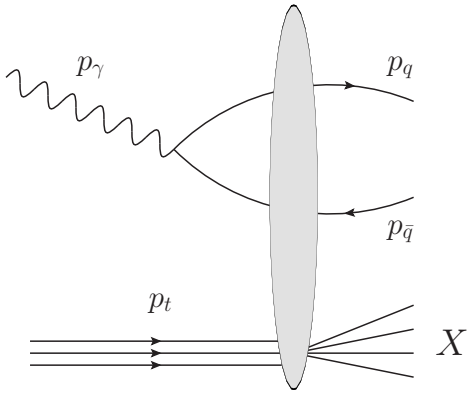
- **Gisin's theorem** $\Longrightarrow$ If ρ describes a pure state, entanglement necessarily entails Bell-nonlocality [Gisin (1991)]
- Our case: it holds for a **mixed state** [M.F., Hatta (2025)]
- **Color transparency** implies a point of **maximal entanglement**



- GPDs treatment leads to more complex patterns [Hatta, Schoenleber (2025)]

Heavy-quark pair production in DIS

Inclusive heavy-quark pair production in DIS



- Higher number of events
- LO Collinear factorization [Qi, Guo, Xiao (2025)]
- Small-x TMD treatment [M.F., Hatta, Xiao (2026)]

- CGC dijet cross section [Dominguez, Marquet, Xiao, Yuan (2011)]

$$\frac{d\sigma^{L/T}_{\alpha\alpha'\beta\beta'}}{dz d^2p_{q\perp} d^2p_{\bar{q}\perp}} \sim \int \frac{d^2x_{1\perp}}{(2\pi)^2}\int \frac{d^2x_{2\perp}}{(2\pi)^2}\int \frac{d^2x_{1'\perp}}{(2\pi)^2}\int \frac{d^2x_{2'\perp}}{(2\pi)^2} e^{-ip_{q\perp}\cdot x_{11'\perp}} e^{-ip_{\bar{q}\perp}\cdot x_{22'\perp}}$$
$$\times\Big(1 + S^{(4)}(x_{1\perp}, x_{2\perp}; x_{2'\perp}, x_{1'\perp}) - S^{(2)}(x_{1\perp}, x_{2\perp}) - S^{(2)}(x_{2'\perp}, x_{1'\perp})\Big)$$
$$\times\psi^{L/T(\lambda)*}_{\beta\beta'}(x_{1'2'\perp})\psi^{L/T(\lambda)}_{\alpha\alpha'}(x_{12\perp})$$

- **Correlation limit** [Dominguez, Marquet, Xiao, Yuan (2011)]

$$|P_\perp| \simeq |p_{q\perp}| \simeq |p_{\bar{q}\perp}| \gg |p_{q\perp} + p_{\bar{q}\perp}| \simeq |q_\perp| \quad \Rightarrow \quad \text{back-to-back jets}$$

- **Factorization**

$$\text{Hard part}(P_\perp) \otimes \text{Gluon TMD } (q_\perp)$$

General structure of the spin density matrix

- Result

$$\frac{d\sigma^{L/T}_{\alpha\alpha'\beta\beta'}}{dz d^2P_\perp d^2q_\perp} \sim \frac{\partial}{\partial P'^i_\perp}\frac{\partial}{\partial P^j_\perp}\psi^{L/T(\lambda)*}_{\beta\beta'}(P'_\perp)\psi^{L/T(\lambda)}_{\alpha\alpha'}(P_\perp)\Big|_{P'_\perp=P_\perp}$$
$$\times\left[\frac{\delta_{ij}}{2}G_0(q_\perp)+\left(\frac{q^i_\perp q^j_\perp}{q^2_\perp}-\frac{\delta_{ij}}{2}\right)G_2(q_\perp)\right]$$

- Target: G_0: unpolarized W.W. G_2: linearly polarized W.W.
- Hard factor structure: $\partial_i\psi^*\,\partial_j\psi \;\Rightarrow\; \boxed{\mathbb{I}\otimes\mathbb{I}+C_{ab}\sigma^a\otimes\sigma^b}$
- Total DIS spin density matrix

$$\frac{d\sigma_{\alpha\alpha'\beta\beta'}}{dz d^2P_\perp d^2q_\perp} \sim \mathcal{A}\xi^\dagger_\alpha\eta^\dagger_{\alpha'}\left(\mathbb{I}\otimes\mathbb{I}+\mathcal{C}_{ab}\sigma^a\otimes\sigma^b\right)\xi_\beta\eta_{\beta'} \qquad \xi^\dagger\eta^\dagger(\sigma^a\otimes\sigma^b)\xi\eta\equiv\xi^\dagger\sigma^a\xi\eta^\dagger\sigma^b\eta$$

$$\mathcal{A}=G_0\left(A_T^{(0)}+\varepsilon A_L^{(0)}\right)+\cos(2\phi_{P,q})G_2\left(A_T^{(2)}+\varepsilon A_L^{(2)}\right)$$

$$\mathcal{C}_{ab}=\frac{1}{\mathcal{A}}\left[G_0\left(\tilde{C}_T^{(0)}+\varepsilon\tilde{C}_L^{(0)}\right)+\cos(2\phi_{P,q})G_2\left(\tilde{C}_T^{(2)}+\varepsilon\tilde{C}_L^{(2)}\right)\right]_{ab}$$

spin density matrix in dijet production

- $A^{(0)}_{T/L}$ and $A^{(2)}_{T/L}$ reproduce the known results for the total cross sections ✓

  [Dominguez, Marquet, Xiao, Yuan (2011)]
- Averaging over the angle $\phi_{P,q}$ we obtain same result of CF ✓

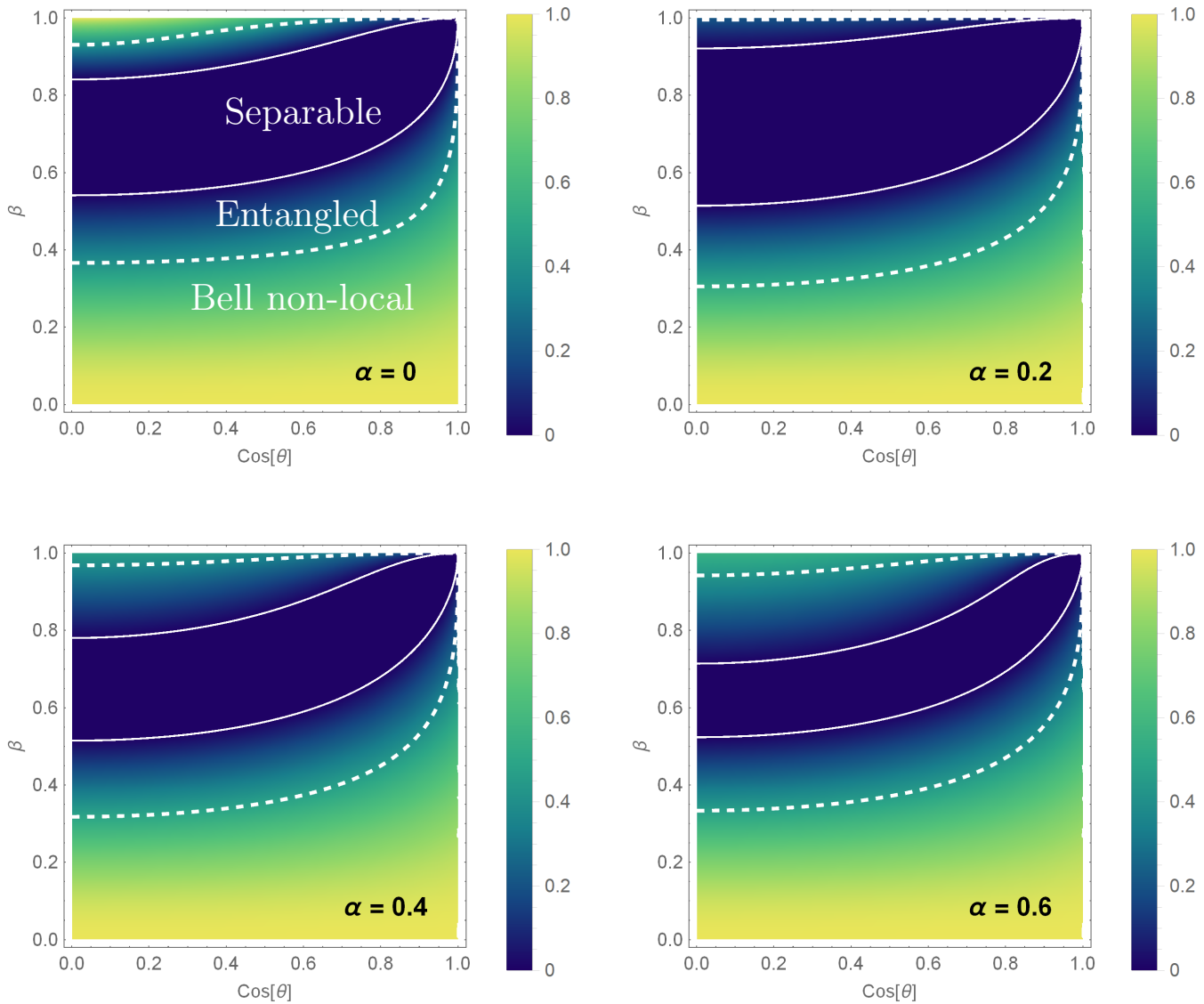
  [Qi, Guo, Xiao (2025)]
- Differential with respect to $\phi_{P,q}$ $\Longrightarrow$ Target description comes into play
- Large nuclei or very small-x $\longrightarrow$ **MV description** [McLerran, Venugopalan]
- G_0 and G_2 given in [Dumitru, Skokov (2016)]

$$G_0(q_\perp) = 8\pi S_\perp \frac{C_F}{N_c} \int_0^\infty d|r_\perp| \frac{J_0(|q_\perp||r_\perp|)}{|r_\perp|} \left[1 - \exp\left(-\frac{Q_s^2|r_\perp|^2}{4} \ln \frac{1}{|r_\perp|^2\Lambda_{\rm IR}^2}\right)\right]$$

$$G_2(q_\perp) = 8\pi S_\perp \frac{C_F}{N_c} \int_0^\infty d|r_\perp| \frac{J_2(|q_\perp||r_\perp|)}{|r_\perp| \ln \frac{1}{|r_\perp|^2\Lambda_{\rm IR}^2}} \left[1 - \exp\left(-\frac{Q_s^2|r_\perp|^2}{4} \ln \frac{1}{|r_\perp|^2\Lambda_{\rm IR}^2}\right)\right]$$

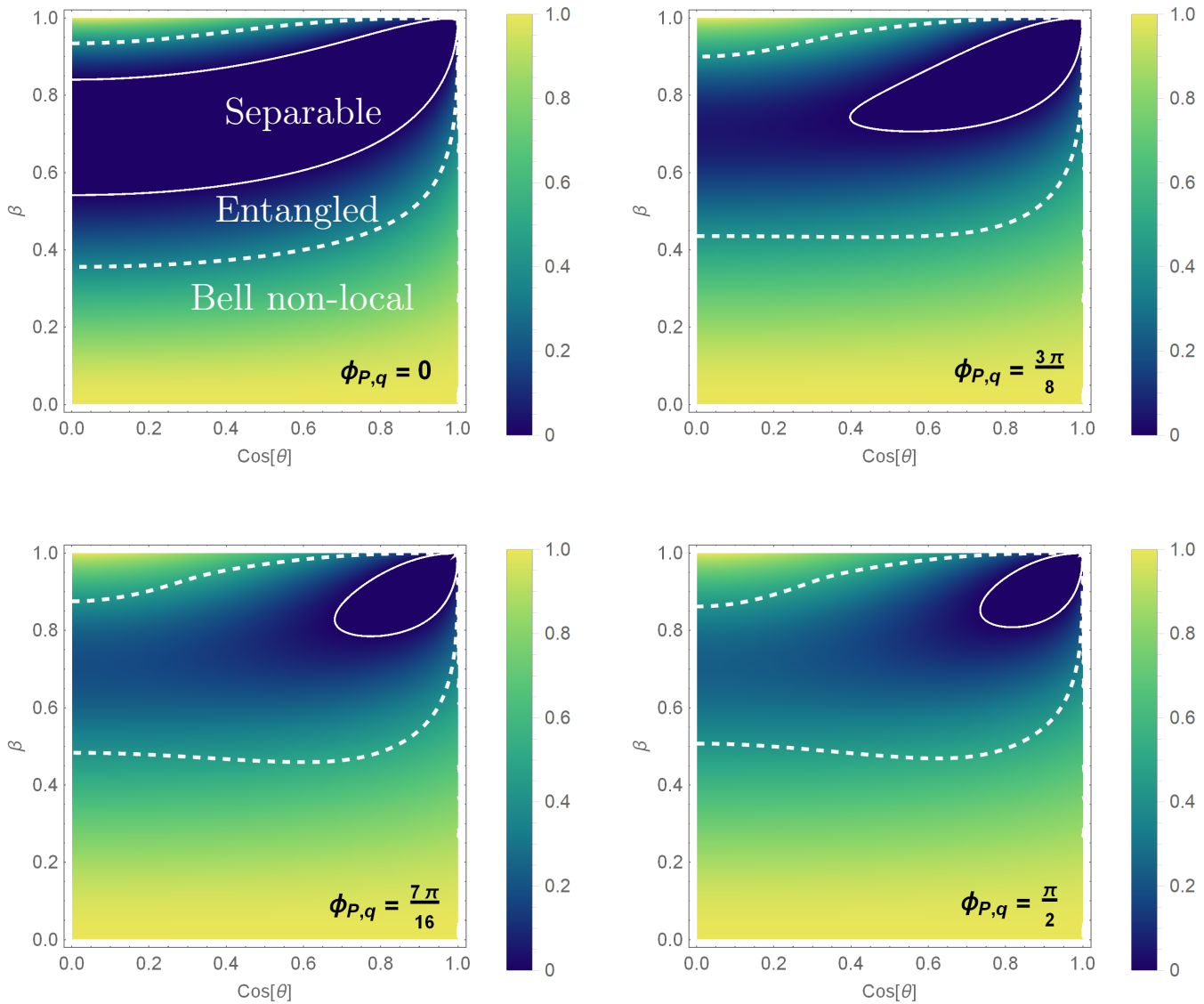
Concurrence

- **Concurrence** density plot ($\phi_{P,q}$-averaged and varying $\alpha = Q^2/M^2$)



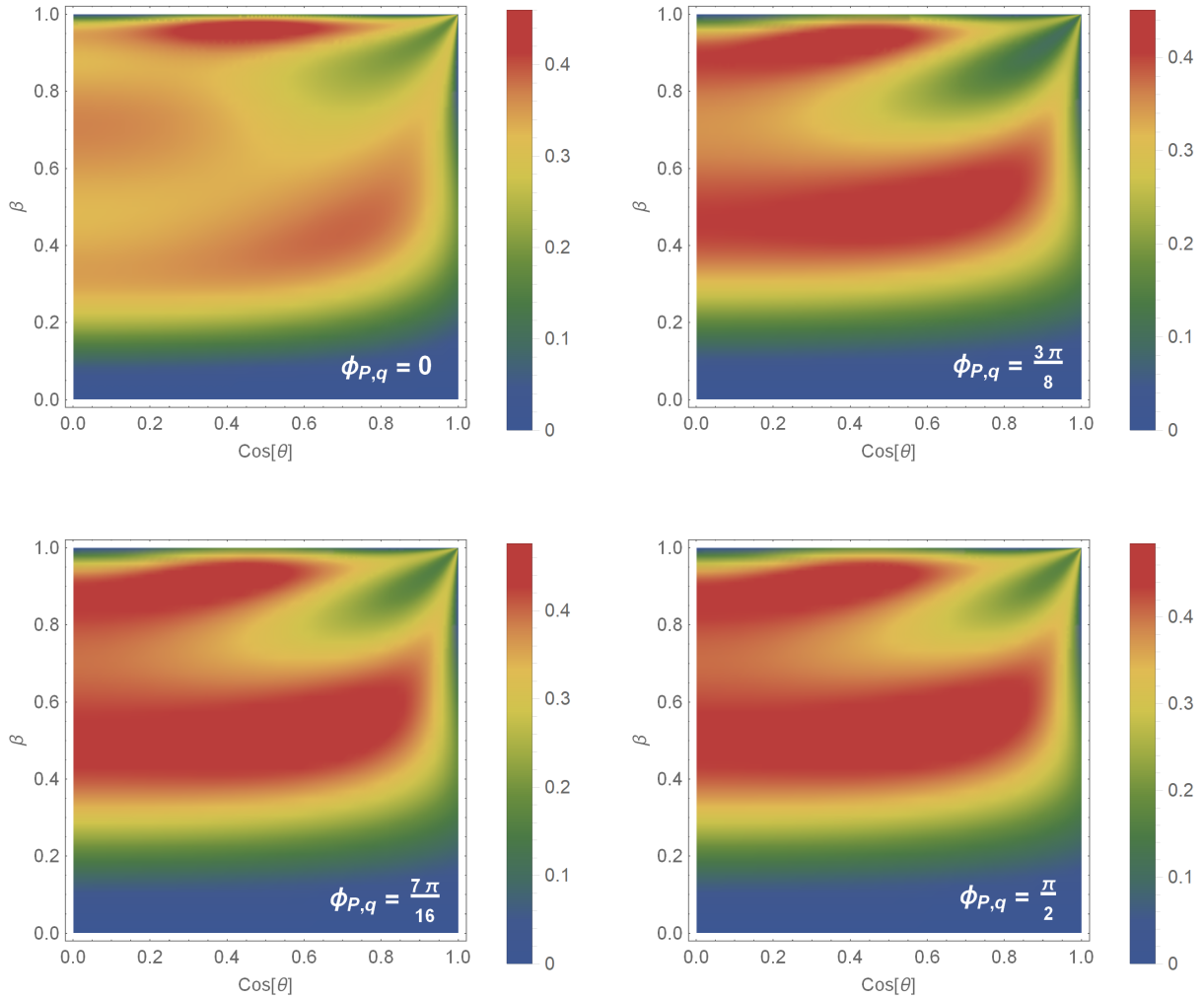
Concurrence

- **Concurrence** density plot ($\alpha = 0$, $q_\perp/Q_s = 3$ and varying $\phi_{P,q}$)



Magic

- **Magic** density plot ($\alpha = 0$ and varying $\phi_{P,q}$)



- The upper bound $S_{\mathrm{R}}^{\mathrm{stab}} \leq \ln \frac{9}{5} \simeq 0.588$ is not reached

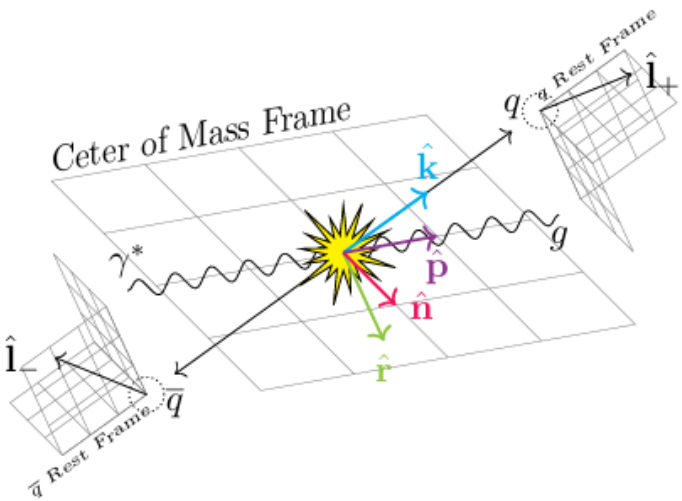
Accessing spin correlations for c and b quarks

- **Current strategy**: $Q \to$ decay products (e.g. $t \to b + W^+ \to b + \ell^+ + \nu_\ell$)
- Possible channels for c and b quarks [Kats, Uzan (2024)]

$$b \xrightarrow{7.0\%} \Lambda_b \xrightarrow{11\%} X_c^+ + \mu^- + \bar{\nu}_\mu$$

$$c \xrightarrow{6.4\%} \Lambda_c^+ \xrightarrow{3.5\%} \Lambda + \mu^+ + \nu_\mu$$

- Spin information from angular distributions



$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\theta_+ d\cos\theta_-} = \frac{1 + \alpha_+\alpha_- r_T^2 \cos\theta_+ \cos\theta_- C_{nn}}{4}$$

↑ decay angles ↑ spin correlation

- How does the quark-level correlation survive in observables?
- In practice, a number of issues: spin transfer in hadronization, low branching ratio. Much work to do!
- Ways to reconstruct spin correlation without resorting to weak decays? [Cheng, Yan (2025)]

Summary

- **Quantum correlations** are a fundamental feature of QM
- The interplay between **Quantum Information** and **QCD/collider physics** can provide new perspectives and insights for both fields
- We found several interesting theoretical insights about small-x and dense QCD looking at spin correlations
  - **Pomeron** leaves characteristic imprints on the entanglement patterns between $Q\bar{Q}$-pair diffractively produced
  - Entanglement in $b\bar{b}$-photoproduction shows sensitivity to the **linearly polarized W.W. gluon distribution**
- Quantum entanglement at the EIC: novel topic **beyond the Yellow report** (high luminosity and polarization capabilities)
- Many challenges and opportunities, both for theorists and experimentalists

Backup

Product states, classical correlations, and entanglement

- **Product state (no correlations):**

$$\rho = \rho_A \otimes \rho_B$$

Example: $|0\rangle\langle 0| \otimes |0\rangle\langle 0| = (|0\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 0|) = |00\rangle\langle 00|$

$\Rightarrow$ **Independent** measurement outcomes

- **Separable state (classical correlations)**:

$$\rho = \sum_i p_i \; \rho_A^{(i)} \otimes \rho_B^{(i)}$$

Example:

$$\rho = \frac{1}{2}\Big((|0\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 0|) + |1\rangle \otimes |1\rangle\langle 1| \otimes \langle 1|\Big) = \frac{1}{2}\Big(|00\rangle\langle 00| + |11\rangle\langle 11|\Big)$$

$\Rightarrow$ perfectly **correlated**, but no entanglement

- **Entangled state (quantum correlations):**

$$\rho \neq \sum_i p_i \; \rho_A^{(i)} \otimes \rho_B^{(i)}$$

Example (Bell state):

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$\Rightarrow$ **Correlations not reproducible by classical mixtures**

Spin density matrix in DIS ($\phi_{P,q}$ averaged)

$$C_{Tnn}^{(0)} = -\frac{2z\bar{z}\left(P_\perp^4+\bar{Q}^4\right)}{2m^2P_\perp^2+\left(z^2+\bar{z}^2\right)\left(P_\perp^4+\bar{Q}^4\right)}\,, \tag{30}$$

$$\begin{aligned} C_{Trr}^{(0)} = {} & \frac{-2z\bar{z}}{\left(m^2+P_\perp^2\right)\left(m^2(1-2z)^2+P_\perp^2\right)\left(\left(z^2+\bar{z}^2\right)\left(\bar{Q}^4+P_\perp^4\right)+2m^2P_\perp^2\right)} \\ & \times\Big\{m^4\left((1-2z)^2\bar{Q}^4-(3-2z)(1+2z)P_\perp^4\right) \\ & +2m^2P_\perp^2\left(2\bar{Q}^2P_\perp^2+(z^2+\bar{z}^2)\bar{Q}^4+(z^2+\bar{z}^2-2)P_\perp^4\right)-P_\perp^4\left(\bar{Q}^4+P_\perp^4\right)\Big\}\,, \end{aligned} \tag{31}$$

$$\begin{aligned} C_{Tkk}^{(0)} = {} & \frac{1}{\left(m^2+P_\perp^2\right)\left(m^2(1-2z)^2+P_\perp^2\right)\left(\left(z^2+\bar{z}^2\right)\left(\bar{Q}^4+P_\perp^4\right)+2m^2P_\perp^2\right)} \\ & \times\Big\{-m^4(1-2z)^2\left(4\bar{Q}^2P_\perp^2+(z^2+\bar{z}^2)\bar{Q}^4-(3+2\bar{z}z)P_\perp^4\right) \\ & +2m^2P_\perp^2\left(-2(z^2+\bar{z}^2)\bar{Q}^2P_\perp^2-4\bar{z}^2z^2\bar{Q}^4+\left(1-4\left(z^4-2z^3+z\right)\right)P_\perp^4\right) \\ & +(z^2+\bar{z}^2)P_\perp^4\left(\bar{Q}^4+P_\perp^4\right)+2m^6(1-2z)^2P_\perp^2\Big\}\,, \end{aligned} \tag{32}$$

$$\begin{aligned} C_{Trk}^{(0)} = {} & \frac{2m(1-z)|P_\perp|}{\left(m^2+P_\perp^2\right)\left(m^2(1-2z)^2+P_\perp^2\right)\left(\left(z^2+\bar{z}^2\right)\left(\bar{Q}^4+P_\perp^4\right)+2m^2P_\perp^2\right)} \\ & \times\Big\{-m^4(1-2z)\left((1-2z)\bar{Q}^2+(1+2z)P_\perp^2\right)+m^2\left(-4z^2\bar{Q}^2P_\perp^2+(1-2z)^2\bar{Q}^4+\left(8z^2-4z-1\right)P_\perp^4\right) \\ & +\left(4z^2-2z+1\right)\bar{Q}^4P_\perp^2+\bar{Q}^2P_\perp^4-2z(1-2z)P_\perp^6\Big\}\,, \end{aligned} \tag{33}$$

$$\begin{aligned} C_{Tkr}^{(0)} = {} & \frac{-2zm|P_\perp|}{\left(m^2+P_\perp^2\right)\left(m^2(1-2z)^2+P_\perp^2\right)\left(\left(z^2+\bar{z}^2\right)\left(\bar{Q}^4+P_\perp^4\right)+2m^2P_\perp^2\right)} \\ & \times\Big\{m^4(2z-1)\left((1-2z)\bar{Q}^2+(2z-3)P_\perp^2\right)+m^2\left(-4(1-z)^2\bar{Q}^2P_\perp^2+(1-2z)^2\bar{Q}^4+(3-4z(3-2z))P_\perp^4\right) \\ & +\left(4z^2-6z+3\right)\bar{Q}^4P_\perp^2+\bar{Q}^2P_\perp^4+\left(4z^2-6z+2\right)P_\perp^6\Big\}\,, \end{aligned} \tag{34}$$

Spin density matrix in diffractive DIS

$$\widetilde{C}^T_{nn} = -\frac{4\alpha_{em}e_q^2}{N_c} z\bar{z}k_\perp^2 (T_1+T_2)^2,$$

$$\begin{aligned}\widetilde{C}^T_{rr} &= \frac{\alpha_{em}e_q^2}{2z\bar{z}N_c}\Big[2T_1(T_1+T_2)k_\perp m\sqrt{1-\beta^2}\sin\theta(\beta+(1-2z)\cos\theta)+T_1^2m^2\left(1-\beta^2\right)\sin^2\theta \\ &\quad -(T_1+T_2)^2k_\perp^2\Big(\left(1-\beta^2\right)\sin^2\theta+2z(1-z)\left(\left(2-\beta^2\right)\cos^2\theta-1\right)\Big)\Big],\end{aligned}$$

$$\begin{aligned}\widetilde{C}^T_{kk} &= \frac{\alpha_{em}e_q^2}{2z\bar{z}N_c}\Big[2T_1(T_1+T_2)k_\perp m\sqrt{1-\beta^2}\sin\theta(\beta-(1-2z)\cos\theta)+T_1^2m^2\left(\cos^2\theta-\beta^2\right) \\ &\quad +(T_1+T_2)^2k_\perp^2\Big(\beta^2-2z(1-z)+\cos^2\theta\big(2z(1-z)\left(2-\beta^2\right)-1\big)\Big)\Big],\end{aligned}$$

$$\begin{aligned}\widetilde{C}^T_{rk} &= \frac{\alpha_{em}e_q^2}{2z\bar{z}N_c}\Big[2T_1(T_1+T_2)k_\perp m\left(z-\beta^2+\cos^2\theta\left(1-\left(2-\beta^2\right)z\right)\right) \\ &\quad +\sqrt{1-\beta^2}\sin\theta\left(T_1^2m^2(\cos\theta-\beta)+(T_1+T_2)^2k_\perp^2\left(\beta-(1-2z)^2\cos\theta\right)\right)\Big],\end{aligned}$$

$$\begin{aligned}\widetilde{C}^T_{kr} &= \frac{\alpha_{em}e_q^2}{2z\bar{z}N_c}\Big[2T_1(T_1+T_2)k_\perp m\left(z-\left(1-\beta^2\right)\sin^2\theta-\left(2-\beta^2\right)z\cos^2\theta\right) \\ &\quad +\sqrt{1-\beta^2}\sin\theta\left(T_1^2m^2(\beta+\cos\theta)-(T_1+T_2)^2k_\perp^2\left(\beta+(1-2z)^2\cos\theta\right)\right)\Big].\end{aligned}$$