Semirelativistic Bound-State Equations: Trivial Considerations

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Introduction: Spinless Salpeter equation

The recent years have witnessed a rise of attempts to study bound states by (semi-)relativistic equations of motion, such as the Klein–Gordon equation, the Dirac equation, or, as straightforward generalization of the Schrödinger equation, the spinless Salpeter equation, with several merits and drawbacks (see, e.g., Refs. [1] for details), inferred by nonrelativistic reduction (cf., e.g., Refs. [2]) of the Bethe–Salpeter equation [3]. For two particles of equal mass, m, interacting via a potential $V(\boldsymbol{x})$ depending on the relative coordinate $\boldsymbol{x}$, the spinless Salpeter equation is the eigenvalue equation of the Hamiltonian

$$H \equiv T(\boldsymbol{p}) + V(\boldsymbol{x}) \ , \qquad T(\boldsymbol{p}) = 2\sqrt{\boldsymbol{p}^2 + m^2} \ . \tag{1}$$

In view of the interest noted, we revisit these problems for central potentials $V(\boldsymbol{x}) = V(r)$, $r \equiv |\boldsymbol{x}|$, by recalling, or exploiting, a few well-known results.

Approximate solutions: Strict constraints

The existential question: Maximum number of bound states

In contrast to the Coulomb potential, lots of potentials (e.g., the Yukawa or the Woods–Saxon potential) admit only a finite number N of bound states: this number is a crucial characteristic. For the generic Schrödinger operator

$$H = \frac{\boldsymbol{p}^2}{2\,\mu} + V(r) \ , \qquad \mu > 0 \ , \qquad V(r) \leq 0 \ ,$$

the maybe most easy-to-evaluate upper limit to N is that by Bargmann [4]:

$$N \lneqq \frac{I\,(I+1)}{2}\ , \qquad I \equiv 2\,\mu \int_0^\infty \mathrm{d}r\, r\, |V(r)|\ .$$

For semirelativistic Hamiltonians of the shape (1), an upper limit to N is [5]

$$N \leq \frac{C}{12\,\pi} \int_0^\infty \mathrm{d}r\, r^2\, [|V(r)|\,(|V(r)| + 4\,m)]^{3/2}\ , \qquad \begin{array}{l} C = 6.0748\ (m = 0)\ , \\ C = 14.108\ (m > 0)\ . \end{array}$$

Narrowing down solutions: Rigorous bounds on eigenvalues

Due to the concavity of the square root as function of $\boldsymbol{p}^2$, the operator (1) is bounded from above by its Schrödinger limit, thus also all its eigenvalues [6]:

$$H \leq 2\,m + \frac{\boldsymbol{p}^2}{m} + V(\boldsymbol{x})\ .$$

The Rayleigh–Ritz variational method applies to self-adjoint Hilbert-space operators, H, bounded from below, with eigenvalues $E_0 \leq E_1 \leq E_2 \leq \cdots$: the d likewise ordered eigenvalues of H restricted to any d-dimensional trial subspace of the domain of H form upper bounds to the lowest d eigenvalues of H below the onset of its essential spectrum. It is favourable to know one's basis of this trial space analytically in both position and momentum spaces. We achieve this by our choice [7] of orthonormal basis defined in terms of the generalized-Laguerre polynomials $L_k^{(\gamma)}(x)$ for parameter γ and utilizing two variational parameters, μ, with unit mass dimension, and β, dimensionless:

$$\psi_{k,\ell m}(\boldsymbol{x}) \propto r^{\ell+\beta-1} \exp(-\mu\, r)\, L_k^{(2\ell+2\beta)}(2\,\mu\, r)\, \mathcal{Y}_{\ell m}(\Omega_{\boldsymbol{x}})\ ,$$

$$L_k^{(\gamma)}(x) \equiv \sum_{t=0}^{k} (-1)^t \binom{k+\gamma}{k-t} \frac{x^t}{t!}\ , \qquad k = 0, 1, 2, \ldots\ .$$

At the lower end of the spectrum, $T(\boldsymbol{p}) \geq 2\,m \geq 0$ entails $E_0 \geq \inf_{\boldsymbol{x}} V(\boldsymbol{x})$.

Accuracy and reliability of solutions: Master virial theorem

Both quality and accuracy [8] of approximate solutions to some bound-state equation are easily scrutinized by a relativistic generalization [9] of the virial theorem: all eigenstates $|\chi\rangle$ of some $T(\boldsymbol{p})+V(\boldsymbol{x})$ satisfy the master relation

$$\left\langle \chi \middle| \boldsymbol{p} \cdot \frac{\partial T}{\partial \boldsymbol{p}}(\boldsymbol{p}) \middle| \chi \right\rangle = \left\langle \chi \middle| \boldsymbol{x} \cdot \frac{\partial V}{\partial \boldsymbol{x}}(\boldsymbol{x}) \middle| \chi \right\rangle .$$

Desperately seeking analytic results: Seductions and pitfalls

Aiming at analytic approximations at any price triggers hectic activity [10]: Frequently, close encounters with the nonlocality of the Hamiltonian (1) are avoided by expanding $T(\boldsymbol{p})$ up to $O(\boldsymbol{p}^4/m^4)$, to deal with the Hamiltonians

$$H_{\rm p} \equiv 2\,m + \frac{\boldsymbol{p}^2}{m} - \frac{\boldsymbol{p}^4}{4\,m^3} + V(\boldsymbol{x}) \ .$$

However, the expectation value of each such $H_{\rm p}$ over, e.g., the trial function $\phi(r) \propto \exp(-\mu\,r)$ reveals that the operator $H_{\rm p}$ is not bounded from below:

$$\langle H_{\rm p}\rangle = 2\,m + \frac{\mu^2}{m} - \frac{5\,\mu^4}{4\,m^3} + \langle V(\boldsymbol{x})\rangle \ \Rightarrow \ \lim_{\mu\to\infty}\langle H_{\rm p}\rangle = -\infty \ \Rightarrow \ E_0 \le -\infty \ .$$

Consequently, searches for ground states are doomed to fail. However, some perturbative approach to $\boldsymbol{p}^4/4\,m^3$, if adopted correctly, might save the day. Expansions over potential-inspired functions [11] mitigate the singularity of the Laplacian's centrifugal term $\sim r^{-2}$, but alter the full effective potential.

Applications: Real Woods–Saxon problem

For a first try, we apply [12] our concepts to a rather tame potential, familiar from nuclear physics, the Woods–Saxon potential [13], but for real coupling:

$$V(r) = -\frac{V_0}{1+\exp\left(\frac{r-R}{a}\right)} \ , \qquad V_0 > 0 \ , \qquad R \ge 0 \ , \qquad a > 0 \ .$$

Both nonrelativistic and variational upper limits to the binding energies (in GeV) of semirelativistic Woods–Saxon bound states with radial and orbital angular momentum quantum numbers n_r and ℓ are easily obtained, e.g., for variational setup $\mu = 1$ GeV, $\beta = 1$, $d = 25$ and realistic parameter values $m = 940.271$ MeV, $V_0 = 67.70352$ MeV, $R = 7.6136$ fm, $a = 0.65$ fm [14]:

n_r	ℓ	Spinless Salpeter equation	Schrödinger equation
0	0	-0.06032	-0.06030
	1	-0.05309	-0.05305
1	0	-0.04119	-0.04108
	1	-0.02967	-0.02946
2	0	-0.01527	-0.01545
	1	-0.00233	-0.00362

The lower bound $E_0 \geq \inf_r V(r) = V(0) = -67.70296$ MeV $\gtrapprox -V_0$ to the binding energy is evident. In the interval $V(0) < E_k \leq 0$ $(k = 0, 1, \ldots, N)$ defined by this bound, at most $N \leq 850$ and $N \leq 1201$ (for relativistic and nonrelativistic kinematics, respectively) eigenstates can be accommodated. Moreover, for our variational ground state, we get $\langle \boldsymbol{p}^2/m^2 \rangle \approx 6 \times 10^{-3}$: the system is highly nonrelativistic, so it hardly warrants a relativistic analysis.

Summary and conclusions

Even though the spinless Salpeter equation resists being solved analytically, elementary techniques allow us to draw a clear picture of the solutions to be expected. Nevertheless, not all offered solutions respect the picture's frame.

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