On the mathematical error of Aspect/Bell and its resolution

Frank Lad

University of Canterbury, Department of Mathematics and Statistics

You may download these overheads and my manuscript with this same title from https://www.researchgate.net/

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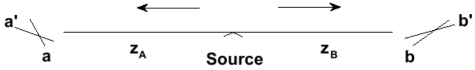
Topics for this talk:

- Review the setup of Bell's inequality in an optical context
  and its violation as currently understood
- Identify neglected symmetric functional relations
- Review de Finetti's "fundamental theorem of prevision"
- Apply the FTP to the QM-motivated assertions
  yielding a 4-D coherent prevision polytope
- Review, assess, and correct Aspect's empirical work
  and perhaps Yack a bit

The Journeys of a pair of prepared photons

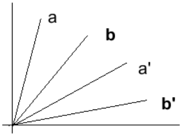


and their detection via angled polarizers

$$A(\mathbf{a}^*, \lambda) = +1 \quad \text{when parallel detection}$$
$$A(\mathbf{a}^*, \lambda) = -1 \quad \text{when perpendicular detection, and}$$

and similarly for $B(\cdot, \cdot) = \pm 1$

Relative angle settings of detectors yielding the most egregious purported violation of Bell's inequality



(a,b) = $-\pi/8$

(a,b') = $-3\pi/8$

(a',b) = $\pi/8$

(a',b') = $-\pi/8$

BTW ... Double these angles ... $-\pi/4, -3\pi/4, \pi/4$ and $-\pi/4$

Why ? We'll see !

QM-motivated probabilities

$$P[(A(\mathbf{a}^*) = +1)(B(\mathbf{b}^*) = +1)]$$
$$= \; P[(A(\mathbf{a}^*) = -1)(B(\mathbf{b}^*) = -1)] \; = \; \frac{1}{2}\, cos^2(\mathbf{a}^*, \mathbf{b}^*) \; ,$$

and

$$P[(A(\mathbf{a}^*) = +1)(B(\mathbf{b}^*) = -1)]$$
$$= \; P[(A(\mathbf{a}^*) = -1)(B(\mathbf{b}^*) = +1)] \; = \; \frac{1}{2}\, sin^2(\mathbf{a}^*, \mathbf{b}^*) \; .$$

N.B. These imply $E[A(\mathbf{a}^*)B(\mathbf{b}^*)] \; = \; cos\, 2(\mathbf{a}^*, \mathbf{b}^*)$,

and BTW $P[A(\mathbf{a}^*) = +1] = P[B(\mathbf{b}^*) = +1] \; = \; 1/2\,.$

The Metaphysics

$s(\lambda, \mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}') \equiv$ as per Aspect/Bell/CHSH

$$A(\lambda, \mathbf{a})\, B(\lambda, \mathbf{b}) \;-\; A(\lambda, \mathbf{a})\, B(\lambda, \mathbf{b}') \;+\; A(\lambda, \mathbf{a}')\, B(\lambda, \mathbf{b}) \;+\; A(\lambda, \mathbf{a}')\, B(\lambda, \mathbf{b}')$$

for $\lambda \in \Lambda$

$$= A(\lambda, \mathbf{a})\, [B(\lambda, \mathbf{b}) - B(\lambda, \mathbf{b}')] \;+\; A(\lambda, \mathbf{a}')\, [B(\lambda, \mathbf{b}) + B(\lambda, \mathbf{b}')]$$

$$= B(\lambda, \mathbf{b})\, [A(\lambda, \mathbf{a}) + A(\lambda, \mathbf{a}')] \;-\; B(\lambda, \mathbf{b}')\, [A(\lambda, \mathbf{a}) - A(\lambda, \mathbf{a}')]$$

$$\in \{-2, +2\}$$

So $E[s(\lambda)] \;=\; E[A(\lambda, \mathbf{a})B(\lambda, \mathbf{b})] \;-\; E[A(\lambda, \mathbf{a})B(\lambda, \mathbf{b}')]$

$$+\; E[A(\lambda, \mathbf{a}')B(\lambda, \mathbf{b})] \;+\; E[A(\lambda, \mathbf{a}')B(\lambda, \mathbf{b}')]$$

$$= E[A(\mathbf{a})B(\mathbf{b})] \;-\; E[A(\mathbf{a})B(\mathbf{b}')] \;+\; E[A(\mathbf{a}')B(\mathbf{b})] \;+\; E[A(\mathbf{a}')B(\mathbf{b}')]$$

Applying the QM probs and expectations

to all four egregious angles yields

$$cos\, 2(\mathbf{a}, \mathbf{b}) \;=\; cos\, 2(\mathbf{a}', \mathbf{b}) \;=\; cos\, 2(\mathbf{a}', \mathbf{b}') \;=\; 1/\sqrt{2}$$

and $\quad cos\, 2(\mathbf{a}, \mathbf{b}') \;=\; -1/\sqrt{2}$

So it seems $\quad E[s(\lambda, \mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}')] \;=\; 2\sqrt{2} \;>\; 2$!!!

Hmmmm ... Let's see !

BTW for later ...

$\frac{1}{2} cos^2(\mathbf{a}, \mathbf{b}) = \frac{1}{2} cos^2(\mathbf{a}', \mathbf{b}) = \frac{1}{2} cos^2(\mathbf{a}', \mathbf{b}') = \frac{1}{2} sin^2(\mathbf{a}, \mathbf{b}') \approx .4268$
and $\frac{1}{2} sin^2(\mathbf{a}, \mathbf{b}) = \frac{1}{2} sin^2(\mathbf{a}', \mathbf{b}) = \frac{1}{2} sin^2(\mathbf{a}', \mathbf{b}') = \frac{1}{2} cos^2(\mathbf{a}, \mathbf{b}') \approx .0732$

Let's consider the "realm matrix" of *all*

(im)possible observations ... smile ...

We'll look in banks of *columns* at

the observable quantities $A(\mathbf{a}), B(\mathbf{b}), A(\mathbf{a}'), B(\mathbf{b}')$;

their products

$A(\mathbf{a})B(\mathbf{b}),\ A(\mathbf{a})B(\mathbf{b}'),\ A(\mathbf{a}')B(\mathbf{b}),\ A(\mathbf{a}')B(\mathbf{b}')$;

a mysterious product $\mathcal{A}(\mathbf{a}')\mathcal{B}(\mathbf{b}')$;

and four symmetric function quantities

$\Sigma_{/(\mathbf{a},\mathbf{b})},\ \Sigma_{/(\mathbf{a},\mathbf{b}')},\ \Sigma_{/(\mathbf{a}',\mathbf{b})},\ \Sigma_{/(\mathbf{a}',\mathbf{b}')}$ . ... PLUS YACK

$$
\mathbf{R}\begin{pmatrix}
A(\mathbf{a})\\
B(\mathbf{b})\\
A(\mathbf{a}')\\
B(\mathbf{b}')\\
*****\\
A(\mathbf{a})B(\mathbf{b})\\
A(\mathbf{a})B(\mathbf{b}')\\
A(\mathbf{a}')B(\mathbf{b})\\
A(\mathbf{a}')B(\mathbf{b}')\\
*****\\
\mathcal{A}(\mathbf{a}')\mathcal{B}(\mathbf{b}')\\
*****\\
\sum_{/(\mathbf{a},\mathbf{b})}\\
\sum_{/(\mathbf{a},\mathbf{b}')}\\
\sum_{/(\mathbf{a}',\mathbf{b})}\\
\sum_{/(\mathbf{a}',\mathbf{b}')}\\
*****\\
s(\lambda)\\
s_{\mathcal{A}/\mathcal{B}}(\mathbf{a}',\mathbf{b}')
\end{pmatrix} =
$$

$$
\mathbf{R}\begin{pmatrix}
s(\lambda)\\
s_{\mathcal{A}/\mathcal{B}}(\mathbf{a}',\mathbf{b}')\\
1\\
*****\\
(A(\mathbf{a})=+1)(B(\mathbf{b})=+1)\\
(A(\mathbf{a})=-1)(B(\mathbf{b})=-1)\\
(A(\mathbf{a})=+1)(B(\mathbf{b})=-1)\\
*****\\
(A(\mathbf{a})=+1)(B(\mathbf{b}')=+1)\\
(A(\mathbf{a})=-1)(B(\mathbf{b}')=-1)\\
(A(\mathbf{a})=+1)(B(\mathbf{b}')=-1)\\
*****\\
(A(\mathbf{a}')=+1)(B(\mathbf{b})=+1)\\
(A(\mathbf{a}')=-1)(B(\mathbf{b})=-1)\\
(A(\mathbf{a}')=+1)(B(\mathbf{b})=-1)\\
*****\\
(A(\mathbf{a}')=+1)(B(\mathbf{b}')=+1)\\
(A(\mathbf{a}')=-1)(B(\mathbf{b}')=-1)\\
(A(\mathbf{a}')=+1)(B(\mathbf{b}')=-1)
\end{pmatrix} =
$$

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
\\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\\
3 & -1 & -1 & -1 & 1 & 1 & -3 & 1 & 1 & -3 & 1 & 1 & -1 & -1 & -1 & 3 \\
3 & 1 & -1 & 1 & 1 & -1 & -3 & -1 & -1 & -3 & -1 & 1 & 1 & -1 & 1 & 3 \\
3 & -1 & 1 & 1 & -1 & -1 & -3 & 1 & 1 & -3 & -1 & -1 & 1 & 1 & -1 & 3 \\
3 & 1 & 1 & -1 & -1 & 1 & -3 & -1 & -1 & -3 & 1 & -1 & -1 & 1 & 1 & 3 \\
\\
2 & 2 & -2 & 2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 & 2 & 2 & -2 & 2 & 2 \\
2 & 4 & 0 & 2 & 2 & 0 & 0 & -2 & -4 & -2 & -2 & 0 & 0 & -2 & 2 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
2 & 2 & -2 & 2 & 2 & -2 & -2 & -2 & -2 & -2 & -2 & 2 & 2 & -2 & 2 & 2 \\
2 & 4 & 0 & 2 & 2 & 0 & 0 & -2 & -4 & -2 & -2 & 0 & 0 & -2 & 2 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\[1em]
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\[1em]
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\[1em]
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\[1em]
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

After the yack you know ...

$$\Sigma_{/(\mathbf{a}',\mathbf{b}')} \equiv \Sigma\Big(A(\mathbf{a})B(\mathbf{b}),\ A(\mathbf{a})B(\mathbf{b}'),\ A(\mathbf{a}')B(\mathbf{b})\Big)$$

$$\equiv A(\mathbf{a})B(\mathbf{b}) + A(\mathbf{a})B(\mathbf{b}') + A(\mathbf{a}')B(\mathbf{b})$$

and similarly for other quantities named $\Sigma_{/(\mathbf{a}^*,\mathbf{b}^*)}$

and

$$A(\mathbf{a}')B(\mathbf{b}') = (\Sigma_{/(\mathbf{a}',\mathbf{b}')} = 3 \textit{ or } -1) - (\Sigma_{/(\mathbf{a}',\mathbf{b}')} = -3 \textit{ or } +1)$$

and similarly for other quantities named $A(\mathbf{a}^*)B(\mathbf{b}^*)$

These are completely symmetric functional relations.

The neglected functional relations imply ...

$$\text{Well} \quad E[s(\lambda)] \;=\; E[A(\lambda,\mathbf{a})B(\lambda,\mathbf{b})] \;-\; E[A(\lambda,\mathbf{a})B(\lambda,\mathbf{b}')] \\ \;+\; E[A(\lambda,\mathbf{a}')B(\lambda,\mathbf{b})] \;+\; E[A(\lambda,\mathbf{a}')B(\lambda,\mathbf{b}')]$$

... sure enough, BUT ... this equals

$$= \; E[A(\mathbf{a})B(\mathbf{b})] \;-\; E[A(\mathbf{a})B(\mathbf{b}')] \;+\; E[A(\mathbf{a}')B(\mathbf{b})] \\ \;+\; E[(\Sigma_{/(\mathbf{a}',\mathbf{b}')} \;=\; 3 \; or - 1) \;-\; (\Sigma_{/(\mathbf{a}',\mathbf{b}')} \;=\; -3 \; or + 1)]$$

... enter Bruno de Finetti and FTP

The fundamental theorem of probability (prevision)

$\mathbf{X}_{N+1}$ ... any quantities whatsoever

$$\mathbf{R}(\mathbf{X}_{N+1}) \ = \ \binom{\mathbf{R}_{N,K}}{\mathbf{r}_{N+1}}$$

If you assert $P(\mathbf{X}_N) \ = \ \mathbf{p}_N$

then the further assertion of $P(X_{N+1})$ coheres, iff

it lies within $\{min\, \mathbf{r}_{N+1}\, \mathbf{q}_K,\ \ max\, \mathbf{r}_{N+1}\, \mathbf{q}_K\}$

subject to restrictions that $\mathbf{R}_{N,K}\, \mathbf{q}_K \ = \ \mathbf{p}_N$

along with $\mathbf{1}_K^T \mathbf{q}_K \ = \ 1$ and $\mathbf{q}_K \geq \mathbf{0}_K$.

If there is no feasible solution

then $P(\mathbf{X}_N) \ = \ \mathbf{p}_N$ itself is incoherent.

What do coherent assertions of QM probabilities specify ?

Look at the realm matrix to see the setup
of the LP problems ...

QM probs for a polarization pair are coherent *for any angle setting*

QM probs for the *same photon pair* are coherent for any two angles

QM probs for the *same photon pair* are coherent for any three angles

QM probs for the *same photon pair* at **all four angle settings** are
INCOHERENT !

What *do* assertions for any three angles imply for the fourth ???

Results of 8 LP problems ... in a Table

... with Yack provided right here ...

Table 1: Bounding values of coherent QM expectations for $s(\lambda)$

LP problem	$E[s(\lambda)]$	$P_{++}(\mathbf{a}^*, \mathbf{b}^*)$	$P_{+-}(\mathbf{a}^*, \mathbf{b}^*)$	$E[A(\mathbf{a}^*)\, B(\mathbf{b}^*)$
min $E[s(\lambda)](\mathbf{a}, \mathbf{b}')$	1.1213	.5	0	1.0
max $E[s(\lambda)](\mathbf{a}, \mathbf{b}')$	2.0	.2803	.2197	.1213
min $E[s(\lambda)](\mathbf{a}', \mathbf{b}')$	1.1213	0	.5	-1.0
max $E[s(\lambda)](\mathbf{a}', \mathbf{b}')$	2.0	.2197	.2803	$-.1213$
min $E[s(\lambda)](\mathbf{a}, \mathbf{b})$	1.1213	0	.5	-1.0
max $E[s(\lambda)](\mathbf{a}, \mathbf{b})$	2.0	.2197	.2803	$-.1213$
min $E[s(\lambda)](\mathbf{a}', \mathbf{b})$	1.1213	0	.5	-1.0
max $E[s(\lambda)](\mathbf{a}', \mathbf{b})$	2.0	.2197	.2803	$-.1213$

Their solution vectors are the cols of $\quad \mathbf{q_{16}VertexMat} =$

$$\begin{pmatrix}
.3902 & .2803 & 0 & .0732 & 0 & .0732 & 0 & .0732 \\
0 & .0732 & .3902 & .2803 & 0 & .0732 & 0 & .0732 \\
.0366 & 0 & .0366 & 0 & 0 & 0 & .0366 & 0 \\
0 & .0732 & 0 & .0732 & 0 & .0732 & .3902 & .2803 \\
0 & .0732 & 0 & .0732 & .3902 & .2803 & 0 & .0732 \\
.0366 & 0 & .0366 & 0 & .0366 & 0 & 0 & 0 \\
0 & 0 & .0366 & 0 & .0366 & 0 & .0366 & 0 \\
.0366 & 0 & 0 & 0 & .0366 & 0 & .0366 & 0 \\
.0366 & 0 & 0 & 0 & .0366 & 0 & .0366 & 0 \\
0 & 0 & .0366 & 0 & .0366 & 0 & .0366 & 0 \\
.0366 & 0 & .0366 & 0 & .0366 & 0 & 0 & 0 \\
0 & .0732 & 0 & .0732 & .3902 & .2803 & 0 & .0732 \\
0 & .0732 & 0 & .0732 & 0 & .0732 & .3902 & .2803 \\
.0366 & 0 & .0366 & 0 & 0 & 0 & .0366 & 0 \\
0 & .0732 & .3902 & .2803 & 0 & .0732 & 0 & .0732 \\
.3902 & .2803 & 0 & .0732 & 0 & .0732 & 0 & .0732
\end{pmatrix}$$

Projection of the Prevision Polytope ... in a Table

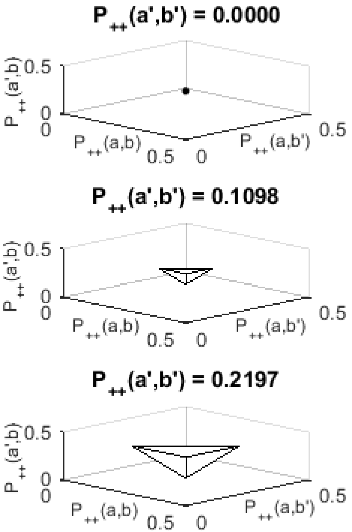
... with even more Yack ...

Table 2: Vertex vectors of the coherent QM probability polytope

$P_{++}(\mathbf{a},\mathbf{b})$	0.4268	0.4268	0.4268	0.4268	0.0000	0.2197	0.4268	0.4268
$P_{++}(\mathbf{a},\mathbf{b}')$	0.5000	0.2803	0.0732	0.0732	0.0732	0.0732	0.0732	0.0732
$P_{++}(\mathbf{a}',\mathbf{b})$	0.4268	0.4268	0.4268	0.4268	0.4268	0.4268	0.0000	0.2197
$P_{++}(\mathbf{a}',\mathbf{b}')$	0.4268	0.4268	0.0000	0.2197	0.4268	0.4268	0.4268	0.4268
$E[s(\lambda)]$	1.1213	2.0000	1.1213	2.0000	1.1213	2.0000	1.1213	2.0000

A movie of this 4-D polytope passing through 3-D space.
Rachael Tappenden, director

In SLO MO, Slices of the 4-D $\mathbf{P}_{++}$ polytope



What to make of Aspect's Empirical Estimation Results?

Estimate each product moment by method of moments ...

$$\hat{E}(\mathbf{a},\mathbf{b}) \quad = \quad \frac{[N_{++}(\mathbf{a},\mathbf{b}) - N_{+-}(\mathbf{a},\mathbf{b}) - N_{-+}(\mathbf{a},\mathbf{b}) + N_{--}(\mathbf{a},\mathbf{b})]}{[N_{++}(\mathbf{a},\mathbf{b}) + N_{+-}(\mathbf{a},\mathbf{b}) + N_{-+}(\mathbf{a},\mathbf{b}) + N_{--}(\mathbf{a},\mathbf{b})]} \quad ,$$

using experiments on distinct photon pairs

Well OK ... but don't pretend that all four product pairs are free!

Let's check consequences of recognition using simulation data

Simulation Results ... requiring some Yack

"Bell's Theorem: the Naive View of an Experimentalist" ... A. Aspect, 2002

Table 3: Corrections to Aspect's estimate of $E[s(\lambda)]$

	$(\mathbf{a}', \mathbf{b}')$	$(\mathbf{a}, \mathbf{b}')$	$(\mathbf{a}', \mathbf{b})$	$(\mathbf{a}', \mathbf{b}')$
$\hat{E}[A(\mathbf{a}^*)B(\mathbf{b}^*)]$	0.707232	-0.706186	0.706840	0.707480
Aspect $\hat{E}[s]$	2.827738	2.827738	2.827738	2.827738
$\hat{E}[A(\mathbf{a}^*)B(\mathbf{b}^*)]$ as fnctn	-0.353078	0.354348	-0.354766	-0.353934
Corrected $\hat{E}[s]$	1.767180	1.767204	1.765740	1.766964

Note the tantalizing tease of an "estimate" near to $2.5/\sqrt{2} =$ 1.767766952966369

Hmmm ...

Concluding Comments

** Bell would be pleased today, worries about the boundary

** Einstein's pleading for supplementary or hidden variables

** Partial assertions of interval probabilities

** General relevance of complementary distributions?

$$\mathbf{q}_N \; = \; (N-1)^{-1} \, (\mathbf{1} - \mathbf{p}_N)$$

** References and downloads, see next slide

Well, so it is St Paddy's day ... I'll leave you with a problem from another Irishman, Samuel Beckett

References and Downloads

** Download an MS and slides for this lecture from
https://www.researchgate.net/

** A very insightful article that should be studied by physicists and probabilists is ...
"The role of probability in Statistical Physics", Romano Scozzafava (2000),
Transport Theory and Statistical Physics, **29**, 107-123.

** Something interesting of my own, with colleagues, is ...

"Extropy: complementary dual of Entropy", Frank Lad, Giuseppe Sanfilippo and Gianna Agrò (2015) *Statistical Science*, **30**, 40-58.