

Non-Abelian Anomalies and Hadronic Fluids

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Lect. Notes Phys. 81 (2013); EM, M.Valle, JHEP1411 (2014);
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Issues

- 1 Introduction: Anomalous Transport
 - The Chiral Magnetic Effect
 - Hydrodynamics of Relativistic Fluids
- 2 Equilibrium Partition Function Formalism to Hydrodynamics
 - Equilibrium Partition Function
 - Derivative Expansion at 1st order
- 3 Non-Abelian Anomalies
 - The chiral anomaly
 - Partition function. Currents without Goldstone bosons
 - The Wess-Zumino-Witten partition function

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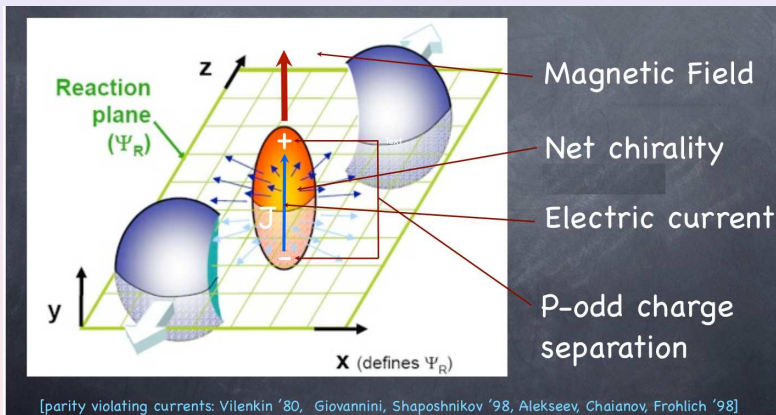
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The Chiral Magnetic Effect (CME)

[Kharzeev, McLerran, Warringa '07]



Strong Magnetic field induces a $\mathcal{P}$ -odd charge separation $\implies$
 $\implies$ Electric current: $\vec{J} = \sigma^B \vec{B}$.

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Hydrodynamics of Relativistic Fluids

[Son,Surowka], [Eling,Neiman,Oz], [Erdmenger et al.], [Banerjee et al.], [Loganayagam],
[Kharzeev, Yee], [Sadovyyev et al.], [Landsteiner, EM, Pena-Benitez], . . .

$$\begin{aligned}
 \langle T^{\mu\nu} \rangle &= \underbrace{(\varepsilon + \mathcal{P})u^\mu u^\nu + \mathcal{P}g^{\mu\nu}}_{\text{Ideal Hydro}} + \underbrace{\langle T^{\mu\nu} \rangle_{\text{diss \& anom}}}_{\text{Dissipative \& Anomalous}}, \\
 \langle J^\mu \rangle &= \underbrace{nu^\mu}_{\text{Ideal Hydro}} + \underbrace{\langle J^\mu \rangle_{\text{diss \& anom}}}_{\text{Dissipative \& Anomalous}}.
 \end{aligned}$$

- Landau frame: $\langle T^{0i} \rangle \sim u^i$

$$\begin{aligned}
 \langle T^{\mu\nu} \rangle_{\text{diss \& anom}} &= -\eta P^{\mu\alpha} P^{\nu\beta} \left(D_\alpha u_\beta + D_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta} D^\lambda u_\lambda \right) - \zeta P^{\mu\nu} D^\alpha u_\alpha + \dots \\
 \langle J^\mu \rangle_{\text{diss \& anom}} &= -\sigma T P^{\mu\nu} D_\nu \left(\frac{\mu}{T} \right) + \sigma E^\mu + \underbrace{\sigma^B B^\mu}_{\text{CME}} + \underbrace{\sigma^\nu \omega_\nu^\mu}_{\text{CVE}} + \dots
 \end{aligned}$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, and **vorticity**: $\omega^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\lambda} u_\nu D_\rho u_\lambda \rightarrow$
Chiral Vortical Effect (CVE).

Parity and Time Reversal Properties

- Hydrodynamics at 1st order in derivative expansion:

$$\text{CME : } \underbrace{\langle \vec{J} \rangle_1}_{\mathcal{P}\text{-odd}, \mathcal{T}\text{-odd}} = \underbrace{\sigma^{\mathcal{B}}}_{\mathcal{P}\text{-odd}, \mathcal{T}\text{-even}} \cdot \underbrace{\vec{B}}_{\mathcal{P}\text{-even}, \mathcal{T}\text{-odd}}$$

$$\text{CVE : } \underbrace{\langle \vec{J} \rangle_1}_{\mathcal{P}\text{-odd}, \mathcal{T}\text{-odd}} = \underbrace{\sigma^{\mathcal{V}}}_{\mathcal{P}\text{-odd}, \mathcal{T}\text{-even}} \cdot \underbrace{\vec{\omega}}_{\mathcal{P}\text{-even}, \mathcal{T}\text{-odd}}$$

$\mathcal{T}$ - even $\implies \sigma^{\mathcal{B}}$ and $\sigma^{\mathcal{V}}$ are non dissipative, i.e. they cannot contribute to entropy production:

$$\frac{\partial}{\partial t} s > 0 \quad (\text{Only } \mathcal{T}\text{-odd contributions in } s)$$

Electric conductivity is dissipative: $\langle \vec{J} \rangle_1 = \sigma \vec{E} \implies \partial_t s > 0$.

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Equilibrium Partition Function

[Banerjee et al '12], [Jensen et al '13], [Bhattacharyya '14], [EM, Valle '14]

- Relativistic Invariant Quantum Field Theory on the manifold

$$ds^2 = -e^{2\sigma(\vec{x})}(dt + a_i(\vec{x})dx^i)^2 + g_{ij}(x)dx^i dx^j$$

and time independent background $U(1)$ gauge connection:

$$\mathcal{A} = \mathcal{A}_0(\vec{x})dx^0 + \mathcal{A}_i(\vec{x})dx^i.$$

- Partition function of the system:

$$Z = \text{Tr} e^{-\frac{H - \mu_0 Q}{T_0}}$$

→ Dependence of Z on σ , g_{ij} and a_i ?

- Most general partition function consistent with:
 - 3-dim diffeomorphism invariance.
 - Kaluza-Klein (KK) invariance: $t \rightarrow t + \phi(\vec{x})$, $\vec{x} \rightarrow \vec{x}$.
 - $U(1)$ time-independent gauge invariance (up to an anomaly).

Equilibrium Partition Function

- Stress Tensor and $U(1)$ current $\rightarrow$ under t -indep variations

$$\delta \log Z = \frac{1}{T_0} \int d^3x \sqrt{-g_3} \left(-\frac{1}{2} T_{\mu\nu} \delta g^{\mu\nu} + J^\mu \delta \mathcal{A}_\mu \right)$$

$$\rightarrow T_{\mu\nu} = -2T_0 \frac{\delta \log Z}{\delta g^{\mu\nu}}, \quad J^\mu = T_0 \frac{\delta \log Z}{\delta \mathcal{A}_\mu}.$$

- In particular, for $\log Z = \mathcal{W}(e^\sigma, A_0, a_i, A_i, g^{ij}, T_0, \mu_0)$, where

$$A_0 = \mathcal{A}_0, \quad A_i = \mathcal{A}_i - a_i \mathcal{A}_0,$$

are KK invariant quantities, one gets the consistent currents

$$\begin{aligned} \langle J^i \rangle_{\text{cons}} &= \frac{T_0}{\sqrt{-G}} \frac{\delta \mathcal{W}}{\delta A_i}, & \langle J_0 \rangle_{\text{cons}} &= -\frac{T_0 e^{2\sigma}}{\sqrt{-G}} \frac{\delta \mathcal{W}}{\delta A_0}, \\ \langle T_0^i \rangle &= \frac{T_0}{\sqrt{-G}} \left(\frac{\delta \mathcal{W}}{\delta a_i} - A_0 \frac{\delta \mathcal{W}}{\delta A_i} \right), & \langle T_{00} \rangle &= -\frac{T_0 e^{2\sigma}}{\sqrt{-G}} \frac{\delta \mathcal{W}}{\delta \sigma}. \end{aligned}$$

- $\rightarrow \mathcal{W}$ is a generating functional for the hydrodynamic constitutive relations.

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Equilibrium Partition Function at first order

- **Partition function at 1st order** in derivative expansion [Banerjee et al '12; EM, M.Valle '14]:

$$\mathcal{W}^{(1)} = \int d^3x \sqrt{g_3} \left[\alpha_1(\sigma, A_0) \epsilon^{ijk} A_i A_{jk} + \alpha_2(\sigma, A_0) \epsilon^{ijk} A_i f_{jk} + \alpha_3(\sigma, A_0) \epsilon^{ijk} a_i f_{jk} \right]$$

where $A_{ij} = \partial_i A_j - \partial_j A_i$, $f_{ij} = \partial_i a_j - \partial_j a_i$.

- **Ideal gas of Dirac fermions** $\rightarrow$ from a computation of $\langle T_0^i \rangle$ and $\langle J^i \rangle$, one gets

$$\alpha_1(\sigma, A_0) = \frac{C}{6T_0} A_0, \quad \alpha_2(\sigma, A_0) = \frac{1}{2} \left(\frac{C}{6T_0} A_0^2 + C_2 T_0 \right), \quad \alpha_3(\sigma, A_0) = 0.$$

where: $\begin{cases} C = -\frac{1}{4\pi^2} & (\text{chiral anomaly}): [\text{Son, Surowka '09}, [\text{Erdmenger et al '09}], \dots \\ C_2 = \frac{1}{24} & (\text{gauge-gravitational anomaly}): [\text{Landsteiner, EM, Pena-Benitez '11}] \end{cases}$

- **Coefficients related to chiral magnetic σ^B and chiral vortical σ^V conductivities.**

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The chiral anomaly

- Under a general shift $\mathcal{A}_\mu^a \rightarrow \mathcal{A}_\mu^a + \delta \mathcal{A}_\mu^a$:

$$\delta \Gamma[\mathcal{A}] = \int d^D x \, \delta \mathcal{A}_\mu^a J_a^\mu(x)_{\text{cons}} . \quad (1)$$

- The anomaly is given by the *“failure of the effective action to be invariant under axial gauge transformations”*:

$$\mathcal{A}_\mu \longrightarrow U \mathcal{A}_\mu U^{-1} - i \partial_\mu U U^{-1}, \quad U = \exp(i \Lambda_a^{\text{Axial}} t_a),$$

$$\delta_{\text{gauge}} \Gamma[\mathcal{A}] = - \int d^D x \, \Lambda^{\text{Axial}, a} G_a[\mathcal{A}].$$

- Particularizing (1) to $\delta \mathcal{A}_\mu^a = (D_\mu \Lambda^{\text{Axial}})^a \rightarrow$
 $\rightarrow$ (non)-conservation law for the consistent current:

$$D_\mu J_a^\mu(x)_{\text{cons}} = G_a[\mathcal{A}(x)].$$

The non-abelian anomaly

- Bardeen form of the non-abelian anomaly for the symmetry group $U(N_f) \times U(N_f)$ [W.A. Bardeen, PR184, '69]:

$$G_a[\mathcal{V}, \mathcal{A}] = \frac{iN_c}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left\{ \lambda_a \left[\mathcal{V}_{\mu\nu} \mathcal{V}_{\rho\sigma} + \frac{1}{3} \mathcal{A}_{\mu\nu} \mathcal{A}_{\rho\sigma} - \frac{32}{3} \mathcal{A}_\mu \mathcal{A}_\nu \mathcal{A}_\rho \mathcal{A}_\sigma + \frac{8}{3} i (\mathcal{A}_\mu \mathcal{A}_\nu \mathcal{V}_{\rho\sigma} + \mathcal{A}_\mu \mathcal{V}_{\rho\sigma} \mathcal{A}_\nu + \mathcal{V}_{\rho\sigma} \mathcal{A}_\mu \mathcal{A}_\nu) \right] \right\}.$$

where

$$\mathcal{V}_{\mu\nu} = \partial_\mu \mathcal{V}_\nu - \partial_\nu \mathcal{V}_\mu - i[\mathcal{V}_\mu, \mathcal{V}_\nu] - i[\mathcal{A}_\mu, \mathcal{A}_\nu],$$

$$\mathcal{A}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i[\mathcal{V}_\mu, \mathcal{A}_\nu] - i[\mathcal{A}_\mu, \mathcal{V}_\nu].$$

- This includes *triangle*, *square* and *pentagon* one-loop diagrams.
- The anomaly arises from the breaking of gauge invariance under 'axial' gauge transformations of the effective action $\Gamma_0[\mathcal{V}, \mathcal{A}]$:

$$\mathcal{Y}_a(x) \Gamma_0[\mathcal{V}, \mathcal{A}] = 0, \quad \mathcal{X}_a(x) \Gamma_0[\mathcal{V}, \mathcal{A}] = G_a[\mathcal{V}, \mathcal{A}]. \quad (2)$$

$(\mathcal{Y}_a(x), \mathcal{X}_a(x)) \equiv$ Local generator of (vector, axial) transform.

- Computation of $\Gamma_0[\mathcal{V}, \mathcal{A}]$ from a solution of (2) by trial and error.

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The anomalous functional

- Solution of the local functional $\Gamma_0[V, A, G]$ [Mañes,EM,Valle,MAVZ'18]:

$$\begin{aligned} \Gamma_0[V, A, G] = & -\frac{N_c}{32\pi^2} \int dt d^3x \sqrt{g} \epsilon^{ijk} \text{Tr} \left\{ \frac{32}{3} i V_0 A_i A_j A_k \right. \\ & + \frac{4}{3} (A_0 A_i + A_i A_0) A_{jk} + 4 (V_0 A_i + A_i V_0) V_{jk} \\ & \left. + \frac{8}{3} (A_0^2 + 3 V_0^2) A_i \partial_j a_k \right\} + C_2 T_0^2 \int dt d^3x \sqrt{g} \epsilon^{ijk} \text{Tr} A_i \partial_j a_k \end{aligned}$$

- V_μ and A_μ are KK inv fields: $A_0 = \mathcal{A}_0$, $A_i = \mathcal{A}_i - a_i \mathcal{A}_0$, etc.
- Γ_0 can be determined also from **differential geometry methods**: Chern-Simons effective action $\rightarrow$ dimensional reduction [Jensen, Laganayagam, Yarom '14; Mañes,EM,Valle,MAVZ '18].
- C_2 related to the **mixed gauge-gravitational anomaly**
 $\sim \text{Tr}\{t_a\} \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\lambda\kappa} R_{\rho\sigma}{}^{\lambda\kappa} \rightarrow$ One should take into account the Riemann tensor contribution in the anomaly polynomial
 $\sim \text{Tr} \mathcal{F}_A R^\mu{}_\nu R^\nu{}_\mu$ [Nair,Ray,Roy PRD86 '12].

Currents in equilibrium

- The **covariant currents** are defined by adding to the consistent currents the Bardeen-Zumino (BZ) polynomials:

$$\mathbf{J}_{\text{cov}}^\mu = \mathbf{J}_{\text{cons}}^\mu + \mathbf{J}_{\text{BZ}}^\mu,$$

with
$$\mathbf{J}_{\text{BZ}A}^\mu = \frac{N_c}{48\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left\{ \lambda_a (\mathcal{A}_\nu \mathcal{A}_{\rho\sigma} + \mathcal{A}_{\rho\sigma} \mathcal{A}_\nu) \right\},$$

$$\mathbf{J}_{\text{BZ}V}^\mu = \frac{N_c}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left\{ \lambda_a (\mathcal{A}_\nu \mathcal{V}_{\rho\sigma} + \mathcal{V}_{\rho\sigma} \mathcal{A}_\nu + \frac{8}{3} i \mathcal{A}_\nu \mathcal{A}_\rho \mathcal{A}_\sigma) \right\}.$$

- From $W_0 = i\Gamma_0$ the **covariant currents** and **stress tensor** are:

$$\langle \mathbf{J}_{5a}^i \rangle_{\text{cov}} = -\frac{N_c}{8\pi^2} e^\sigma \epsilon^{ijk} \text{Tr} \left\{ t_a [(A_0 A_{jk} + A_{jk} A_0) + (V_0 V_{jk} + V_{jk} V_0) + 2(A_0^2 + V_0^2) \partial_j a_k] \right\}, \quad \langle \mathbf{J}_{5a0} \rangle_{\text{cov}} = 0, \quad t_a = \frac{\lambda_a}{2},$$

$$\langle T_0^i \rangle = \frac{N_c}{8\pi^2} e^{-\sigma} \epsilon^{ijk} \text{Tr} \left\{ (A_0^2 + V_0^2) (F_A)_{jk} + (V_0 A_0 + V_0 A_0) (F_V)_{jk} + \left(\frac{2}{3} A_0^3 + 2 A_0 V_0^2 \right) \partial_j a_k \right\}, \quad \langle T_{00} \rangle = \langle T^{ij} \rangle = 0.$$

Constitutive relations

- Maximal number of chemical potentials to be consistently introduced = **dimension of the Cartan subalgebra**. Let us consider the background ($N_f = 2$):

$$V_\mu(x) = V_{0\mu}(x)t_0 + V_{3\mu}(x)t_3, \quad t_0 = \frac{1}{2}1_{2 \times 2}, \quad t_3 = \frac{1}{2}\sigma_3$$

$$A_0 = A_{50}t_0, \quad A_i = 0, \quad \mathcal{A}_i = a_i A_{50}t_0.$$

- Equilibrium **velocity field** and equilibrium **baryonic**, **isospin** and **axial** chemical potentials:

$$u_\mu = -e^\sigma(1, a_i), \quad \mu_0 = \mathcal{V}_{00}e^{-\sigma}, \quad \mu_3 = \mathcal{V}_{30}e^{-\sigma}, \quad \mu_5 = \mathcal{A}_{50}e^{-\sigma}.$$

μ_5 controls chiral imbalance [Gatto, Ruggieri '12; Planells et al, '13].

- Distinguish between three currents:

$$J_{\text{electromagnetic}}^\mu = e\bar{\Psi}\gamma^\mu Q\Psi = \frac{e}{3}\bar{\Psi}\gamma^\mu t_0\Psi + e\bar{\Psi}\gamma^\mu t_3\Psi,$$

$$J_{\text{baryonic}}^\mu = \frac{2}{3}e\bar{\Psi}\gamma^\mu t_0\Psi,$$

$$J_{\text{isospin}}^\mu = e\bar{\Psi}\gamma^\mu t_3\Psi.$$

Constitutive relations

- Constitutive relations $\rightarrow$ Out-of-equilibrium expressions $\rightarrow$

Covariantization of the equilibrium currents:

$$\begin{aligned}
 J_{0\text{ cov}}^\mu &= -\frac{N_c}{16\pi^2} \mu_5 \epsilon^{\mu\nu\lambda\rho} u_\nu \mathcal{V}_{0\lambda\rho}, & J_{3\text{ cov}}^\mu &= -\frac{N_c}{16\pi^2} \mu_5 \epsilon^{\mu\nu\lambda\rho} u_\nu \mathcal{V}_{3\lambda\rho}, \\
 J_{50\text{ cov}}^\mu &= -\frac{N_c}{16\pi^2} \mu_0 \epsilon^{\mu\nu\lambda\rho} u_\nu \mathcal{V}_{0\lambda\rho} - \frac{N_c}{16\pi^2} \mu_3 \epsilon^{\mu\nu\lambda\rho} u_\nu \mathcal{V}_{3\lambda\rho} \\
 &\quad + \frac{N_c}{16\pi^2} (\mu_5^2 - \mu_0^2 - \mu_3^2) \epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho, \\
 J_{53\text{ cov}}^\mu &= -\frac{N_c}{16\pi^2} \mu_3 \epsilon^{\mu\nu\lambda\rho} u_\nu \mathcal{V}_{0\lambda\rho} - \frac{N_c}{16\pi^2} \mu_0 \epsilon^{\mu\nu\lambda\rho} u_\nu \mathcal{V}_{3\lambda\rho} - \frac{N_c}{8\pi^2} \mu_0 \mu_3 \epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho.
 \end{aligned}$$

- Covariant form of the stress tensor $T^{\mu\nu} = u^\mu q^\nu + u^\nu q^\mu$:

$$\begin{aligned}
 q^\mu &= -\frac{N_c}{16\pi^2} \mu_5 \mu_0 \epsilon^{\mu\nu\lambda\rho} u_\nu \mathcal{V}_{0\lambda\rho} - \frac{N_c}{16\pi^2} \mu_5 \mu_3 \epsilon^{\mu\nu\lambda\rho} u_\nu \mathcal{V}_{3\lambda\rho} \\
 &\quad + \frac{N_c}{48\pi^2} \mu_5 (\mu_5^2 - 3\mu_0^2 - 3\mu_3^2) \epsilon^{\mu\nu\lambda\rho} u_\nu \partial_\lambda u_\rho.
 \end{aligned}$$

- Chiral Magnetic and Chiral Vortical contributions.

See also [Neiman, Oz, JHEP03 '11].

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Wess-Zumino-Witten partition function

[Kaiser '01], [Son, Stephanov '08], [Fukushima, Mameda '12], [Brauner, Kadam '17], . . .

- Effects of the anomaly when **symmetry is spontaneously broken**.
- The WZW action describes the **interactions of the Goldstone bosons among them and with the background fields**.
- Application to QCD in the confined phase → **Hadronic fluids** → Fluid of pions, kaons, . . . interacting with external EM fields.
- WZW action:

$$\Gamma^{WZW}[\xi, A] = \Gamma_0[A] - \Gamma_0[A_{-\xi}]$$

$\Gamma_0 \equiv$ anomalous functional in absence of symmetry breaking (computed above).

$A_{-\xi} \equiv$ gauge field transformed with gauge parameters $\Lambda_a = -\xi_a$.

[Wess, Zumino '71; Witten '83; Mañes '85; Chu, Ho, Zumino '96].

- $\xi_a \equiv$ Goldstone bosons

$$U(\xi) = \exp \left(2i \sum_a \xi_a t_a \right).$$

Goldstone bosons

- Spontaneous breaking of axial symmetry:

$$U(2)_L \times U(2)_R \rightarrow U(2)_V.$$

- The matrix of Goldstone bosons (GB)

$$U(\xi) = \exp\left(2i \sum_{a=1}^3 \xi_a t_a\right),$$

includes three GBs from the broken $SU(2)_A$ symmetry. The fourth GB ξ_0 is absent, as the $U(1)_A$ sym is violated by non-pert effects.

- In terms of conventionally normalized GB fields $\pi^0, \pi^\pm$:

$$2 \sum_a \xi_a t_a = \frac{\sqrt{2}}{f_\pi} \begin{pmatrix} \frac{1}{\sqrt{2}} \pi^0 & \pi^+ \\ \pi^- & -\frac{1}{\sqrt{2}} \pi^0 \end{pmatrix},$$

where $f_\pi \approx 92 \text{ MeV}$ is the pion decay constant.

- Lagrangian to zero order

$$\mathcal{L} = -\frac{f_\pi^2}{4} G^{\mu\nu} \text{Tr} \{ D_\mu U (D_\nu U)^\dagger \}, \quad \xi_\mu := D_\mu U = \partial_\mu U - i[\mathcal{V}_3{}_\mu t_3, U].$$

The lowest order description

- The lowest order description is

$$T_0 W = \int d^3x \sqrt{g} e^\sigma (P_0(\mu_0, \mu_3, T) + \mathcal{L}) ,$$

with

$$\mathcal{L} = \frac{f_\pi^2}{4} \left(e^{-2\sigma} \text{Tr}(\xi_0 \xi_0^\dagger) - g^{ij} \text{Tr}(\Phi_i \Phi_j^\dagger) \right) ,$$

and

$$\xi_0 = -i \mathcal{V}_{30}[t_3, U] , \quad \text{and} \quad \Phi_i = \partial_i U - i \mathcal{V}_{3i}[t_3, U] .$$

- Constitutive relations** at the lowest order:

$$(J_0)_\mu)_{\text{perfect}} = n_0 u_\mu ,$$

$$(J_3)_\mu)_{\text{perfect}} = n_3 u_\mu + i \frac{f_\pi^2}{4} \text{Tr} \{ [t_3, U] \partial_\mu U^\dagger + [t_3, U^\dagger] \partial_\mu U \} + \frac{f_\pi^2}{2} \mathcal{V}_{3\mu} \text{Tr} \{ [t_3, U] [t_3, U^\dagger] \}$$

$$T_{\text{perfect}}^{\mu\nu} = (\varepsilon + P) u^\mu u^\nu + P G^{\mu\nu} + \frac{f_\pi^2}{2} G^{\mu\rho} G^{\nu\lambda} \text{Tr} \{ D_{(\rho} U (D_{\lambda)} U)^\dagger \} .$$

- $u^\mu = e^{-\sigma} (1, 0, 0, 0)$ → Equilibrium currents.

Wess-Zumino-Witten partition function

- Spontaneous breaking of axial symmetry:

$$U(2)_L \times U(2)_R \rightarrow U(2)_V.$$

Suitable gauge transformation is an axial transformation with parameters $\Lambda_a^{\text{Axial}}(x) = -\xi_a(x)$.

- WZW partition function:

$$\begin{aligned} T_0 W^{\text{WZW}} = & \frac{N_c}{8\pi^2} \int d^3x \sqrt{g} V_{00} \epsilon^{ijk} \left[-\frac{1}{2} \text{Tr}\{ \partial_i (R_j + L_j) t_3 \} V_{3k} + \frac{1}{6} i \text{Tr}\{ L_i L_j L_k \} \right] \\ & + \frac{N_c}{16\pi^2} \int d^3x \sqrt{g} \epsilon^{ijk} \text{Tr}\{ (R_i + L_i) t_3 \} \\ & \times (V_{00} \partial_j V_{3k} + V_{30} \partial_j V_{0k} + V_{00} V_{30} \partial_j a_k) \\ & + \frac{N_c}{48\pi^2} A_{50} \int d^3x \sqrt{g} \epsilon^{ijk} \left[\text{Tr}\{ (L_i - R_i) t_3 \} + 2 \text{Tr}\{ t_3^2 - U t_3 U^{-1} t_3 \} V_{3i} \right] \\ & \times (\partial_j V_{3k} + V_{30} \partial_j a_k) \end{aligned}$$

where $L_j = i \partial_j U U^{-1}$, $R_j = i U^{-1} \partial_j U$.

Constitutive relations in hydrodynamics

- Contribution to the constitutive relations:

$$T^{\mu\nu} = T_{\text{perfect}}^{\mu\nu} + \underbrace{\pi^{\mu\nu}}_{\text{Dissipative \& Anomalous}}, \quad J^\mu = J_{\text{perfect}}^\mu + \underbrace{\delta J^\mu}_{\text{Dissipative \& Anomalous}}.$$

- Consistent expansion 

$$T_{\text{perfect}}^{\mu\nu}(\mu_{a0} + \delta\mu_a, T_0 + \delta T, \dots), \quad J_{\text{perfect}}^\mu(\mu_{a0} + \delta\mu_a, T_0 + \delta T, \dots).$$

- Choose the frame in which the first derivative corrections to the fluid quantities vanish, i.e. $\delta\mu_a^{(1)} = 0$, $\delta T^{(1)} = 0$ and $\delta u^\mu{}^{(1)} = 0$
- In this frame

$$\pi^{\mu\nu} = 0, \\ \delta J^\mu = - \underbrace{(\delta J^\nu u_\nu)}_{\text{Longitudinal}} u^\mu + \underbrace{P^\mu{}_\nu \delta J^\nu}_{\text{Transverse}},$$

where $P^{\mu\nu} = G^{\mu\nu} + u^\mu u^\nu$.

Constitutive relations at first order in derivatives

- Written in terms of 6 pseudo-scalars:

$$S_1 = u_\lambda \epsilon^{\lambda\nu\rho\sigma} \mathcal{I}_\nu \mathcal{V}_{3\rho\sigma}, \quad S_2 = u_\lambda \epsilon^{\lambda\nu\rho\sigma} \mathcal{I}_\nu \partial_\rho u_\sigma,$$

$$S_3 = u_\lambda \epsilon^{\lambda\nu\rho\sigma} \left[-\frac{1}{2} \partial_\nu (\text{Tr}\{(L_\rho + R_\rho) t_3\}) \mathcal{V}_{3\sigma} + \frac{i}{6} \text{Tr}\{L_\nu L_\rho L_\sigma\} \right]$$

$$S_{4(a)} = u_\lambda \epsilon^{\lambda\nu\rho\sigma} \mathcal{T}_\nu \mathcal{V}_{a\rho\sigma}, \quad a = 0, 3, \quad S_5 = u_\lambda \epsilon^{\lambda\nu\rho\sigma} \mathcal{T}_\nu \partial_\rho u_\sigma,$$

- and 8 pseudo-vectors:

$$P_1^\mu = \epsilon^{\mu\nu\lambda\sigma} u_\nu \mathcal{I}_\lambda \partial_\sigma \left(\frac{\mu_3}{T} \right), \quad P_2^\mu = \epsilon^{\mu\nu\lambda\sigma} u_\nu \partial_\lambda \mathcal{I}_\sigma,$$

$$P_{3(a)}^\mu = \epsilon^{\mu\nu\lambda\sigma} u_\nu \mathcal{T}_\lambda \partial_\sigma \left(\frac{\mu_a}{T} \right), \quad P_4^\mu = \epsilon^{\mu\nu\lambda\sigma} u_\nu \partial_\lambda \mathcal{T}_\sigma,$$

$$P_{5(a)}^\mu = \epsilon^{\mu\nu\lambda\sigma} u_\nu \mathcal{V}_{a\lambda\sigma}, \quad P_6^\mu = \epsilon^{\mu\nu\lambda\sigma} u_\nu \partial_\lambda u_\sigma.$$

- We define:

$$\mathcal{H} = \text{Tr}\{t_3^2 - U t_3 U^{-1} t_3\}, \quad \mathcal{I}_\mu \equiv \text{Tr}\{(L_\mu + R_\mu) t_3\},$$

$$\mathcal{T}_\mu \equiv \text{Tr}\{(L_\mu - R_\mu) t_3 + 2(t_3^2 - U t_3 U^{-1} t_3) \mathcal{V}_{3\mu}\}.$$

Constitutive relations at first order in derivatives

- **Baryonic currents** [In color the BZ contributions: **CME** and **CVE**]:

$$u^\mu \delta J_{0\mu \text{ cov}} = \frac{N_c}{16\pi^2} [S_1 + 2S_3] ,$$

$$P^\mu{}_\nu \delta J_{0\text{ cov}}^\nu = \frac{N_c}{16\pi^2} [-2TP_1^\mu + \mu_3 P_2^\mu] - \frac{N_c}{16\pi^2} \mu_5 P_{5(0)}^\mu ,$$

$$u^\mu \delta J_{50\mu \text{ cov}} = \frac{N_c}{96\pi^2} S_{4(3)} ,$$

$$P^\mu{}_\nu \delta J_{50\text{ cov}}^\nu = -\frac{N_c}{48\pi^2} [TP_{3(3)}^\mu + \mu_3 \mathcal{H} P_{5(3)}^\mu] + \frac{N_c}{24\pi^2} \mu_5^2 P_6^\mu .$$

- **Isospin currents**:

$$u^\mu \delta J_{3\mu \text{ cov}} = -\frac{N_c}{48\pi^2} \mu_5 S_5 ,$$

$$P^\mu{}_\nu \delta J_{3\text{ cov}}^\nu = \frac{N_c}{48\pi^2} \mu_5 \left[P_4^\mu + (\mathcal{H} - 3) P_{5(3)}^\mu + 2\mu_3 \mathcal{H} P_6^\mu \right] .$$

$$u^\mu \delta J_{53\mu \text{ cov}} = \frac{N_c}{32\pi^2} S_{4(0)} - \frac{N_c}{48\pi^2} \mu_5 S_2 ,$$

$$P^\mu{}_\nu \delta J_{53\text{ cov}}^\nu = -\frac{N_c}{16\pi^2} \left[TP_{3(0)}^\mu + \mu_3 \mathcal{H} P_{5(0)}^\mu \right] + \frac{N_c}{48\pi^2} \mu_5 P_2^\mu .$$

Constitutive relations at first order in derivatives

Properties:

- The only dependence in the background is through the physical electromagnetic fields $\mathcal{V}_{a\mu}$, so that the dependence in a_i has disappeared $\rightarrow$ Constitutive relations should be expressed in terms of quantities of the fluid: fluid velocity, external fields, etc.
- Chiral electric effect in vector current [Neiman, Oz, JHEP09 '11]

$$P^\mu{}_\nu \delta J_{0\text{cov}}^\nu = -\frac{N_c}{8\pi^2} T \epsilon^{\mu\nu\lambda\sigma} u_\nu \text{Tr}\{(L_\lambda + R_\lambda)t_3\} \partial_\sigma \left(\frac{\mu_3}{T}\right) + \dots$$

- Chiral electric effect (CEE) in axial currents:

$$P^\mu{}_\nu \delta J_{50\text{cov}}^\nu = -\frac{N_c}{48\pi^2} T \epsilon^{\mu\nu\lambda\sigma} u_\nu \text{Tr}\{(L_\lambda - R_\lambda)t_3 + 2(t_3^2 - U t_3 U^{-1} t_3) \mathcal{V}_{3\lambda}\} \partial_\sigma \left(\frac{\mu_3}{T}\right)$$

- We provide explicit values for the corresponding transport coefficients of the CEE. Derived from an equilibrium partition function $\rightarrow$ CEE cannot lead to entropy production $\rightarrow$ CEE is non-dissipative.

Conclusions

- We have studied **non-dissipative transport effects** up to 1st order in the **hydrodynamic expansion** in **non-abelian theories**.
- Effects are induced by **external magnetic fields**, **vortices** and **curvature** in a relativistic fluid $\Rightarrow$ **Anomalous Transport**.
- **Equilibrium partition function method** can only account for **non-dissipative effects**: *time reversal properties*.
- Dissipative effects (shear viscosity, electric conductivity, ...) $\rightarrow$ other methods: Kubo formulae, Fluid/gravity correspondence, ...
- **Anomalous effects in presence of Spontaneous Breaking** of non-abelian gauge symmetries:
 - Interactions of Goldstone bosons with themselves and with external electromagnetic fields.
 - Application to **QCD in confined phase**: Fluid of pions, kaons, etc.
- **Future directions**:
 - Application to other sectors of the SM $\rightarrow$ ElectroWeak sector.
 - Application to **condensed matter systems with triangle anomalies** $\rightarrow$ Weyl semi-metals [Basar, Kharzeev, Yee '14], [Landsteiner '14].

Thank You!