SPONTANEOUS BREAKING OF U(N) SYMMETRY IN INVARIANT MATRIX MODELS

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Support by:

MARIE CURIE ACTIONS

SEVENTH FRAMEWORK PROGRAMME

arXiv:1412.6523

arXiv:1503.03341

ma per seguir virtute e canoscenza
SISSA

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Outlook

- Localization/extendedness of wavefunctions is a basis-dependent property
- However, eigen-energy statistics (Poisson/Wigner Dyson) characterizes insulating/conducting systems
- Take matrix representation of Hamiltonian
- Seeking for a basis independent, general structure of Anderson insulators

Results

- The U(N) symmetry matrix models are endowed with can be spontaneously broken
- Thermodynamic limit also takes symmetry's rank to infinity
- Eigenvectors encode non-trivial information!
- Certain models break U(N) in a critical way: similarity with Metal/Insulator Transition
- These models are in the family of CS/ABJM

Outline

1. Introduction on Matrix Models

2. Spontaneous Symmetry Breaking:
   - Geometrical argument
   - Symmetry Breaking term
   - Numerical finite size detection

3. Weakly Confined Matrix Models
   - Spectral Statistics (known)
   - Energy landscape (new)

4. Conclusions & Outlook

Part 1

Introduction
on
Matrix Models

Matrix Models

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-W(\mathbf{M})}$$

- Matrix fields only truly strongly-interacting
- Natural implementation of Holography
- Zero-dimensional field theory in localization limit
- Several applications: nuclear theory, mesoscopic conduction, 2-D quantum gravity, string theory, statistical physics, econophysics, neuroscience, chaos theory, number theory, integrability…

Matrix Models

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-W(\mathbf{M})} \qquad \mathcal{F} = \ln \mathcal{Z}$$

- W(M): matrix-valued action
- Same mathematical formulation for different applications: interdisciplinary & cross-fertilization
- Reflects a large universality
- Matrices can be link between points; fields in adjoint or bi-fundamental, representation of operators in many-body theory (Hamiltonians, Scattering…)…

Random Matrices

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-W(\mathbf{M})} \qquad \mathcal{F} = \ln \mathcal{Z}$$

- If W(M) real: statistical model
- Consider M as a Hamiltonian:
  - Interaction between every degree of freedom
  - Matrix entries randomly from a distribution
- Describe quantum "chaotic" systems
- Universality determined by symmetry:

  Orthogonal, Unitary, Symplectic,… ensembles

Invariant Ensembles

- Action invariant under rotations: $W(\mathbf{M}) = \mathrm{Tr} V(\mathbf{M})$

- Switch to eigenvalues/eigenvectors: $\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr} V(\mathbf{M})} = \int \mathcal{D}\mathbf{U} \int d^N\lambda\, \Delta^\beta(\{\lambda\})\, e^{-\sum_j V(\lambda_j)}$$

Eigenvectors uniformly distributed over the N-dimensional sphere (Hilbert space): independent from V(x)

Van der Monde Determinant:

$$\Delta(\{\lambda\}) = \prod_{j>l}^{N} (\lambda_j - \lambda_l)$$

(from Jacobian)

Coulomb Gas Picture

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int d^N\lambda \, \Delta^\beta(\{\lambda\}) \, e^{-\sum_j V(\lambda_j)}$$

- Jacobian introduces interaction between eigenvalues
- Effective Coulomb gas:

$$\mathcal{L} = -\beta \sum_{j>l} \ln |E_j - E_l| + \sum_j V(E_j)$$

- Eigenvalues as 1-D particles with
  - logarithmic interaction
  - external confining potential V(x)
- Eigenvalue distribution from equilibrium configuration

$\beta = 1, 2, 4$

for

Orthogonal

Unitary

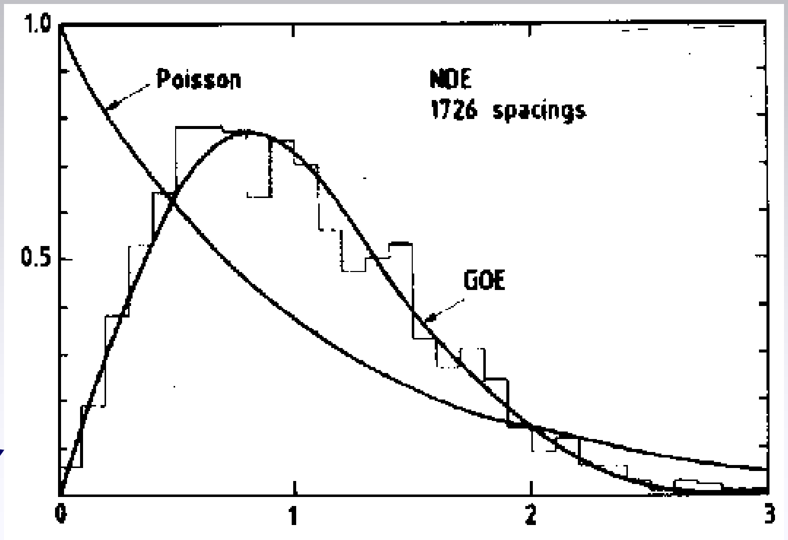
Symplectic

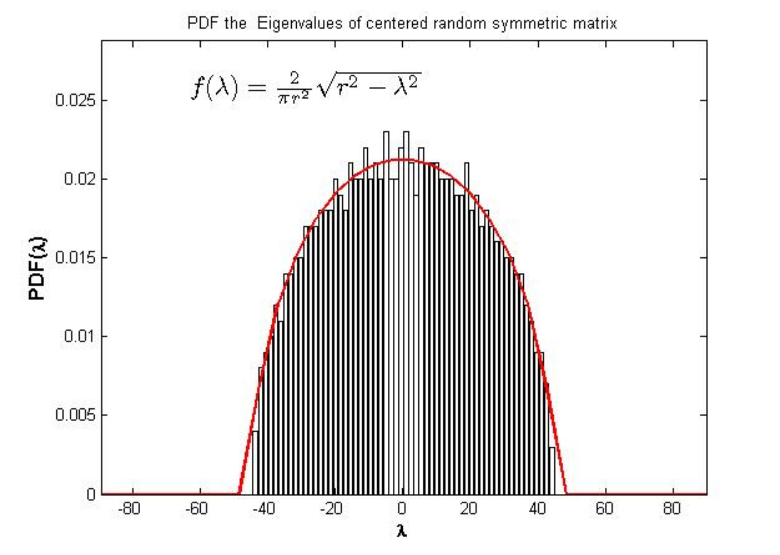
Wigner-Dyson Universality

$$\mathcal{L} = -\beta \sum_{j>l} \ln|E_j - E_l| + \sum_j V(E_j)$$

- Distribution of the distance between n.n. eigenvalues (level spacing) universal:

$$P(s) \propto s^\beta e^{-A(\beta)s^2}$$





- Universality captured by Gaussian ensemble: $V(x) = \dfrac{x^2}{2}$
- Valid for any polynomial V(x)

Invariant Ensembles

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int d^N\lambda \, \Delta^\beta(\{\lambda\}) \, e^{-\sum_j V(\lambda_j)}$$

- Wigner Dyson distribution & level repulsion:
  Jacobian introduces interaction between eigenvalues
- Extended states/conducting phases:
  uniform distribution means eigenvectors typically have all non-vanishing entries
- Eigenvalues interact through their eigenvectors:

$$\boxed{\textbf{WD} \Leftrightarrow \textbf{extended states}}$$

Non-Invariant Ensembles

- To study localization problems, introduce non-invariant random matrix ensembles (Random Banded Matrices)

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\sum_{j,l} A_{jl} \, |H_{jl}|^2} \Rightarrow \langle M_{jl}^2 \rangle = A_{jl}^{-1}$$

$$A_{nm} = \mathrm{e}^{|n-m|/B}$$

→ Localized states (Poisson statistics)

(Mirlin et al. '96)

$$A_{nm} = 1 + \frac{(n-m)^2}{B^2}$$

→ Multi-Fractal states (Critical Statistics)

(Evers & Mirlin, '00)

Invariant vs. non-Invariant Ensembles

- Invariant: basis independent

  → Wigner-Dyson eigenvalue statistics
  de-Haar measure for eigenvector

  ⇒ delocalized systems
  analytical techniques

- Non-Invariant: basis dependent

  → Poisson/critical eigenvalue statistics
  eigenvector connected with eigenvalue

  ⇒ localized/critical systems
  mostly numerical approaches

Loophole: Spontaneous Breaking of Rotational Invariance

- Invariant models are endowed with superior (non-perturbative) analytical techniques
- Spontaneous breaking of rotational invariance:

$\Rightarrow$ Eigenvectors contain non-trivial information

$\Rightarrow$ Invariant machinery for localization problems!

- Recall a ferromagnet:
  - From partition function, rotational invariance $\rightarrow$ no spontaneous magnetization
  - Need symmetry breaking term

Part 2

Spontaneous Breaking of Rotational Symmetry in Invariant Multi-Cuts Matrix Model

Multi-Cut Solutions

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int d^N\lambda \, e^{-\sum_j V(\lambda_j) + 2\sum_{j>l} \ln|\lambda_j - \lambda_l|}$$

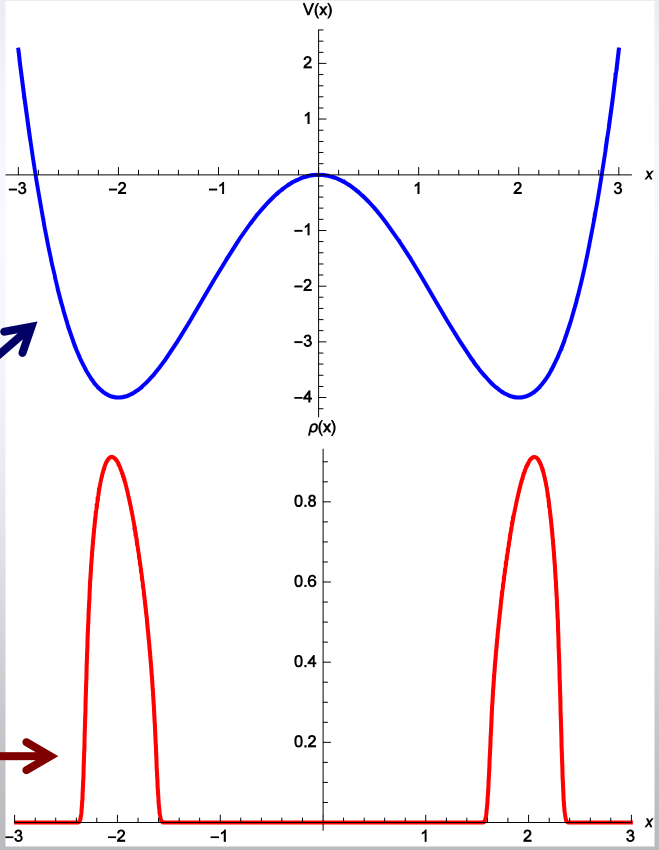
- V(x) with several, well separated, minima

  ⇒ disconnected support for eigenvalues (multi-cuts)

- For example: double well potential

$$V_{2W}(x) = \frac{1}{4}x^4 - \frac{t}{2}x^2$$

(2-cuts for $t > 2$)



Level Density: $\rho(x) = \sum_j^N \delta(x - \lambda_j)$

Understanding the matrix SSB

- Geometrical argument: line element

$$\boxed{\beta = 2\,,\mathrm{Unitary}}$$

$$ds^2 = \mathrm{Tr}\,(dM)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + 2 \sum_{j>l}^{N} (\lambda_j - \lambda_l)^2 \, |dA_{jl}|^2$$

$$d\mathbf{A} \equiv \mathbf{U}^\dagger d\mathbf{U}$$

$$\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$

- Angular degrees of freedom live on spheres of radii $r_{jl} = |\lambda_j - \lambda_l|$

  - Two lengths scales:
    - Eigenvalues spacing: $\mathcal{O}\left(\frac{1}{N}\right)$
    - Support of distribution: $\mathcal{O}\,(1)$
  - Small arc lengths:
    - $r_{jl} \sim \mathcal{O}\left(\frac{1}{N}\right) \to dA_{jl} \sim \mathcal{O}\,(1)$
    - $r_{jl} \sim \mathcal{O}\,(1) \to dA_{jl} \sim \mathcal{O}\left(\frac{1}{N}\right)$

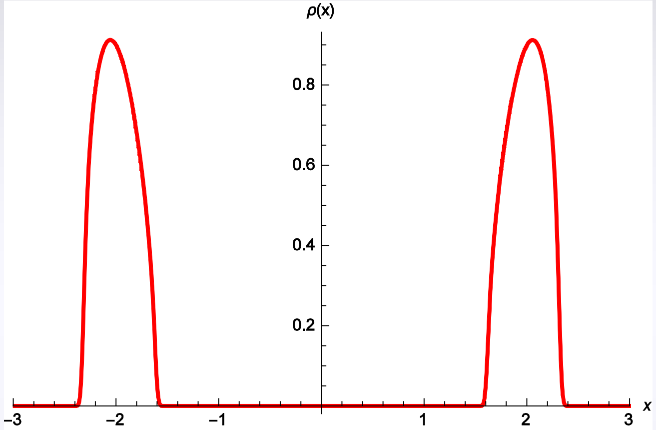
Multi-Cuts SSB

- Level repulsion resolves degeneracy:

  $\Rightarrow$ each of the n cuts contains m_j eigenvalues

- Gap between cuts breaks rotational invariance: $\boxed{U(N) \overset{N\to\infty}{\to} \prod_{j=1}^{n} U\left(m_j\right)}$

- Three Arguments:
  - ★ Brownian motion;
  - ★ Symmetry Breaking Term;
  - ★ Numerical finite size analysis



Double well

$U(N) \overset{N\to\infty}{\to} U(N/2) \times U(N/2)$

(assume N even)

F.F. arXiv:1412.6523

Symmetry Breaking Term

$$\boxed{W(J) = \ln \int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN |\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|}}$$

$$\boxed{\begin{array}{c} \mathrm{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U} \\ \mathrm{S} = \mathbf{V}^\dagger \mathbf{T} \mathbf{V} \end{array}}$$

- Calculate (dis-)order parameter:
  - $\frac{d}{dJ} \lim_{N \to \infty} W(J) \Big|_{J=0} = 0 \longrightarrow$ **Symmetry is Broken!**
  - Finite N: $\frac{dW(J)}{dJ}\Big|_{J=0} = \langle |\mathrm{Tr}\,(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})| \rangle \neq 0$ Eigenvectors misaligned

Symmetry Breaking Term

$$\boxed{W(J) = \ln \int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN |\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|}}$$

$$\mathrm{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^\dagger \mathbf{T} \mathbf{V}$$

- Calculate (dis-)order parameter:
  - $\frac{d}{dJ} \lim_{N\to\infty} W(J) \Big|_{J=0} = 0$
  - Finite N: $\frac{dW(J)}{dJ}\Big|_{J=0} \neq 0$

Instantons:
- Pairs of eigenvalues tunneling between wells
- Restore broken symmetries

$$\int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN |\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|} \propto \mathcal{Z}_0 + \mathcal{Z}_1 \left(e^{-2JN|\lambda_j - \lambda'_l|} \right) + \ldots$$

Multi-Cuts SSB: Conclusions

- Gap in the eigenvalue distribution
  → Deviation from WD universality
  → Spontaneous breaking of rotational symmetry
  → Eigenvectors localized in patch of Hilbert space spanned by the other eigenvectors in the same cut
- Broken symmetries restored by instantons
- Abstract characterization of localization without reference to basis: IPR not sufficient, need response under perturbation (application to MBL?)

PART 3

Weakly Confined Matrix Models & The Metal/Insulator Transition

Weakly Confined Invariant Models

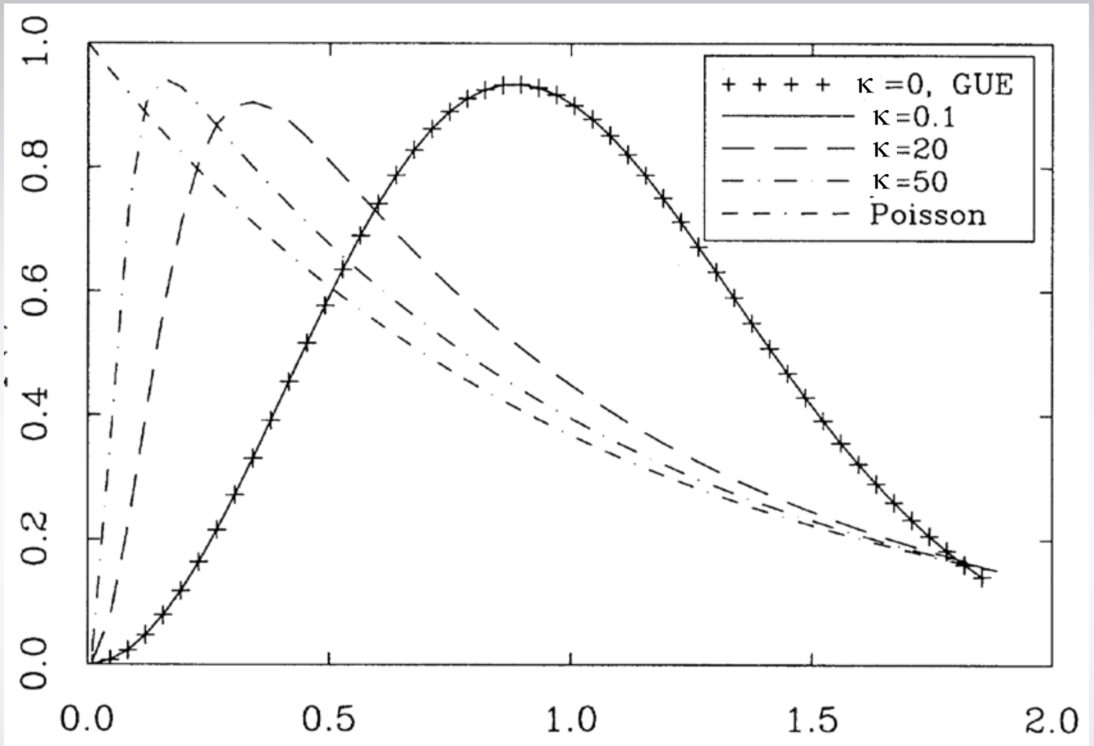
$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr} V(\mathbf{M})} \, , \; V(\lambda) \overset{|\lambda| \to \infty}{\simeq} \frac{1}{2\kappa} \ln^2 |\lambda|$$

- Soft confinement sets them apart from usual polynomial potentials
  → WD universality does not apply
  → Indeterminate moment problem
- Arise in localization limit of Chern-Simons/ABJM: $\kappa \propto \frac{i}{g_s}$
  (Marino '02; Kapustin et a. '10; …)
- Solvable through orthogonal polynomials:
  q-deformed Hermite/Laguerre Polynomials
  (Muttalib et al. '93; Tierz'04)

Weakly Confined Matrix Models

$$V(\lambda) \overset{|\lambda|\to\infty}{\simeq} \frac{1}{2\kappa} \ln^2|\lambda|$$

- Intermediat level spacing statistics
- Same eigenvalue correlations as power law (critical) Random Banded Matrices

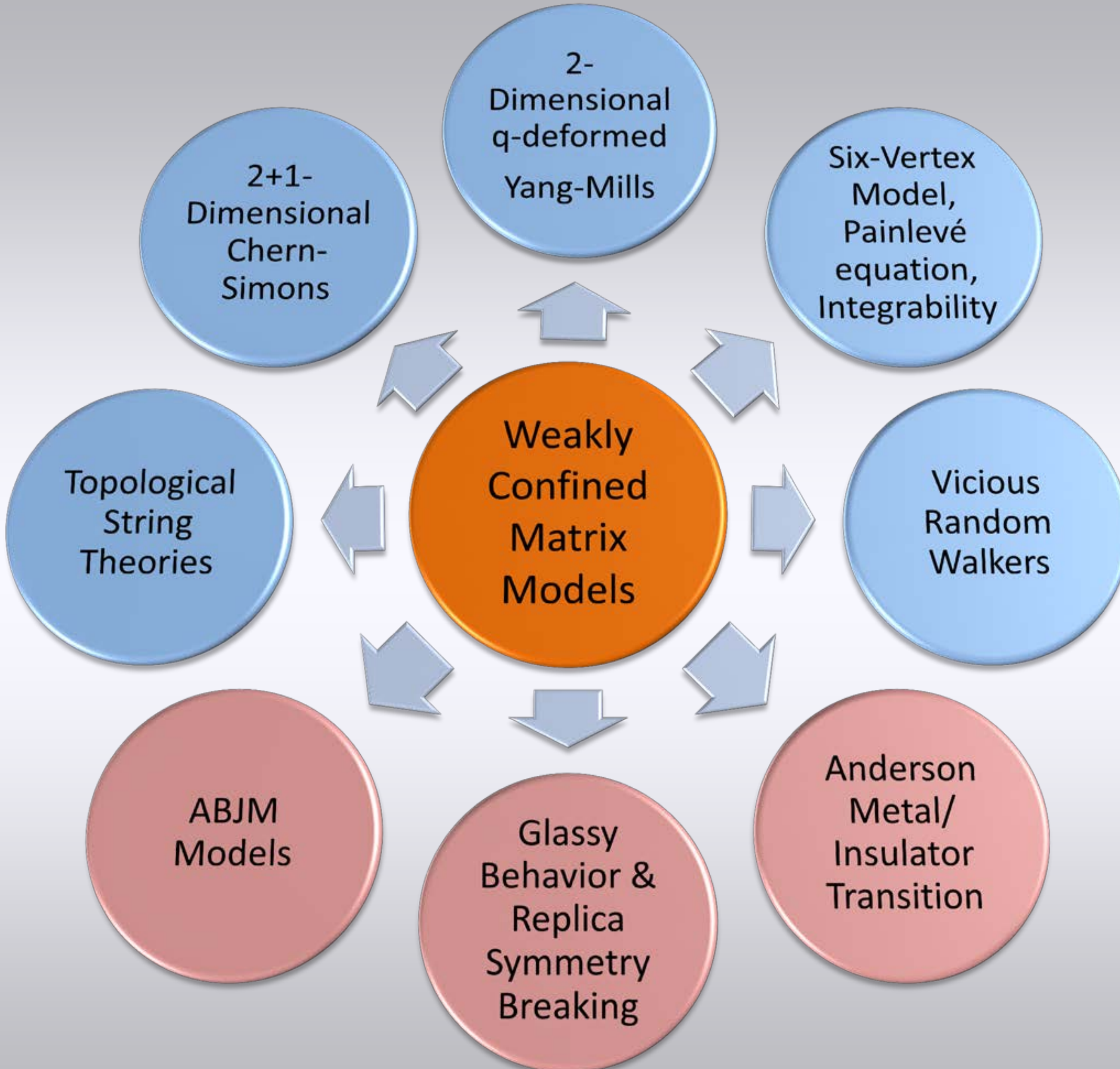


(Muttalib et al. ‘93)

- Critical level statistics signals fractal eigenstates?
- Critical Spontaneous Breaking of U(N) Invariance?

(Canali, Kravtsov, ‘95)

Weakly Confined Matrix Models & their applica-tions



WCMM Energy Landscape

- Take exactly log-normal ensemble (positive eigenvalues)

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int_{\lambda>0} d^N\lambda\, \Delta\left(\{\lambda\}\right) e^{-\frac{1}{2\kappa}\sum_j \ln^2 \lambda_j}$$

- Exponential mapping: $\boxed{\lambda_j = e^{\kappa x_j}}$

$$\mathcal{Z} \propto \int d^N x_j \prod_{n<m} \left(e^{\kappa x_n} - e^{\kappa x_m}\right)^2 \; e^{-\frac{\kappa}{2}\sum_{l=1}^{N}\left[x_l^2 - 2x_l\right]}$$

- Each term of the Van der Monde shifts the equilibrium of the parabolic potential: different effective potential felt by each eigenvalue for each term for the VdM

WCMM Energy Landscape

$$\mathcal{Z} \propto \int d^N x \prod_{n<m} \left(e^{\kappa x_n} - e^{\kappa x_m}\right)^2 \; e^{-\frac{\kappa}{2}\sum_{l=1}^{N}\left[x_l^2 - 2x_l\right]}$$

- Consider simplex of eigenvalues in increasing order

$$\mathcal{Z} \propto N! \int d^N x \prod_{n<m} \left[1 - e^{\kappa(x_n - x_m)}\right]^2 \; e^{-\frac{\kappa}{2}\sum_{l=1}^{N}\left[x_l^2 - 2(2l-1)x_l\right]}$$

- In the $\kappa \to \infty$ limit, all terms left inside the VdM vanish
- Eigenvalue crystalize on a lattice

(Bogomolny et al. '97)

$$\lim_{\kappa\to\infty} \mathcal{Z} \propto N! e^{\frac{\kappa}{6} N(4N^2-1)} \int d^N x e^{-\frac{\kappa}{2}\sum_{l=1}^{N}(x_l + 1 - 2l)^2}$$

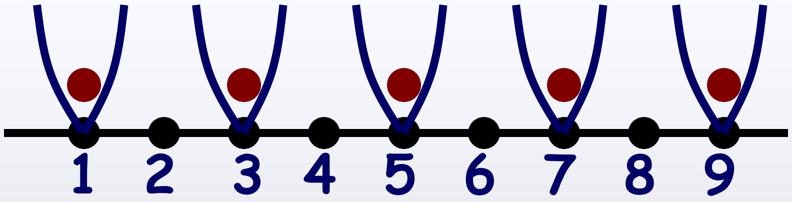
Eigenvalue Crystallization

$$\mathcal{Z} \propto N! \int d^N x \prod_{n<m} \left[1 - e^{\kappa(x_n - x_m)}\right]^2 e^{-\frac{\kappa}{2}\sum_{l=1}^{N}\left[x_l^2 - 2(2l-1)x_l\right]}$$

- Eigenvalue crystallization for $\kappa \to \infty$

(Bogomolny et al. '97)

$$\mathcal{Z} \propto \int d^N x \left[e^{-\frac{\kappa}{2}\sum_{l=1}^{N}(x_l+1-2l)^2} + \dots\right]$$



- Corresponds to SSB: $\boxed{U(N) \to U(1)^N}$

(Exponential separation between eigenvalues completely freezes eigenstates dynamics)

(Pato, '00)

WCMM Energy Landscape

$$\mathcal{Z} \propto N! \int d^N x \prod_{n<m} \left[1 - e^{\kappa(x_n - x_m)}\right]^2 e^{-\frac{\kappa}{2}\sum_{l=1}^{N}\left[x_l^2 - 2(2l-1)x_l\right]}$$

- Finite κ corrections organized in powers of $q = e^{-\kappa}$

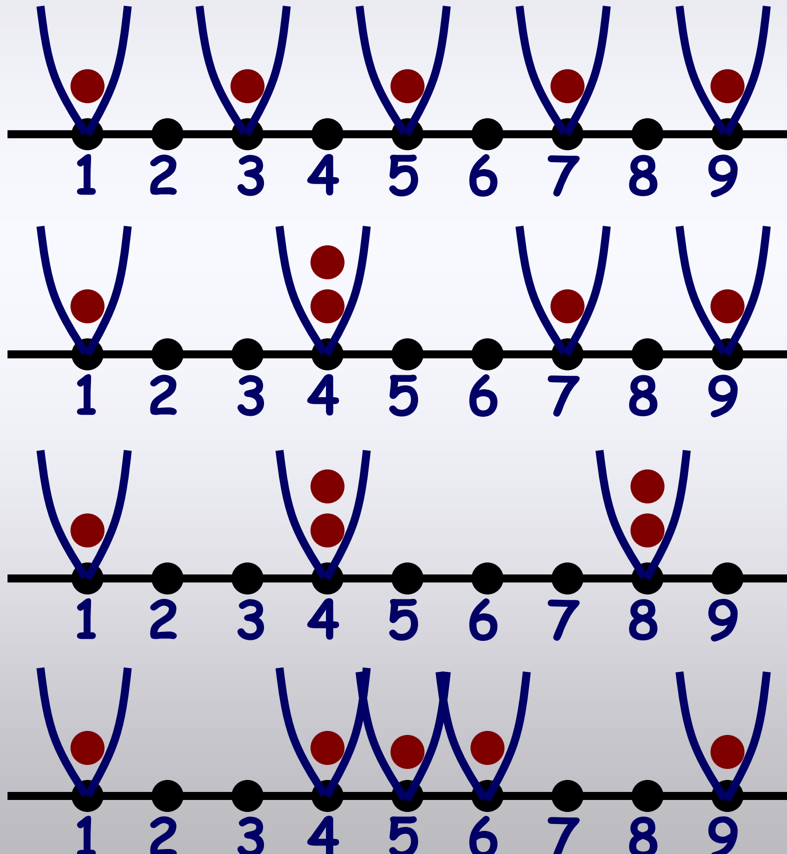
q^0

$-q^1$ $\times(N-1)$

$+q^2$ $\times\frac{(N-2)(N-3)}{2}$

$+q^2 - q^2$ $\times(N-2)$

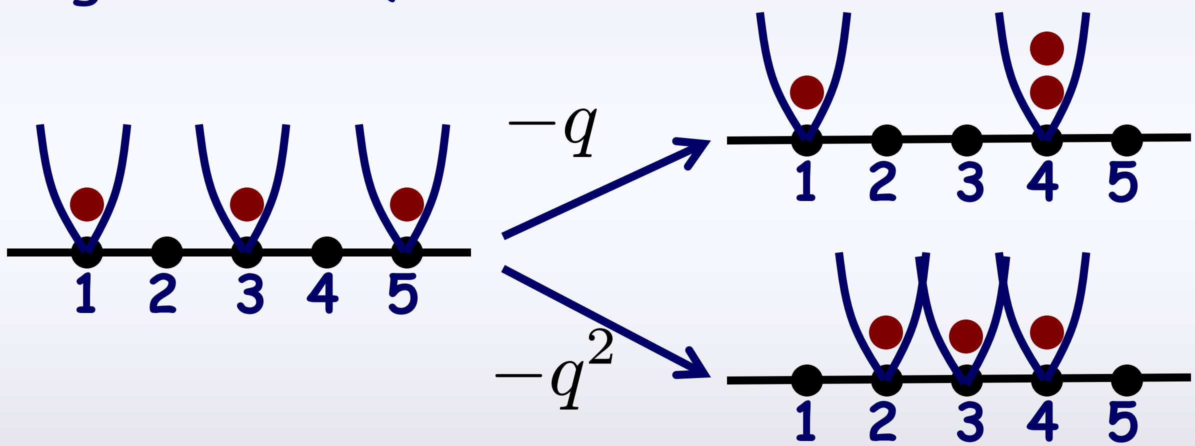
$\dots$



WCMM Energy Landscape

$$\mathcal{Z} \propto N! \int d^N x \prod_{n<m} \left[1 - e^{\kappa(x_n - x_m)}\right]^2 e^{-\frac{\kappa}{2}\sum_{l=1}^{N}\left[x_l^2 - 2(2l-1)x_l\right]}$$

- Each term in VdM shifts the equilibrium point of 2 eigenvalues (one notch closer to one-another)



- Each term of the VdM generates a new equilibrium configuration (saddle point of the partition function)
- Saddles connected by instantons with weight $-q^n$

WCMM Energy Landscape

- Full partition function known (orthogonal polynomials)

$$\mathcal{Z} \propto e^{\frac{\kappa}{6}N(4N^2-1)} (2\pi\kappa)^{N/2} N! \prod_{n=1}^{N-1} (1-q^n)^{N-n}$$

- Natural interpretation in terms of instantons
  - 1 instanton connecting a pair of eigenvalues N-1 apart
  - 2 instantons connecting a pair N-2 apart
  - …
  - N-1 instantons connecting a pair of N.N. eigenvalues
- Metal/Insulator Transition as glassy phase?

WCMM Energy Landscape

$$\mathcal{Z} \propto e^{\frac{\kappa}{6} N(4N^2-1)} \, (2\pi\kappa)^{N/2} \, N! \prod_{n=1}^{N-1} (1-q^n)^{N-n}$$

- Exponential number of saddle points
- Every equilibrium configuration which preserves the center of mass is realized by the action of the instantons
- Each equilibrium configuration has the same leading energy: they differ only for the powers of q
- Instantons restore broken symmetries: from the $U(1)^N$ configuration at $\kappa \to \infty$, to the full $U(N)$ when all instantons bring each eigenvalue to the same equilibrium point

Multi-fractal Spectrum

- Fresh, preliminary results
- Unitary matrix from anti-Hermitian matrix: $\mathbf{U} = e^{i\mathbf{A}}$

$$ds^2 = \mathrm{Tr}\,(dM)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + 2\sum_{j>l}^{N} (\lambda_j - \lambda_l)^2\,|dA_{jl}|^2$$

$$d\mathbf{A} \equiv \mathbf{U}^\dagger d\mathbf{U}$$

- With probability $\begin{cases} q^{j-l-1} & \text{sample uniformly } A_{jl} \\ 1-q^{j-l-1} & \text{take } A_{jl} = 0 \end{cases}$
- Inverse Participation Ratio of $\mathbf{U}$ scales with fractional power of N

$\rightarrow$ Multi-fractal spectrum from invariant matrix model!

Conclusions

- Invariant Matrix Models usually applied only to extended/conducting states: eigenvectors discarded
- Deviation of eigenvalue statistics from Wigner-Dyson signals loos of ergodicity: gap between eigenvalues mutually localize their eigenvectors: U(N) broken
- WCMM has complex energy landscape $\rightarrow$ critical SSB
- Novel SSB? New eigenvector-related observables?
- Replica Symmetry Breaking as SSB

Thank you!

Outlook

- Matching WCMM critical SSB with Metal/Insulator Transition multi-fractal spectrum?
- Critical exponents of SSB
- Direct characterization of eigenvector behavior
- Connection between SSB & Replica Symmetry Breaking
- WCMM & Matrix models arise in string theory: meaning of the $N \to \infty$ $U(N)$ symmetry breaking?
- …

Thank you!

Disorder & Localization

- Anderson Model: $\mathcal{H} = \sum_j \epsilon_j c_j^\dagger c_j + \sum_{\langle j,l \rangle} \left[c_l^\dagger c_j + c_j^\dagger c_l \right]$
- Tight-binding model (nearest neighbor hopping)
- Random on-site energies: $\epsilon_j \in [-W, +W]$
- 1 & 2 Dimensions: localized for any $W \neq 0$
- Higher D:
  - Small W: conducting (weak localization, Random Matrices)
  - $W > W_c$: insulating (localized at low energies)
- Hard problem (uncontrolled perturbation expansion)

Metal/Insulator Transition

$$\mathcal{H} = \sum_j \epsilon_j c_j^\dagger c_j + \sum_{\langle j,l \rangle} \left[c_l^\dagger c_j + c_j^\dagger c_l \right]$$

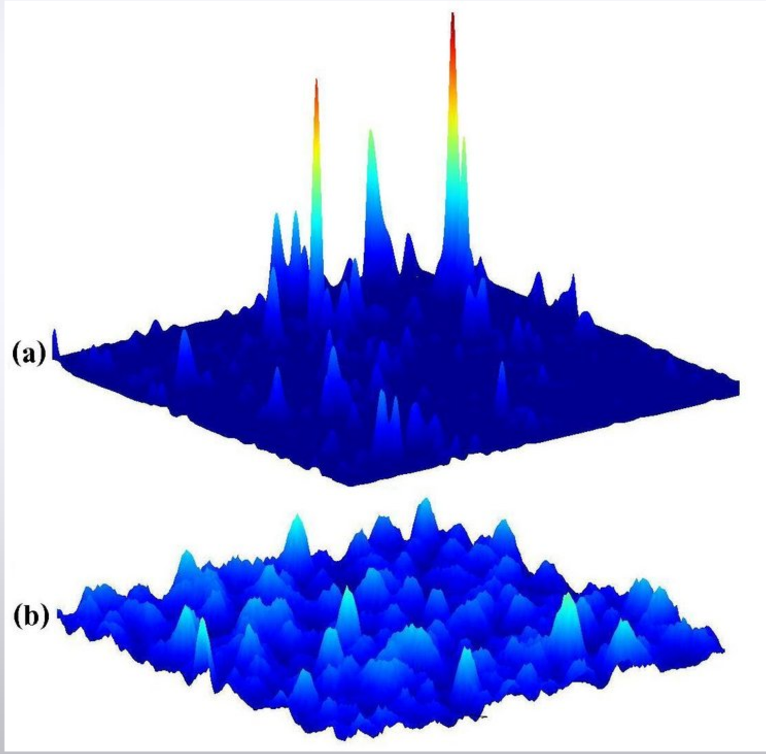
$$\epsilon_j \in [-W, +W] \qquad W > W_c \qquad D \geq 3$$

- Low energy modes: localized
- High energy states: extended
- At $E = E_m$:

Mobility Edge separating the two

Intermediate state

(multifractal)



Van Tiggelen group (PRL 2009)

Multi-fractal Spectrum

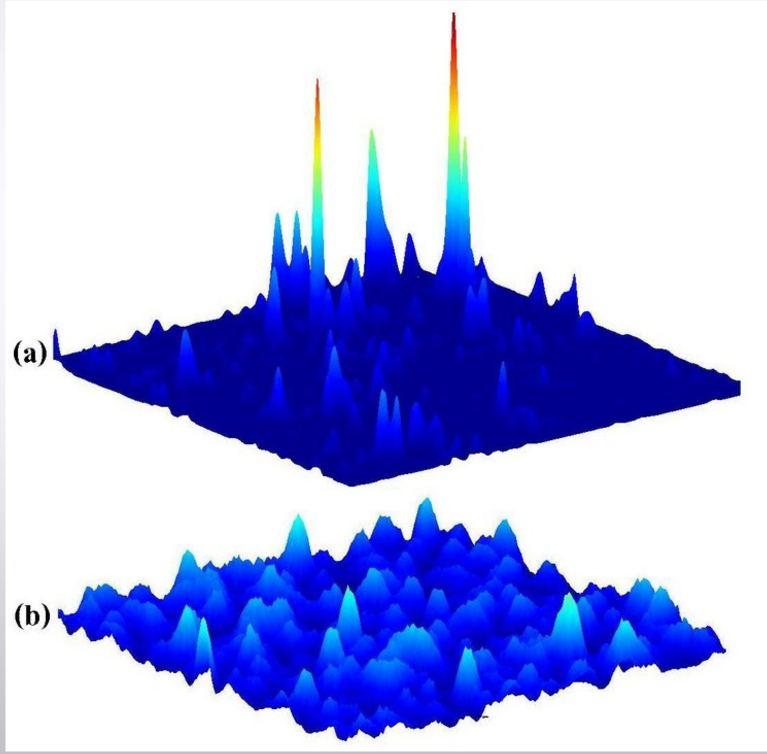
- To characterize localization: $\mathrm{IPR}_q = \sum_j^N |\Psi_j|^{2q} \;, N \propto L^d$
  - Extended: $\mathrm{IPR}_q \simeq N^{1-q} = L^{-d(q-1)}$
  - Localized: $\mathrm{IPR}_q \simeq \mathrm{const}$
  - Critical state:

$$\mathrm{IPR}_q \simeq L^{-d_q(q-1)}$$

$$= \int N^{-q\alpha + f(\alpha)} d\alpha$$

$0 < d_q < d$: fractal dimensions

$f(\alpha)$: multi-fractal spectrum

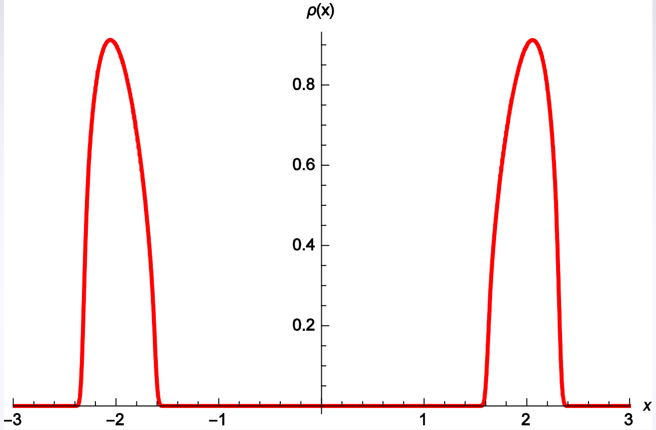


Van Tiggelen group (PRL 2009)

Brownian Motion Picture

- Level repulsion resolves degeneracy:

  $\Rightarrow$ each of the n cuts contains m_j eigenvalues



- Gap between cuts breaks rotational

  invariance: $U(N) \overset{N\to\infty}{\to} \prod_{j=1}^{n} U(m_j)$

- Dyson Brownian Motion for

  equilibrium distribution shows scale separation:

$$d\lambda_j = -\frac{dV(\lambda_j)}{d\lambda_j}\,dt + \frac{1}{N}\sum_{l\neq j}\frac{dt}{\lambda_j-\lambda_l} + \frac{1}{\sqrt{N}}\,dB_j(t)$$

$$d\vec{U}_j(t) = -\frac{1}{2N}\sum_{l\neq j}\frac{dt}{(\lambda_j-\lambda_l)^2}\,\vec{U}_j + \frac{1}{\sqrt{N}}\sum_{l\neq j}\frac{dW_{jl}(t)}{\lambda_j-\lambda_l}\,\vec{U}_l$$

dB_j, dW_{jl} delta-corr. stochastic sources

Generating a Random Matrix

- Gaussian Models: $\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr}\mathbf{M}^2} = \int \prod dM_{jl} e^{-\sum_{jl} M_{jl}^2}$

  $\rightarrow$ each matrix entries sampled independently

- One-Cut Models: $\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr}V(\mathbf{M})} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr}\sum_k g_k \mathbf{M}^k}$

  $\rightarrow$ entries correlated: generated as perturbation of Gaussian case in a Metropolis scheme

- Multi-Cut Solutions: Gaussian case unstable

  $\rightarrow$ start from initial seed and evolve it to equilibrium

  $\rightarrow$ SSB: final configuration has memory of eigenvectors of initial seed

Symmetry Breaking Term

- To detect SSB introduce symmetry breaking term
- Most natural one is $\mathrm{Tr}\left([\mathbf{M},\mathbf{S}]\right)^2$, but too hard to handle

$\mathbf{S}$: given Hermitian Matrix
Favors alignment of eigenvectors

- We introduce: $$W(J) = \ln \int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN|\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|}$$

J : source strength

$$\mathrm{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^\dagger \mathbf{T} \mathbf{V}$$

Absolute value can be removed by sorting eigenvalues in increasing order

Symmetry Breaking: Double Well

$$W(J) = \ln \int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN |\mathrm{Tr}(\mathbf{\Lambda}\, \mathbf{T} - \mathbf{M}\, \mathbf{S})|}$$

$$\boxed{\begin{aligned} \mathrm{M} &= \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U} \\ \mathrm{S} &= \mathbf{V}^\dagger \mathbf{T} \mathbf{V} \end{aligned}}$$

- Double well: $U(N) \overset{N\to\infty}{\to} U(N/2) \times U(N/2)$

  (assume N even)

- Take $\mathbf{S}$ with 2 sets of N/2-degenerate eigenvalues: $t = \pm 1$ to induce correct symmetry breaking

- Calculate (dis-)order parameter:

$$\left. \frac{dW(J)}{dJ} \right|_{J=0} = \langle |\mathrm{Tr}\,(\mathbf{\Lambda}\, \mathbf{T} - \mathbf{M}\, \mathbf{S})| \rangle$$

$= 0$ Symmetry Broken

$\neq 0$ Eigenvectors misaligned

Symmetry Breaking Term

$$W(J) = \ln \int d\mathbf{M} e^{-N\mathrm{Tr}V(\mathbf{M}) + JN|\mathrm{Tr}(\mathbf{\Lambda}\ \mathbf{T} - \mathbf{M}\ \mathbf{S})|}$$

$$\mathrm{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^\dagger \mathbf{T} \mathbf{V}$$

- Use Itzykson-Zuber formula: (Itzykson & Zuber, '80)

$$\int d\mathbf{U} e^{\mathrm{Ttr}\mathbf{AUBU}^\dagger} \propto \frac{\det\left[e^{a_j b_l}\right]}{\Delta(\{a\})\Delta(\{b\})}$$

- After regularization for degenerate eigenvalues:

$$\int d\mathbf{U} e^{JN\mathrm{Tr}\mathbf{M}\ \mathbf{S}} \propto \frac{1}{\Delta(\{\lambda\})} \sum_{\{\alpha\}\cup\{\alpha'\}=\{\lambda\}}{}^{\prime} e^{-JN\sum_j (\alpha_j - \alpha'_j)} \Delta(\{\alpha\})\Delta(\{\alpha'\})$$

Sum over ways to partition eigenvalues of M according to degeneracies of S

Symmetry Breaking Term

$$\int d\mathbf{M} e^{-N\mathrm{Tr}V(\mathbf{M})+JN|\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T}-\mathbf{M}\,\mathbf{S})|} \propto$$

$$\int_{\lambda>0,\lambda'<0} d^{\frac{N}{2}}\lambda d^{\frac{N}{2}}\lambda' e^{-N\sum_j V(\lambda_j)-N\sum_l V(\lambda'_l)} \times \Delta^2\left(\{\lambda\}\right)\Delta^2\left(\{\lambda'\}\right)\prod_{j,l}^{N}(\lambda_j-\lambda_l)\times$$

$$\times\left[1+\sum_{j,l=1}^{N/2} e^{-2JN(\lambda_j-\lambda'_l)}\prod_{p=1}^{N/2}\prod_{q=1}^{N/2}\frac{(\lambda_l-\lambda'_p)(\lambda_j-\lambda'_q)}{(\lambda'_j-\lambda_p)(\lambda'_l-\lambda_q)}+\ldots\right]$$

- Hence: $\left.\dfrac{d}{dJ}\lim_{N\to\infty} W(J)\right|_{J=0} = 0$

- At finite N: $\langle|\mathrm{Tr}\left(\mathbf{\Lambda}\,\mathbf{T}-\mathbf{M}\,\mathbf{S}\right)|\rangle \neq 0$

Instantons:

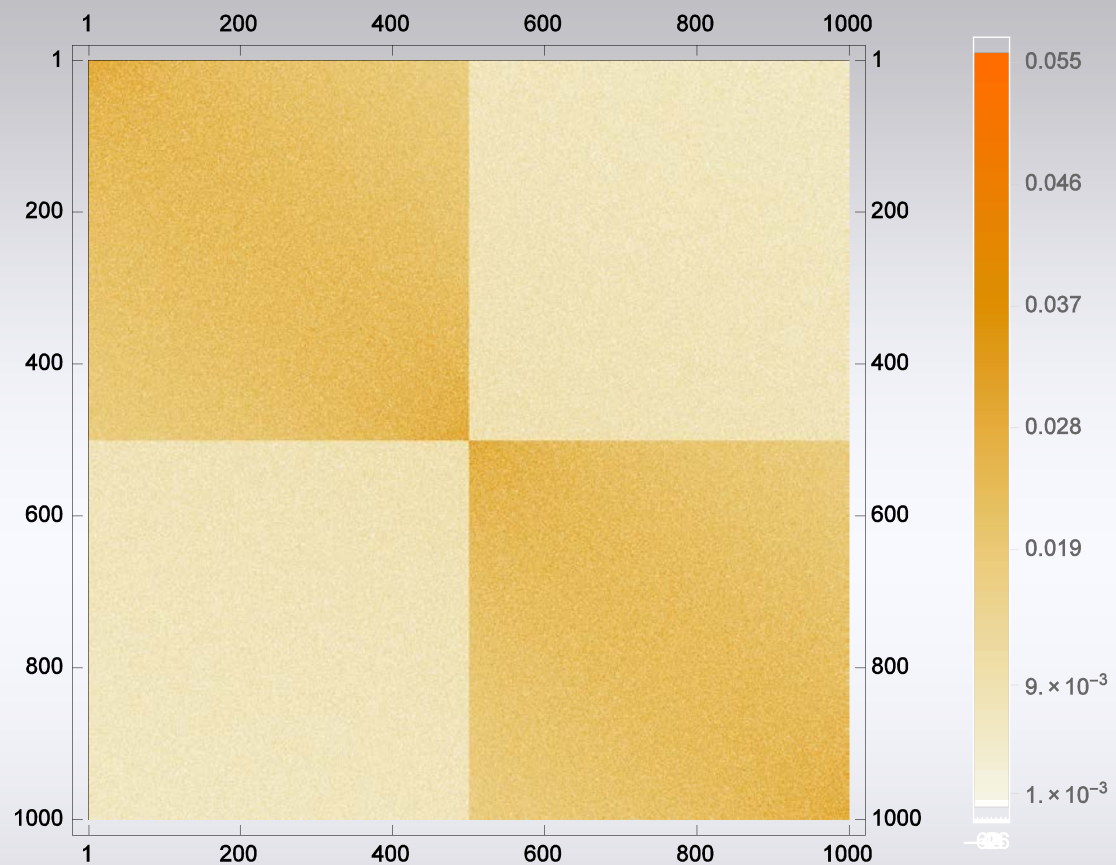
- Pairs of eigenvalues tunneling between wells
- Restore broken symmetries

$$\int d\mathbf{M} e^{-N\mathrm{Tr}V(\mathbf{M})+JN|\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T}-\mathbf{M}\,\mathbf{S})|} \propto \mathcal{Z}_0+\mathcal{Z}_1\left(e^{-2JN|\lambda_j-\lambda'_l|}\right)+\ldots$$

Finite Size Analysis

- Without preferred, reference basis; localization means rigidity of eigenvectors under perturbations
- Take double well matrix model: $\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-N \mathrm{Tr}\left[\frac{1}{4}\mathbf{M}^4 - \frac{t}{2}\mathbf{M}^2\right]}$
- Generate a representative matrix: $\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$
- Apply perturbation $\mathbf{\Delta M}$ (sparse Gaussian Matrix)
- Find eigenvectors of perturbed matrix: $\mathbf{M} + \mathbf{\Delta M} = \mathbf{U}'^\dagger \mathbf{\Lambda}' \mathbf{U}'$
- Consider eigenvectors of perturbed matrix in original eigenvector basis (rotation due to perturbation): $\tilde{\mathbf{U}} = \mathbf{U}' \mathbf{U}^\dagger$

Finite Size Analysis



$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-N\mathrm{Tr}\left[\frac{1}{4}\mathbf{M}^4 - \frac{t}{2}\mathbf{M}^2\right]}$$

Equilibrium conf. from Coulomb gas

$$\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$

Randomly generated

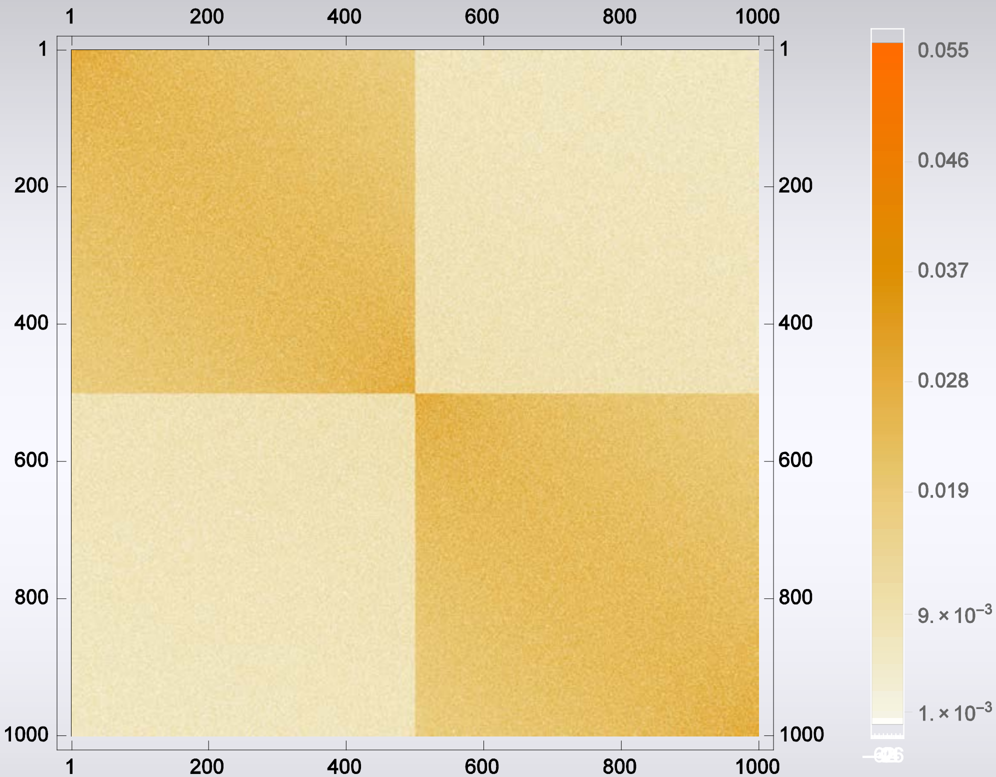
$$\mathbf{M} + \mathbf{\Delta M} = \mathbf{U}'^\dagger \mathbf{\Lambda}' \mathbf{U}'$$

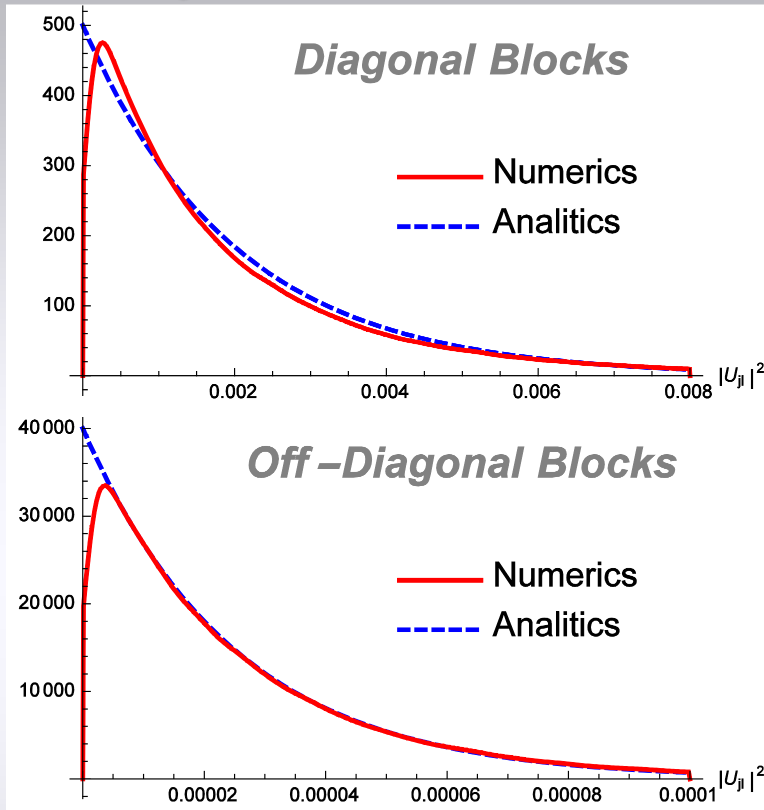
Sparse Gaussian Matrix

Typical Unitary matrix $\tilde{\mathbf{U}} = \mathbf{U}'\mathbf{U}^\dagger$ connecting the eigenvectors before and after the perturbation (t=4; N=1000; sparse matrix with n=200 non zero elements, drawn from Gaussian with zero mean and variance N)

Finite Size Analysis

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-N \mathrm{Tr}\left[\frac{1}{4}\mathbf{M}^4 - \frac{t}{2}\mathbf{M}^2\right]}$$



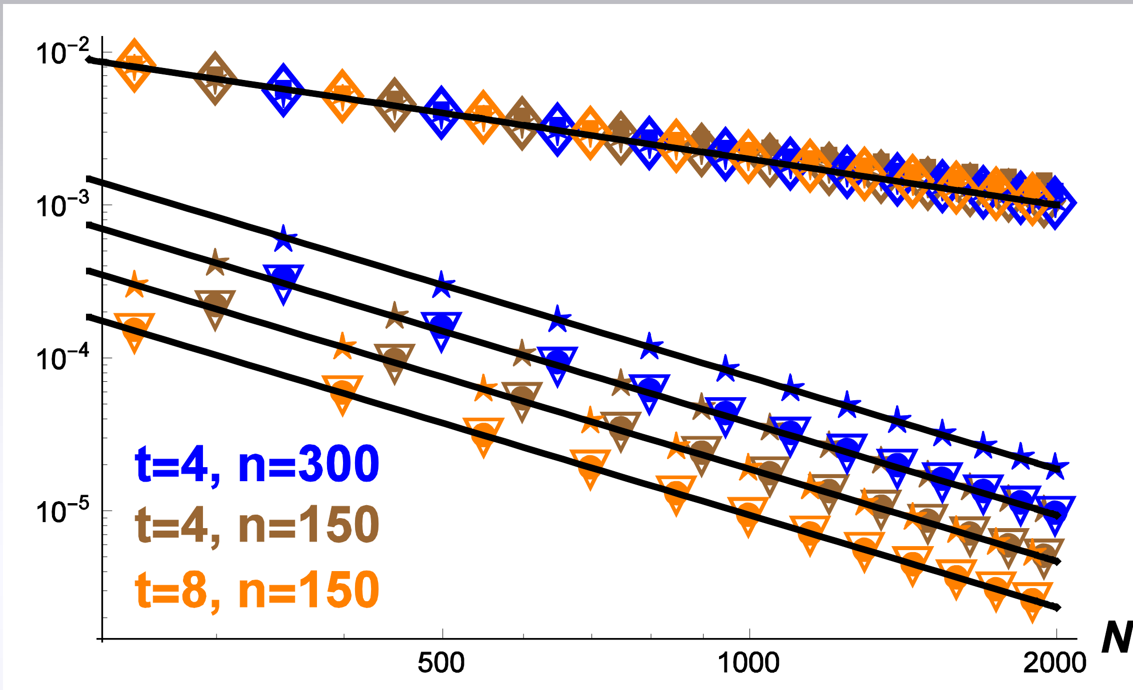


t=4, N=1000, sparse matrix with n=200 non zero elements, drawn from Gaussian with zero mean and variance N)

$$\mathcal{P}\left(\left|\tilde{U}_{ij}\right|^2\right) = \chi \exp\left[-\chi \left|\tilde{U}_{ij}\right|^2\right]$$

$$\chi_{\mathrm{D}} = \frac{N}{2} \qquad \chi_{\mathrm{OD}} = \frac{2tN^2}{n}$$

Finite Size Analysis



$$O_{jl} = \sum_{m}^{N} \left|\tilde{U}_{mj}\right|^2 \left|\tilde{U}_{ml}\right|^2$$

Overlaps between eigenstates

$$\langle|O_{jl}|\rangle_{\mathrm{D}} = \langle\left|\tilde{U}_{jl}\right|\rangle_{\mathrm{D}} = \langle\left|\Delta\tilde{U}_{jl}\right|\rangle_{\mathrm{D}} = \frac{1}{\chi_{\mathrm{D}}} = \frac{2}{N}$$

$$\langle|O_{jl}|\rangle_{\mathrm{OD}} = 2\langle\left|\tilde{U}_{jl}\right|\rangle_{\mathrm{OD}} = 2\langle\left|\Delta\tilde{U}_{jl}\right|\rangle_{\mathrm{OD}} = \frac{2}{\chi_{\mathrm{OD}}} = \frac{n}{tN^2}$$

Off-diagonal blocks suppressed as 1/N compared to diagonal ones

Onset of localizations!

Weakly Confined Matrix Model

- Unfolding to make density constant:

$$\rho(\lambda) \equiv \mathrm{Tr}\left\{\delta\left(\lambda - \mathbf{H}\right)\right\} \xrightarrow{\boxed{\lambda_x = e^{\kappa|x|}\,\mathrm{sign}(x)}} \langle\tilde{\rho}(x)\rangle \equiv \langle\rho(\lambda_x)\rangle \frac{d\lambda_x}{dx} = 1$$

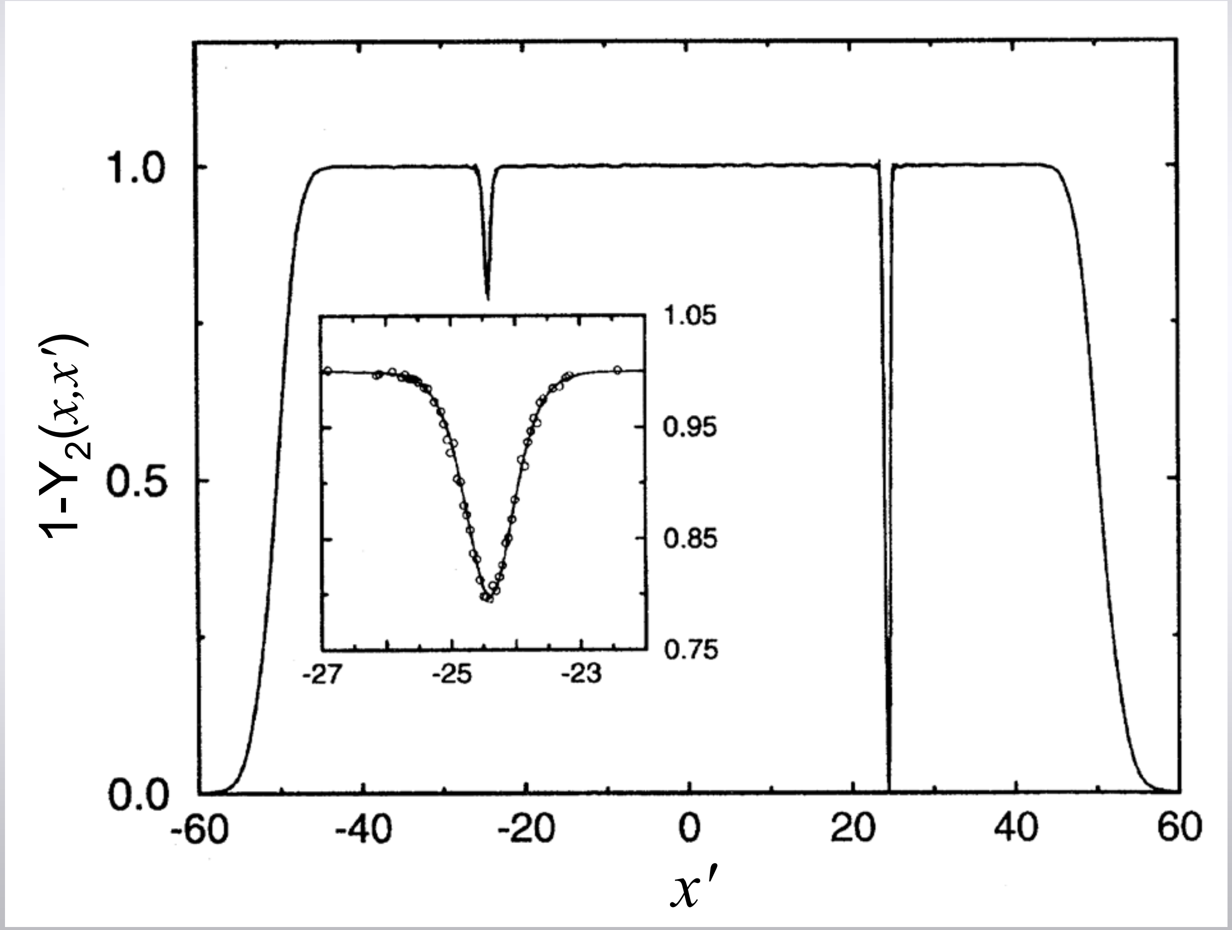
- For $e^{-2\pi^2/\kappa} << 1$ semiclassical analysis (Canali et al '95):

$$\begin{aligned} Y_2(x,x') \quad = \quad & \frac{\kappa^2}{4\pi^2} \frac{\sin^2[\pi(x-x')]}{\sinh^2[\kappa(x-x')/2]}\,\theta(x\,x') \\ & + \frac{\kappa^2}{4\pi^2} \frac{\sin^2[\pi(x-x')]}{\cosh^2[\kappa(x+x')/2]}\,\theta(-\,x\,x') \end{aligned}$$

$$\boxed{Y_2(x,x') \equiv \delta(x-x') - \frac{\langle\rho(E_x)\rho(E_{x'})\rangle - \langle\rho(E_x)\rangle\langle\rho(E_{x'})\rangle}{\langle\rho(E_x)\rangle\langle\rho(E_{x'})\rangle}}$$

Weakly Confined Invariant Ensemble

- Numerical check (Canali et al ’95):



Luttinger theory for RME

$$\rho(x,\tau) = \rho_0 - \frac{1}{\pi}\,\partial_x\Phi + \frac{A_K}{\pi}\cos\left[2\pi\rho_0 x - 2\Phi\right] + \ldots$$

Unfolding:
$\rho_0 = 1$

- Two-Point function (Kravtsov et al. '00):

$$\begin{aligned} Y_2 \quad &= \quad -\frac{1}{\pi^2}\langle\partial_x\Phi(x)\partial_{x'}\Phi(x')\rangle \\ &\quad -\frac{A_K^2}{2\pi^2}\cos(2\pi(x-x'))\langle e^{i2\Phi(x)}e^{-i2\Phi(x')}\rangle + \ldots \end{aligned}$$

- In flat space: $\langle\Phi(x,t)\Phi(x',t')\rangle \propto \ln\left(\Delta x^2 + \Delta t^2\right)$

$$Y_2 \propto \frac{\sin^2\left[\pi(x-x')\right]}{(x-x')^2}$$

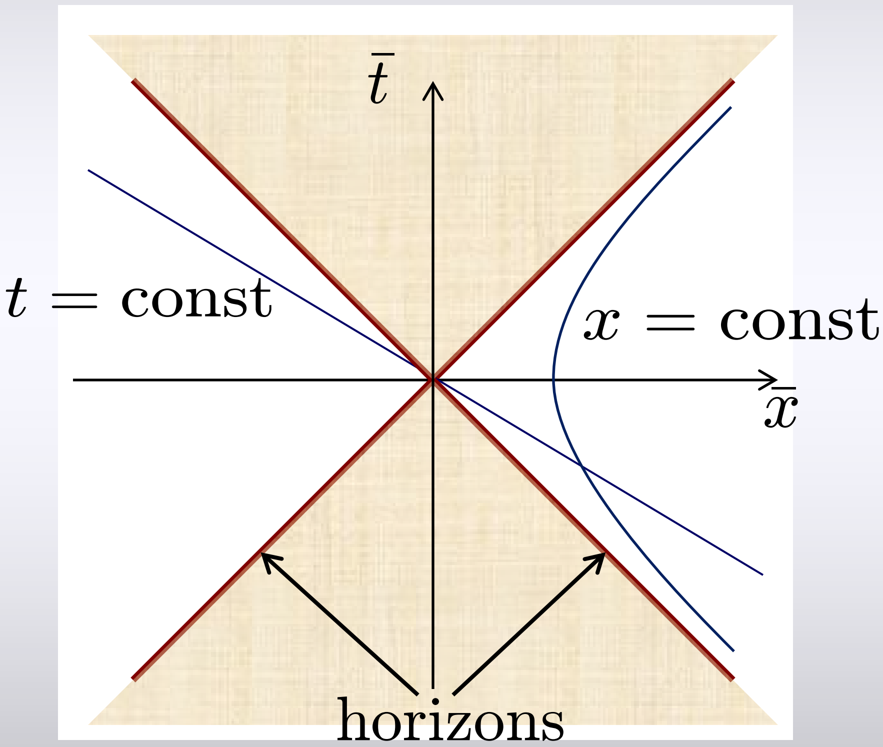
2-Point Function
for Gaussian RME
(K=1: Unitary)

Luttinger theory in Rindler space

$$\begin{cases} \bar t & \equiv & \dfrac{1}{\kappa} \; \sinh \kappa x \;\; \sinh \kappa t \\ \bar x & \equiv & \dfrac{1}{\kappa} \; \sinh \kappa x \;\; \cosh \kappa t \end{cases}$$

Periodic in imaginary time
$\rightarrow$ finite **temperature**

$$\begin{aligned} \mathrm{d}s^2 &= -\sinh^2(\kappa x)\, \mathrm{d}u^+ \, \mathrm{d}u^- \\ &= -\, \mathrm{d}\bar u^+ \, \mathrm{d}\bar u^- \\ \bar u^\pm &\equiv \bar t \pm \bar x \end{aligned}$$



- Far from the origin:

$$\bar u^\pm \simeq \begin{cases} \pm \dfrac{\mathrm{e}^{\pm \kappa u^\pm}}{2\kappa}, & x \gg 1 \\ \mp \dfrac{\mathrm{e}^{\pm \kappa u^\mp}}{2\kappa}, & x \ll -1 \end{cases}$$

Luttinger Liquid in Rindler Space

- Remind two-Point function:

$$Y_2 = -\frac{1}{\pi^2}\langle\partial_x\Phi(x)\partial_{x'}\Phi(x')\rangle - \frac{A_K^2}{2\pi^2}\cos(2\pi(x-x'))\langle \mathrm{e}^{\mathrm{i}2\Phi(x)}\mathrm{e}^{-\mathrm{i}2\Phi(x')}\rangle + \ldots$$

- With the new coordinates: $\left(\bar{x} = \frac{\mathrm{e}^{\kappa|x|}}{2\kappa}\,\mathrm{sgn}(x)\right)$

$$\langle\Phi(x)\Phi(x')\rangle \overset{|x|,|x'|\gg 1}{\propto} \begin{cases} \ln\left[\frac{2}{\kappa}\sinh\frac{\kappa(x-x')}{2}\right], & x\,x' > 0 \\ \ln\left[\frac{2}{\kappa}\cosh\frac{\kappa(x+x')}{2}\right], & x\,x' < 0 \end{cases}$$

Luttinger Liquid in Rindler Space

- We recover exactly the RME correlation (K=1):

$$Y_2^a(x,x') = \frac{\kappa^2}{4\pi^2} \frac{\sin^2[\pi(x-x')]}{\cosh^2[\kappa(x+x')/2]}\,, \qquad \text{for} \quad x\,x' < 0$$

(Anomalous: non-translational invariant)

$$Y_2^n(x,x') = \frac{\kappa^2}{4\pi^2} \frac{\sin^2[\pi(x-x')]}{\sinh^2[\kappa(x-x')/2]}\,, \qquad \text{for} \quad x\,x' > 0$$

(Normal: translational invariant)

Topological String Theory

$$\mathcal{Z} = \int \mathrm{d}\mathbf{M}\, \mathrm{e}^{-\frac{1}{\kappa}\mathrm{Tr}\ln^2(\mathbf{M})}$$

Random Matrix Model
(0+1 dim theory)

U(N) Chern-Simons on S^3
(3 dim theory)

Luttinger Liquid of Levels
(1+1 dim theory)

?

Wess-Zumino-Witten
(1+1 dim theory)

Thank you!