Hadronic light-by-light scattering in the muon $g-2$: a dispersive approach

TECHNISCHE UNIVERSITÄT DARMSTADT

Martin Hoferichter

Institut für Kernphysik, Technische Universität Darmstadt

ExtreMe Matter Institute EMMI, GSI, Darmstadt

Bundesministerium für Bildung und Forschung

EMMI

MesonNet Meeting

Frascati, September 29, 2014

Minerva Stiftung
ARCHES
Award for Research Cooperation and High Excellence in Science

G. Colangelo, MH, M. Procura, P. Stoffer, JHEP 09 (2014) 091, arXiv:1309.6877

G. Colangelo, MH, B. Kubis, M. Procura, P. Stoffer, PLB 738 (2014) 6

Outline

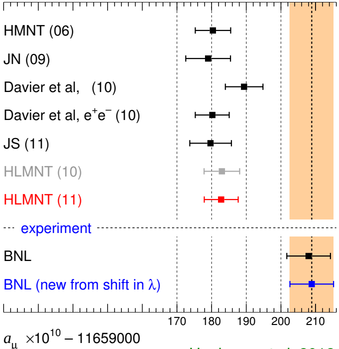
Anomalous magnetic moment of the muon

- **Experimental precision** 0.5 ppm BNL E821 2006

$$a_\mu^{\text{exp}} = (116\,592\,089 \pm 63) \cdot 10^{-11}$$

- **Theory error** of **similar size**
- Deviation from SM prediction around 3σ
- New experiment at **FNAL** (E989) aiming at 0.14 ppm, beam in 2016/2017
- **J-PARC** aiming at 0.1 ppm, new approach with ultra-cold muons, R&D in progress

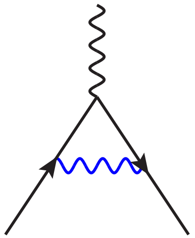
$\Rightarrow$ **Need to improve theory by a factor of 4**



Hagiwara et al. 2012

Overview of SM prediction

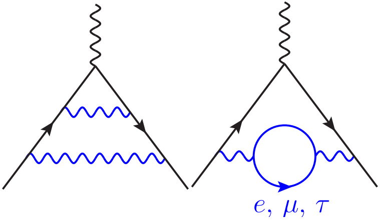
	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.85	0.04
electroweak, total	153.6	1.0
HVP (LO)	6 949.	43.
HVP (NLO)	$-98.$	1.
HLbL (LO)	116.	40.
HVP (NNLO)	12.4	0.1
HLbL (NLO)	3.	2.
theory	116 591 855.	59.



Schwinger 1948

Overview of SM prediction

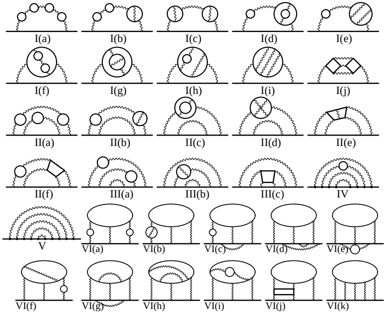
	$a_\mu\ [10^{-11}]$	$\Delta a_\mu\ [10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.85	0.04
electroweak, total	153.6	1.0
HVP (LO)	6 949.	43.
HVP (NLO)	−98.	1.
HLbL (LO)	116.	40.
HVP (NNLO)	12.4	0.1
HLbL (NLO)	3.	2.
theory	116 591 855.	59.



Sommerfield, Petermann 1957

Overview of SM prediction

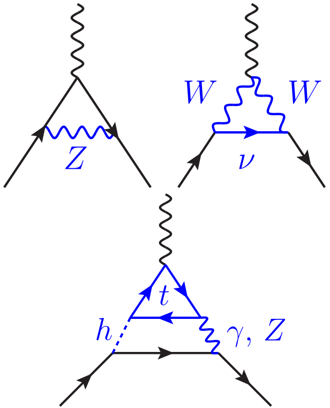
	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.85	0.04
electroweak, total	153.6	1.0
HVP (LO)	6 949.	43.
HVP (NLO)	−98.	1.
HLbL (LO)	116.	40.
HVP (NNLO)	12.4	0.1
HLbL (NLO)	3.	2.
theory	116 591 855.	59.



Kinoshita et al. 2012

Overview of SM prediction

	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.85	0.04
electroweak, total	153.6	1.0
HVP (LO)	6 949.	43.
HVP (NLO)	−98.	1.
HLbL (LO)	116.	40.
HVP (NNLO)	12.4	0.1
HLbL (NLO)	3.	2.
theory	116 591 855.	59.



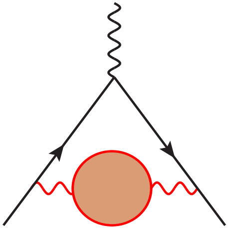
1-loop: Jackiw, Weinberg and others 1972

2-loop: Kukhto et al. 1992, Czarnecki, Krause, Marciano 1995, Degrassi, Giudice 1998, Knecht, Peris, Perrottet, de Rafael 2002, Vainshtein 2003, Heinemeyer, Stöckinger, Weiglein 2004, Gribouk, Czarnecki 2005

Update after Higgs discovery: Gnendiger et al. 2013

Overview of SM prediction

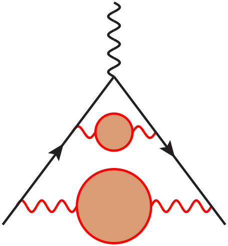
	$a_\mu\ [10^{-11}]$	$\Delta a_\mu\ [10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.85	0.04
electroweak, total	153.6	1.0
HVP (LO)	6 949.	43.
HVP (NLO)	−98.	1.
HLbL (LO)	116.	40.
HVP (NNLO)	12.4	0.1
HLbL (NLO)	3.	2.
theory	116 591 855.	59.



Hagiwara et al. 2011

Overview of SM prediction

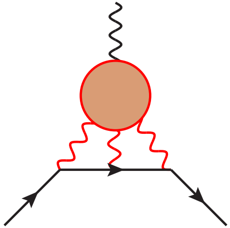
	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.85	0.04
electroweak, total	153.6	1.0
HVP (LO)	6 949.	43.
HVP (NLO)	−98.	1.
HLbL (LO)	116.	40.
HVP (NNLO)	12.4	0.1
HLbL (NLO)	3.	2.
theory	116 591 855.	59.



Calmet et al. 1976, Hagiwara et al. 2011

Overview of SM prediction

	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.85	0.04
electroweak, total	153.6	1.0
HVP (LO)	6 949.	43.
HVP (NLO)	−98.	1.
HLbL (LO)	116.	40.
HVP (NNLO)	12.4	0.1
HLbL (NLO)	3.	2.
theory	116 591 855.	59.



Hayakawa, Kinoshita, Sanda 1995

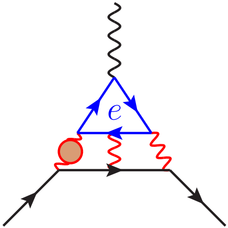
Bijnens, Pallante, Prades 1995

Knecht, Nyffeler 2001

Jegerlehner, Nyffeler 2009

Overview of SM prediction

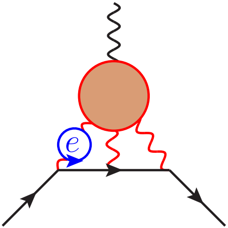
	$a_\mu\ [10^{-11}]$	$\Delta a_\mu\ [10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.85	0.04
electroweak, total	153.6	1.0
HVP (LO)	6 949.	43.
HVP (NLO)	−98.	1.
HLbL (LO)	116.	40.
HVP (NNLO)	12.4	0.1
HLbL (NLO)	3.	2.
theory	116 591 855.	59.



Kurz, Liu, Marquard, Steinhauser 2014

Overview of SM prediction

	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.85	0.04
electroweak, total	153.6	1.0
HVP (LO)	6 949.	43.
HVP (NLO)	−98.	1.
HLbL (LO)	116.	40.
HVP (NNLO)	12.4	0.1
HLbL (NLO)	3.	2.
theory	116 591 855.	59.



Colangelo, MH, Nyffeler, Passera, Stoffer 2014

Overview of SM prediction

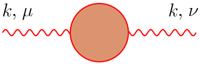
	$a_\mu\,[10^{-11}]$	$\Delta a_\mu\,[10^{-11}]$
experiment	116 592 089.	63.
QED $\mathcal{O}(\alpha)$	116 140 973.21	0.03
QED $\mathcal{O}(\alpha^2)$	413 217.63	0.01
QED $\mathcal{O}(\alpha^3)$	30 141.90	0.00
QED $\mathcal{O}(\alpha^4)$	381.01	0.02
QED $\mathcal{O}(\alpha^5)$	5.09	0.01
QED total	116 584 718.85	0.04
electroweak, total	153.6	1.0
HVP (LO)	6 949.	43.
HVP (NLO)	−98.	1.
HLbL (LO)	116.	40.
HVP (NNLO)	12.4	0.1
HLbL (NLO)	3.	2.
theory	116 591 855.	59.

$$a_\mu^{\text{exp}} - a_\mu^{\text{SM}} = (234 \pm 86) \cdot 10^{-11} [2.7\sigma]$$

⇒**Theory error** comes almost exclusively from **hadronic part**

Hadronic vacuum polarization

- General principles yield **direct connection with experiment**
  - **Gauge invariance**



$$= -i\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)\Pi\left(k^2\right)$$

  - **Analyticity**

$$\Pi_{\text{ren}} = \Pi\left(k^2\right) - \Pi(0) = \frac{k^2}{\pi}\int\limits_{4M_\pi^2}^{\infty} \text{d}s \frac{\text{Im}\,\Pi(s)}{s\left(s-k^2\right)}$$

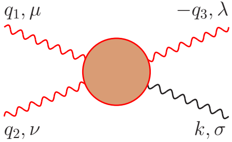
  - **Unitarity**

$$\text{Im}\,\Pi(s) = \frac{s}{4\pi\alpha}\sigma_{\text{tot}}\left(e^+e^- \to \text{hadrons}\right) = \frac{\alpha}{3}R(s)$$

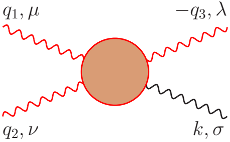
- 1 Lorentz structure, 1 kinematic variable, parameter-free
- **Dedicated e^+e^- program** under way: BaBar, Belle, BESIII, CMD3, KLOE2, SND

HLbL scattering

- Large uncertainty and **model dependence**
- 5 kinematic variables, (at least) 29 Lorentz structures



HLbL scattering

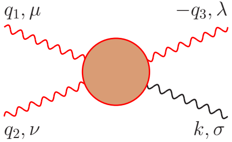


- Large uncertainty and **model dependence**
- 5 kinematic variables, (at least) 29 Lorentz structures
- **Dispersive point of view**
  - Analytic structure: poles and cuts

    $\hookrightarrow$ **residues** and **imaginary parts** $\Rightarrow$ by definition **on-shell** quantities

    $\hookrightarrow$ **form factors** and **scattering amplitudes** from experiment
  - Expansion: mass of intermediate states, partial waves
  - Pseudoscalar poles most important, next $\pi\pi$ cuts

HLbL scattering



- Large uncertainty and **model dependence**
- 5 kinematic variables, (at least) 29 Lorentz structures
- **Dispersive point of view**
  - Analytic structure: poles and cuts

    $\hookrightarrow$ **residues** and **imaginary parts** $\Rightarrow$ by definition **on-shell** quantities

    $\hookrightarrow$ **form factors** and **scattering amplitudes** from experiment
  - Expansion: mass of intermediate states, partial waves
  - Pseudoscalar poles most important, next $\pi\pi$ cuts
- Decompose the tensor according to

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$

  $\hookrightarrow$ accounts for **one-** and **two-pion** intermediate states
- Generalizes immediately to η, η', $K\bar{K}$, but e.g. 3π more difficult

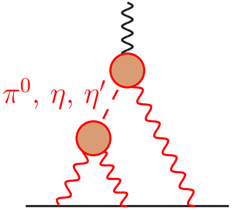
Pion pole: pion transition form factor

Knecht, Nyffeler 2001

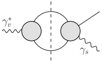
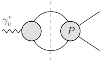
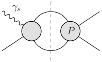
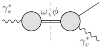
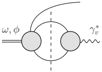
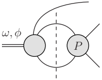
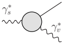
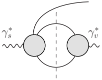
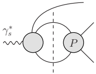
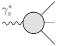
Master formula for pion-pole contribution

$$a_\mu^{\pi^0\text{-pole}} = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 s\big((p+q_1)^2 - m^2\big)\big((p-q_2)^2 - m^2\big)}$$
$$\times \bigg\{ \frac{F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) F_{\pi^0\gamma^*\gamma^*}(s, 0)}{s - M_\pi^2} T_1(q_1, q_2; p) + \frac{F_{\pi^0\gamma^*\gamma^*}(s, q_1^2) F_{\pi^0\gamma^*\gamma^*}(q_2^2, 0)}{q_2^2 - M_\pi^2} T_2(q_1, q_2; p) \bigg\}$$

- Crucial ingredient: **pion transition form factor** $F_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2)$
- **Wick rotation**: only **space-like** s, q_1^2, q_2^2 contribute
- **Dispersive approach**
  - **On-shell** form factor
  - Fix parameters wherever data are available
  - Use **analyticity** to go to the space-like region



Pion transition form factor: unitarity relations

process	unitarity relations	SC 1	SC 2
			$F_{\pi^0\gamma\gamma}$
		$F_{3\pi}$	$\sigma(\gamma\pi\to\pi\pi)$
			$\Gamma_{\pi^0\gamma}$
		$\Gamma_{3\pi}$	$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}s\mathrm{d}t}(\omega,\phi\to 3\pi)$
			$\sigma(e^+e^-\to\pi^0\gamma)$
		$\sigma(e^+e^-\to 3\pi)$	$\sigma(\gamma\pi\to\pi\pi)$ $\frac{\mathrm{d}^2\Gamma}{\mathrm{d}s\mathrm{d}t}(\omega,\phi\to 3\pi)$
		$F_{3\pi}$	$\sigma(e^+e^-\to 3\pi)$

$\gamma\pi\to\pi\pi$

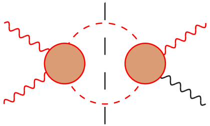
$\omega\to 3\pi$, $\phi\to 3\pi$

$\gamma^*\to 3\pi$

see talks by B. Kubis (π^0)
and A. Wirzba (η, η')

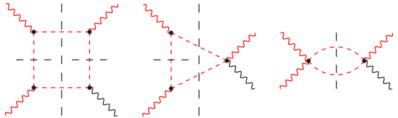
$\pi\pi$ intermediate states: FsQED contribution

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \cdots$$



- Separate terms with **simultaneous cuts**

$$\Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} = F^V_\pi(q_1^2)\, F^V_\pi(q_2^2)\, F^V_\pi(q_3^2) \times \Big[\quad \Big]$$



- Multiplication of sQED diagrams with F^V_π gives correct q^2-dependence

  $\hookrightarrow$ **not an approximation**

- Remaining $\pi\pi$ contribution included in $\bar{\Pi}_{\mu\nu\lambda\sigma}$ has cuts only in one channel

  $\hookrightarrow$ partial-wave expansion, dispersion relations for this part

$\pi\pi$ intermediate states: reconstruction theorem

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_i \Big(A_{i,s}^{\mu\nu\lambda\sigma} \Pi_i(s) + A_{i,t}^{\mu\nu\lambda\sigma} \Pi_i(t) + A_{i,u}^{\mu\nu\lambda\sigma} \Pi_i(u) \Big)$$

- Π_i functions with a **right-hand cut** only

  $\hookrightarrow$ similar to **reconstruction theorem** of $\pi\pi$ scattering Stern, Sazdjian, Fuchs 1993
- Keep discontinuity of lowest partial waves
- Dispersion integrals for Π_i required to have correct **soft-photon zeros**

  $\hookrightarrow$ forces subtraction constants to zero

$\pi\pi$ intermediate states: reconstruction theorem

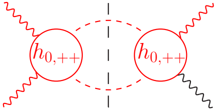
$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_i \left(A^{\mu\nu\lambda\sigma}_{i,s} \Pi_i(s) + A^{\mu\nu\lambda\sigma}_{i,t} \Pi_i(t) + A^{\mu\nu\lambda\sigma}_{i,u} \Pi_i(u) \right)$$

- Π_i functions with a **right-hand cut** only

  $\hookrightarrow$ similar to **reconstruction theorem** of $\pi\pi$ scattering Stern, Sazdjian, Fuchs 1993
- Keep discontinuity of lowest partial waves
- Dispersion integrals for Π_i required to have correct **soft-photon zeros**

  $\hookrightarrow$ forces subtraction constants to zero
- Gives relations such as

$$\Pi_1(s) = \frac{s-q_3^2}{\pi} \int\limits_{4M_\pi^2}^{\infty} \frac{\mathrm{d}s'}{s'-q_3^2} K_1(s',s) \mathrm{Im}\, \bar{h}^0_{++,++}(s';q_1^2,q_2^2;q_3^2,0)$$



$$\mathrm{Im}\, h^0_{++,++}(s;q_1^2,q_2^2;q_3^2,0) = \frac{\sigma(s)}{16\pi} h^*_{0,++}(s;q_1^2,q_2^2) h_{0,++}(s;q_3^2,0)$$

$$\sigma(s) = \sqrt{1-\frac{4M_\pi^2}{s}} \qquad K_1(s',s) = \frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda(s',q_1^2,q_2^2)} \qquad \lambda(x,y,z) = x^2+y^2+z^2-2(xy+xz+yz)$$

$\pi\pi$ intermediate states: non-diagonal terms

$$\bar{\Pi}^{\mu\nu\lambda\sigma} = \sum_i \left(A_{i,s}^{\mu\nu\lambda\sigma} \Pi_i(s) + A_{i,t}^{\mu\nu\lambda\sigma} \Pi_i(t) + A_{i,u}^{\mu\nu\lambda\sigma} \Pi_i(u) \right)$$

- Need to choose $A_i^{\mu\nu\lambda\sigma}$ so that Π_i are **free of kinematic singularities**
- General procedure for finding such a basis Bardeen, Tung 1968, Tarrach 1975
- Results in **non-diagonal terms**

$$\Pi_1(s) = \frac{s - q_3^2}{\pi} \int\limits_{4M_\pi^2}^{\infty} \frac{ds'}{s' - q_3^2} \left(K_1(s', s) \operatorname{Im} \bar{h}^0_{++,++}(s') + \frac{2\xi_1\xi_2}{\lambda(s', q_1^2, q_2^2)} \operatorname{Im} \bar{h}^0_{00,++}(s') \right)$$

- Solved for S-wave, D-wave calculation in progress

Digression: $\gamma^*\gamma^* \to \pi\pi$

- Similar analysis for $\gamma^*\gamma^* \to \pi\pi$: Bardeen–Tung–Tarrach basis

  $\hookrightarrow$ partial-wave dispersion relations (**Roy–Steiner equations**)
- Find similar non-diagonal kernels

Digression: $\gamma^*\gamma^* \to \pi\pi$

- Similar analysis for $\gamma^*\gamma^* \to \pi\pi$: Bardeen–Tung–Tarrach basis

  $\hookrightarrow$ partial-wave dispersion relations (**Roy–Steiner equations**)
- Find similar non-diagonal kernels
- Check within 1-loop ChPT

$$\frac{1}{\pi}\int\limits_{4M_\pi^2}^{\infty} dt' \left\{\left(\frac{1}{t'-t} - \frac{t'-q_1^2-q_2^2}{\lambda(t',q_1^2,q_2^2)}\right)\mathrm{Im}\,h_1(t';q_1^2,q_2^2) + \frac{2q_1^2q_2^2}{\lambda(t',q_1^2,q_2^2)}\mathrm{Im}\,h_2(t';q_1^2,q_2^2)\right\}$$

$$= 1 + 2\left(M_\pi^2 + \frac{tq_1^2q_2^2}{\lambda(t,q_1^2,q_2^2)}\right)C_0(t,q_1^2,q_2^2) + \frac{t(q_1^2+q_2^2)-(q_1^2-q_2^2)^2}{\lambda(t,q_1^2,q_2^2)}\bar{J}(t)$$

$$-\frac{q_1^2(t+q_2^2-q_1^2)}{\lambda(t,q_1^2,q_2^2)}\bar{J}(q_1^2) - \frac{q_2^2(t+q_1^2-q_2^2)}{\lambda(t,q_1^2,q_2^2)}\bar{J}(q_2^2)$$

$$\mathrm{Im}\,h_1(t;q_1^2,q_2^2) = 2\left(M_\pi^2 + \frac{tq_1^2q_2^2}{\lambda(t,q_1^2,q_2^2)}\right)\mathrm{Im}\,C_0(t,q_1^2,q_2^2) + \frac{t(q_1^2+q_2^2)-(q_1^2-q_2^2)^2}{\lambda(t,q_1^2,q_2^2)}\mathrm{Im}\,\bar{J}(t)$$

$$\mathrm{Im}\,h_2(t;q_1^2,q_2^2) = -\frac{1}{\lambda(t,q_1^2,q_2^2)}\left[\left(t^2-(q_1^2-q_2^2)^2\right)\mathrm{Im}\,C_0(t,q_1^2,q_2^2) + 4t\,\mathrm{Im}\,\bar{J}(t)\right]$$

$\hookrightarrow$ non-diagonal kernels crucial for doubly-virtual case

- Another doubly-virtual complication: **anomalous thresholds** in time-like region

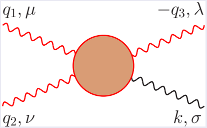
  Colangelo, MH, Procura, Stoffer arXiv:1309.6877

$\pi\pi$ intermediate states: master formula

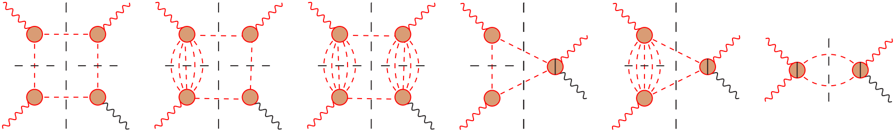
Colangelo, MH, Procura, Stoffer 2014

Master formula for $\pi\pi$ intermediate states

$$a_\mu^{\pi\pi} = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_2}{(2\pi)^4} \frac{\sum_i T_i(q_1, q_2; p) I_i(s, q_1^2, q_2^2)}{q_1^2 q_2^2 s\big((p+q_1)^2 - m^2\big)\big((p-q_2)^2 - m^2\big)}$$



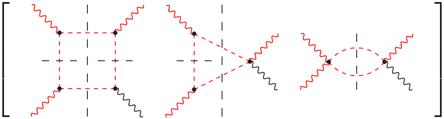
- $I_i(s, q_1^2, q_2^2)$: dispersive integrals over $\gamma^*\gamma^* \to \pi\pi$ **helicity partial waves**
- What is included? How?



$\hookrightarrow$ sorted by analytic structure in the crossed channel

Preliminary numbers: FsQED

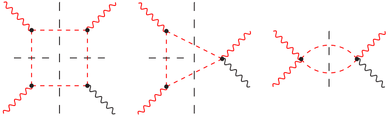
$$\Pi_{\mu\nu\lambda\sigma}^{\text{FsQED}} = F_\pi^V(q_1^2)\,F_\pi^V(q_2^2)\,F_\pi^V(q_3^2) \times$$



- Input for $F_\pi^V(q^2)$: Omnès factor $F_\pi^V(s) = \exp\left\{\frac{s}{\pi}\int\limits_{4M_\pi^2}^{\infty} \mathrm{d}s' \frac{\delta_1^1(s')}{s'(s'-s)}\right\}$
- Results for a_μ^{FsQED} in units of 10^{-11}

phase shift δ_1^1	loop integrals	DR 1	DR 2
CCL	-13.77 ± 0.01	-15.87 ± 0.01	-14.57 ± 0.01
CCL + ρ', ρ''	-14.65 ± 0.01	-16.90 ± 0.02	-15.53 ± 0.01

$$\Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} = F^V_\pi(q_1^2)\,F^V_\pi(q_2^2)\,F^V_\pi(q_3^2) \times \Big[\quad \Big]$$



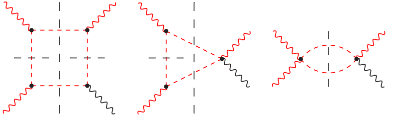
- Input for $F^V_\pi(q^2)$: Omnès factor $F^V_\pi(s) = \exp\left\{ \frac{s}{\pi} \int\limits_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)} \right\}$
- Results for a_μ^{FsQED} in units of 10^{-11}

phase shift δ_1^1	loop integrals	DR 1	DR 2
CCL	-13.77 ± 0.01	-15.87 ± 0.01	-14.57 ± 0.01
CCL + ρ', ρ''	-14.65 ± 0.01	-16.90 ± 0.02	-15.53 ± 0.01

  - Dependence on $F^V_\pi(s)$: analytic continuation can be stabilized using space-like data

Preliminary numbers: FsQED

$$\Pi^{\text{FsQED}}_{\mu\nu\lambda\sigma} = F^V_\pi(q_1^2)\,F^V_\pi(q_2^2)\,F^V_\pi(q_3^2) \times \Big[\quad \Big]$$



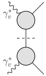
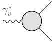
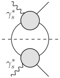
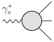
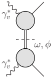
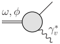
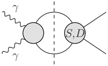
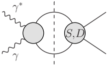
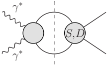
- Input for $F^V_\pi(q^2)$: Omnès factor $F^V_\pi(s) = \exp\left\{\frac{s}{\pi}\int\limits_{4M_\pi^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\}$
- Results for a_μ^{FsQED} in units of 10^{-11}

phase shift δ_1^1	loop integrals	DR 1	DR 2
CCL	-13.77 ± 0.01	-15.87 ± 0.01	-14.57 ± 0.01
CCL + ρ', ρ''	-14.65 ± 0.01	-16.90 ± 0.02	-15.53 ± 0.01

- Dependence on $F^V_\pi(s)$: analytic continuation can be stabilized using space-like data
- Basis $A_i^{\mu\nu\lambda\sigma}$ not unique (but: Π_i need to be **free of kinematic singularities**)

  $\hookrightarrow$ DR 1/2 equivalent for suitable high-energy behavior $\Rightarrow$ theoretical uncertainty
- **Why does this work so well?** it shouldn't: double-spectral regions, only S-waves!

$\gamma^*\gamma^* \to \pi\pi$ partial waves: unitarity relations

process	building blocks and SC
	
	
	
	$\alpha_1 \pm \beta_1,\, \alpha_2 \pm \beta_2$
	$\alpha_1(q^2) \pm \beta_1(q^2)$, ChPT $e^+e^- \to \pi\pi\gamma$ $e^+e^- \to e^+e^-\pi\pi$
	ChPT $(e^+e^- \to \pi\pi\gamma)$ $e^+e^- \to e^+e^-\pi\pi$

left-hand cut

π

2π

$3\pi\,(\sim \omega, \phi)$

unitarity relations

on-shell

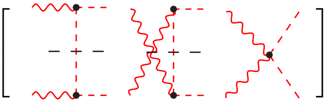
singly-virtual

doubly-virtual

Omnès representation for S-wave

$$h^I_{0,++}(s) = N^I_{0,++}(s) + \frac{\Omega^I_0(s)}{\pi} \int_{4M_\pi^2}^{s_\text{m}} ds' \frac{\sin\delta^I_0(s')}{|\Omega^I_0(s')|} \Bigg[\bigg(\frac{1}{s'-s} - \frac{s'-q_1^2-q_2^2}{\lambda(s',q_1^2,q_2^2)} \bigg) N^I_{0,++}(s') + \frac{2\xi_1\xi_2}{\lambda(s',q_1^2,q_2^2)} N^I_{0,00}(s') \Bigg]$$

- Starting point: **Roy–Steiner** equations for $\gamma^*\gamma^* \to \pi\pi$
- Omnès factors $\Omega^I_0(s) = \exp\left\{ \frac{s}{\pi} \int_{4M_\pi^2}^{s_\text{m}} ds' \frac{\delta^I_0(s')}{s'(s'-s)} \right\}$
- LHC approximated by **pion pole** $N^I_{0,\lambda_1\lambda_2}$ only

$F^V_\pi(q_1^2)\, F^V_\pi(q_2^2) \times$


- **Finite matching point**: $h^I_{0,++}(s) = 0$ above s_m
- Take $\sqrt{s_\text{m}} = 0.98\,\text{GeV}$

  $\hookrightarrow$ no $f_0(980)$ or coupling to $K\bar{K} \Rightarrow$ "σ-contribution"

Preliminary numbers: $\pi\pi$ rescattering for S-waves

- $a_\mu^{\pi\pi}$ in units of 10^{-11}

phase shift δ_1^1	$I=0$ DR 1	$I=0$ DR 2	$I=2$ DR 1	$I=2$ DR 2
CCL	-7.13 ± 0.03	-6.75 ± 0.06	1.82 ± 0.01	1.68 ± 0.01
CCL + ρ',ρ''	-7.79 ± 0.03	-7.38 ± 0.06	2.00 ± 0.01	1.84 ± 0.01

Preliminary numbers: $\pi\pi$ rescattering for S-waves

- $a_\mu^{\pi\pi}$ in units of 10^{-11}

phase shift δ_1^1	$I=0$ DR 1	$I=0$ DR 2	$I=2$ DR 1	$I=2$ DR 2
CCL	-7.13 ± 0.03	-6.75 ± 0.06	1.82 ± 0.01	1.68 ± 0.01
CCL + ρ',ρ''	-7.79 ± 0.03	-7.38 ± 0.06	2.00 ± 0.01	1.84 ± 0.01

- Adding the FsQED contribution

phase shift δ_1^1	FsQED	sum DR 1	sum DR 2
CCL	-13.77 ± 0.01	-19.08 ± 0.03	-18.84 ± 0.06
CCL + ρ',ρ''	-14.65 ± 0.01	-20.44 ± 0.03	-20.19 ± 0.06

Preliminary numbers: $\pi\pi$ rescattering for S-waves

- $a_\mu^{\pi\pi}$ in units of 10^{-11}

phase shift δ_1^1	$I=0$ DR 1	$I=0$ DR 2	$I=2$ DR 1	$I=2$ DR 2
CCL	-7.13 ± 0.03	-6.75 ± 0.06	1.82 ± 0.01	1.68 ± 0.01
CCL + ρ',ρ''	-7.79 ± 0.03	-7.38 ± 0.06	2.00 ± 0.01	1.84 ± 0.01

- Adding the FsQED contribution

phase shift δ_1^1	FsQED	sum DR 1	sum DR 2
CCL	-13.77 ± 0.01	-19.08 ± 0.03	-18.84 ± 0.06
CCL + ρ',ρ''	-14.65 ± 0.01	-20.44 ± 0.03	-20.19 ± 0.06

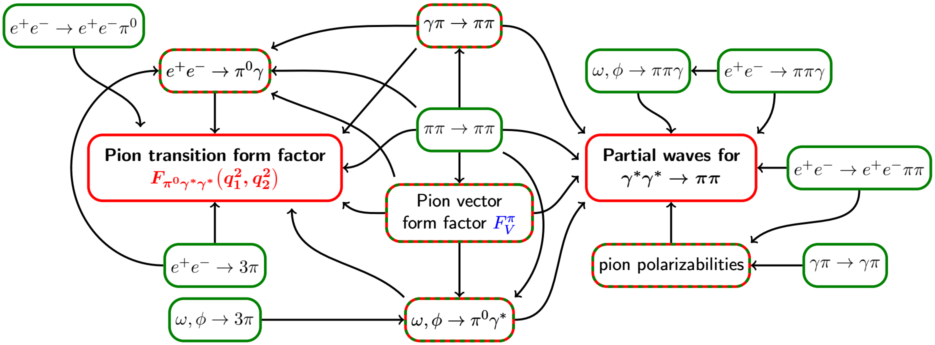
- Comparing to the literature Jegerlehner, Nyffeler 2009

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13
π, K loops + other subleading in N_c	–	–	–	0 ± 10	–	–	–
Axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5
Scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2
Quark loops	21 ± 3	9.7 ± 11.1	–	–	–	$2.3\pm$	21 ± 3
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39

Summary

- **Dispersive framework** for the calculation of the HLbL contribution to a_μ
- Includes **one- and two-pion** intermediate states
- **Master formula** for $\pi\pi$ in terms of $\gamma^*\gamma^* \to \pi\pi$ **partial waves**
- Next steps
  - Pion transition form factor
  - Refinement of $\gamma^*\gamma^* \to \pi\pi$ input
  - Comprehensive treatment of *D*-waves
  - Error analysis: which input quantity has the biggest impact on a_μ?

Outlook: towards a data-driven analysis of HLbL

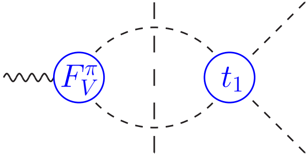


- Reconstruction of $\gamma^*\gamma^* \to \pi\pi, \pi^0$: combine experiment and theory constraints
  $\hookrightarrow$ simplified version for $\pi\pi$ in this talk
- Beyond: η, η', $K\bar{K}$, multi-pion channels (resonances), pQCD constraints, …

Pion vector form factor

- **Dispersive approach**: resum $\pi\pi$ rescattering F_V^π as example
- **Unitarity** for **pion vector form factor**

$$\mathrm{Im}\, F_V^\pi(s) = \theta\big(s-4M_\pi^2\big)\, F_V^\pi(s)\, e^{-i\delta_1(s)} \sin\delta_1(s)$$

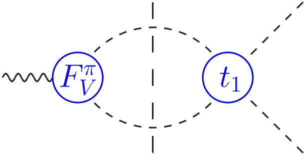


$\hookrightarrow$ **final-state theorem**: phase of F_V^π equals $\pi\pi$ P-wave phase δ_1 Watson 1954

Pion vector form factor

- **Dispersive approach**: resum $\pi\pi$ rescattering F_V^π as example
- **Unitarity** for **pion vector form factor**

$$\operatorname{Im} F_V^\pi(s) = \theta\big(s - 4M_\pi^2\big) F_V^\pi(s) e^{-i\delta_1(s)} \sin\delta_1(s)$$



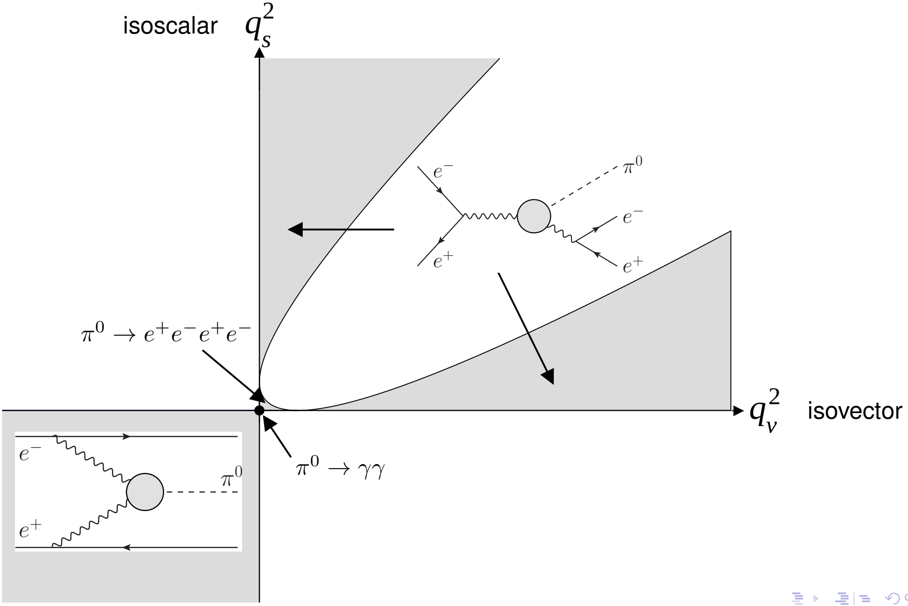
$\hookrightarrow$ **final-state theorem**: phase of F_V^π equals $\pi\pi$ P-wave phase δ_1 Watson 1954

- Solution in terms of **Omnès function** Omnès 1958

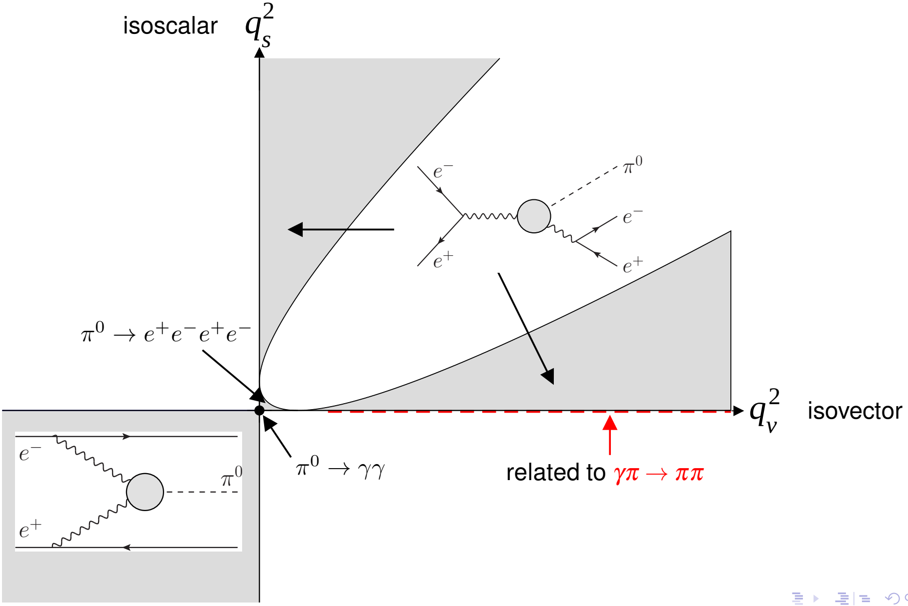
$$F_V^\pi(s) = P(s)\Omega_1(s) \qquad \Omega_1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_1(s')}{s'(s'-s)}\right\}$$

- Asymptotics + normalization $\Rightarrow$ $P(s) = 1$

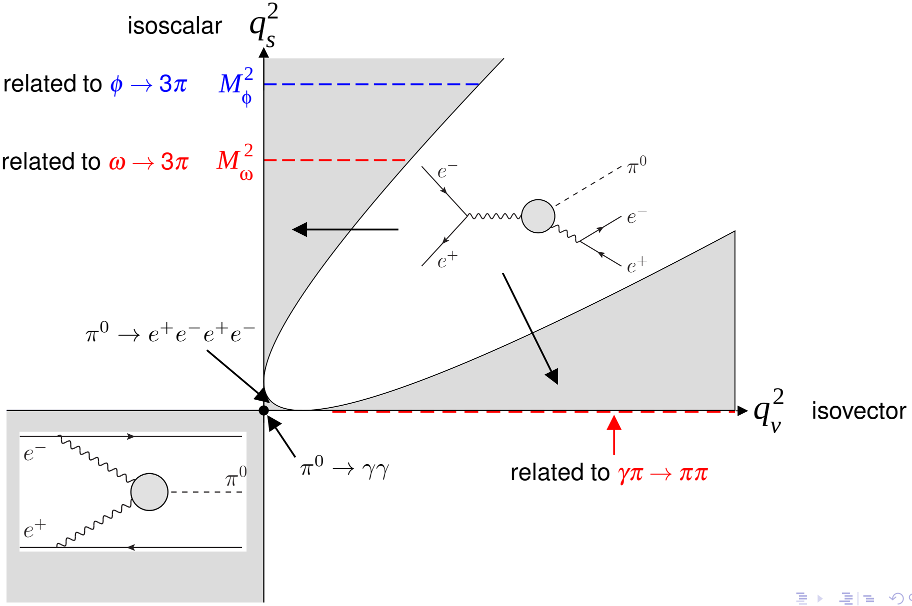
Pion transition form factor: physical regions



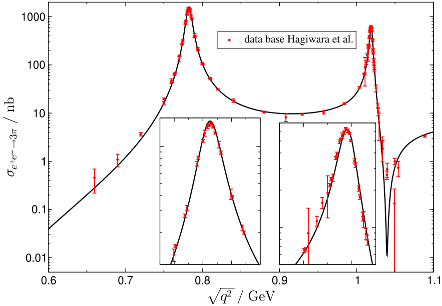
Pion transition form factor: physical regions

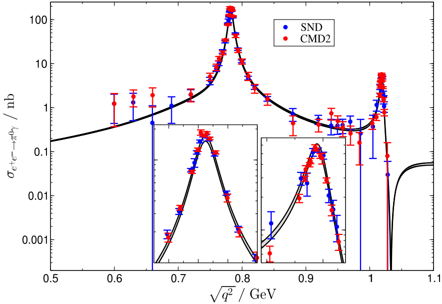


Pion transition form factor: physical regions



Predicting $\sigma(e^+e^- \to \pi^0\gamma)$ from $\sigma(e^+e^- \to 3\pi)$





- fit isoscalar q^2 dependence to

  $\sigma(e^+e^- \to 3\pi)$:

  $$a(q^2) = \frac{F_{3\pi}}{3} + \beta q^2 + \frac{q^4}{\pi}\int\limits_{q^2_{\text{thr}}}^{\infty} \text{d}s \frac{\text{Im}\, a(s)}{s^2(s-q^2)}$$

- $e^+e^- \to \pi^0\gamma$: both isoscalar and isovector
- $\sigma(e^+e^- \to \pi^0\gamma)$ reproduced ✓
- Next steps:
  - analytic continuation into space-like region
  - generalize to **doubly-virtual case** using $e^+e^- \to \pi^0\gamma$ as input

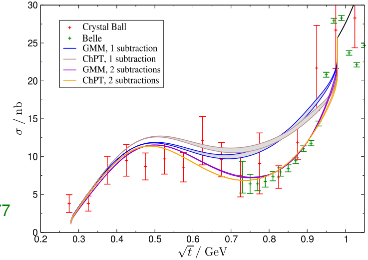
MH, Kubis, Leupold, Niecknig, Schneider (in preparation)

$\gamma^*\gamma^* \to \pi\pi$ partial waves

Roy(–Steiner) equations $=$ Dispersion relations $+$ partial-wave expansion $+$ crossing symmetry $+$ unitarity $+$ gauge invariance

- **On-shell case** $\gamma\gamma \to \pi\pi$ Moussallam 2010, MH, Phillips, Schat 2011 $\hookrightarrow$ precision determination of $\sigma \to \gamma\gamma$ coupling
- **Singly-virtual** $\gamma^*\gamma \to \pi\pi$ Moussallam 2013
- **Doubly-virtual** $\gamma^*\gamma^* \to \pi\pi$: **anomalous thresholds**

  Colangelo, MH, Procura, Stoffer arXiv:1309.6877



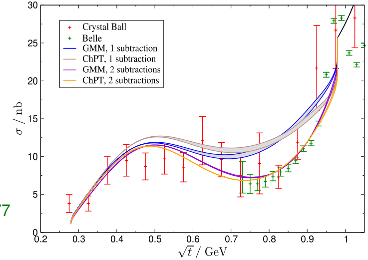
$\gamma^*\gamma^* \to \pi\pi$ partial waves

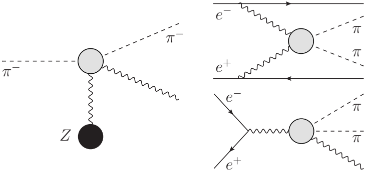
Roy(–Steiner) equations = Dispersion relations + partial-wave expansion

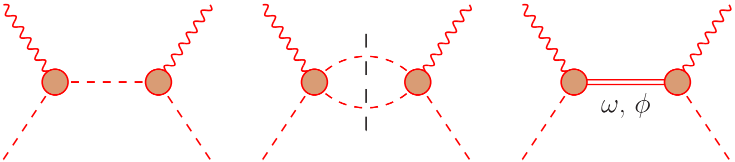
+ crossing symmetry + unitarity + gauge invariance

- **On-shell case** $\gamma\gamma \to \pi\pi$ Moussallam 2010, MH, Phillips, Schat 2011 $\hookrightarrow$ precision determination of $\sigma \to \gamma\gamma$ coupling
- **Singly-virtual** $\gamma^*\gamma \to \pi\pi$ Moussallam 2013
- **Doubly-virtual** $\gamma^*\gamma^* \to \pi\pi$: **anomalous thresholds** Colangelo, MH, Procura, Stoffer arXiv:1309.6877
- Constraints
  - **Low energies**: pion polarizabilities, ChPT
  - **Primakoff**: $\gamma\pi \to \gamma\pi$ (COMPASS), $\gamma\gamma \to \pi\pi$ (JLab)
  - **Scattering**: $e^+e^- \to e^+e^-\pi\pi$, $e^+e^- \to \pi\pi\gamma$
  - **(Transition) Form factors**: F_V^π, $\omega, \phi \to \pi^0\gamma^*$

$\hookrightarrow$ discuss these constraints in the following

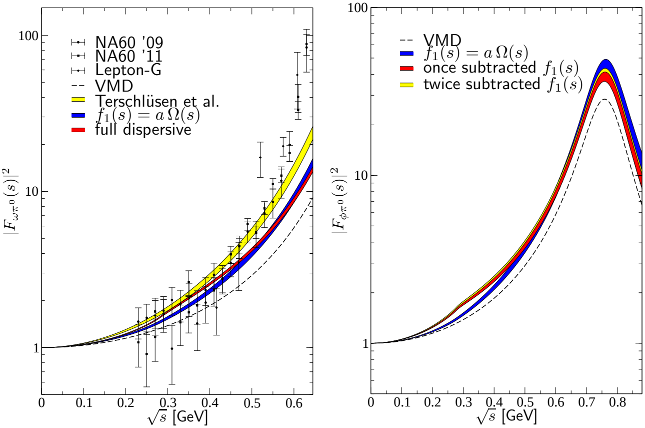






- **Pion pole**: coupling determined by F_V^π as before
- **Multi-pion intermediate states**: approximate in terms of **resonances**
  - $2\pi \sim \rho$: can even be done **exactly** using $\gamma^* \to 3\pi$ amplitude
    $\hookrightarrow$ see pion transition form factor
  - $3\pi \sim \omega, \phi$: narrow-width approximation
    $\hookrightarrow$ **transition form factors** for $\omega, \phi \to \pi^0\gamma^*$
  - Higher intermediate states also potentially relevant: **axials**, **tensors**
    $\hookrightarrow$ **sum rules** to constrain their transition form factors Pauk, Vanderhaeghen 2014

$\omega, \phi \to \pi^0 \gamma^*$ transition form factor



Schneider, Kubis, Niecknig 2012

- Puzzle of steep rise in $F_{\omega\pi^0}$

  $\hookrightarrow$ measurement of $F_{\phi\pi^0}$ would be extremely valuable

- Clarification important for pion transition form factor, but also $\gamma^*\gamma^* \to \pi\pi$

Subtraction functions

Omnès representation for S-wave

$$h_{0,++}(s) = \Delta_{0,++}(s) + \Omega_0(s)\bigg[\frac{1}{2}(s-s_+)a_+(q_1^2,q_2^2) + \frac{1}{2}(s-s_-)a_-(q_1^2,q_2^2) + q_1^2q_2^2 b(q_1^2,q_2^2)$$

$$+\frac{s(s-s_+)}{2\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,++}(s')}{s'(s'-s_+)(s'-s)|\Omega_0(s')|} + \frac{s(s-s_-)}{2\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,++}(s')}{s'(s'-s_-)(s'-s)|\Omega_0(s')|}$$

$$+\frac{2q_1^2q_2^2 s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\sin\delta_0(s')\Delta_{0,00}(s')}{s'(s'-s_+)(s'-s_-)|\Omega_0(s')|}\bigg] \qquad s_\pm = q_1^2+q_2^2 \pm 2\sqrt{q_1^2q_2^2}$$

- Inhomogeneities $\Delta_{0,++}(s), \Delta_{0,00}(s)$ include left-hand cut
- **Subtraction functions**
  - $b(q_1^2,q_2^2)$ and $a_+(q_1^2,q_2^2) - a_-(q_1^2,q_2^2)$ multiply $q_1^2q_2^2$ and $\sqrt{q_1^2q_2^2}$
    $\hookrightarrow$ inherently doubly-virtual observables $\Rightarrow$ need ChPT (or lattice)
  - However: $a(q_1^2,q_2^2) = (a_+(q_1^2,q_2^2) + a_-(q_1^2,q_2^2))/2$ fixed by singly-virtual measurements
    $\hookrightarrow$ compare with chiral prediction, uncertainty estimates for the other functions

- 1-loop result for arbitrary q_i^2, e.g.

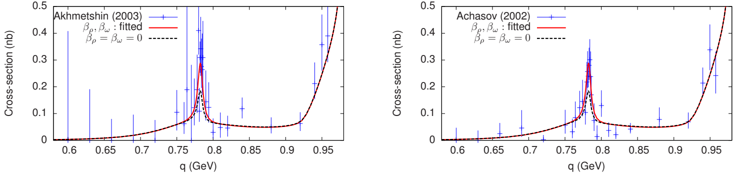
$$a^{\pi^0}(q_1^2,q_2^2) = -\frac{M_\pi^2}{8\pi^2 F_\pi^2 (q_1^2-q_2^2)^2}\bigg\{q_1^2+q_2^2+2\Big(M_\pi^2(q_1^2+q_2^2)+q_1^2q_2^2\Big)C_0(q_1^2,q_2^2)$$
$$+q_1^2\bigg(1+\frac{6q_2^2}{q_1^2-q_2^2}\bigg)\bar{J}(q_1^2)+q_2^2\bigg(1-\frac{6q_1^2}{q_1^2-q_2^2}\bigg)\bar{J}(q_2^2)\bigg\}$$

- Special case: $q_1^2 = q_2^2 = 0$

$$a^{\pi^\pm}(0,0) = \frac{\bar{l}_6-\bar{l}_5}{48\pi^2 F_\pi^2}+\cdots = \frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{\pi^\pm} \qquad b^{\pi^\pm}(0,0)=0$$
$$a^{\pi^0}(0,0) = -\frac{1}{96\pi^2 F_\pi^2}+\cdots = \frac{M_\pi}{2\alpha}(\alpha_1-\beta_1)^{\pi^0} \qquad b^{\pi^0}(0,0) = -\frac{1}{1440\pi^2 F_\pi^2 M_\pi^2}+\cdots$$

$\hookrightarrow$ resum higher chiral orders into **pion polarizabilities**

Subtraction functions: dispersive representation



Moussallam 2013

- Singly-virtual case: phenomenological representation with chiral constraints

  $\hookrightarrow$ parameters fixed from $e^+e^- \to \pi^0\pi^0\gamma$ (CMD2 and SND) Moussallam 2013

- **Dispersive representation**: imaginary part from $2\pi, 3\pi, \ldots$

  $\hookrightarrow$ analytic continuation from time-like to space-like kinematics

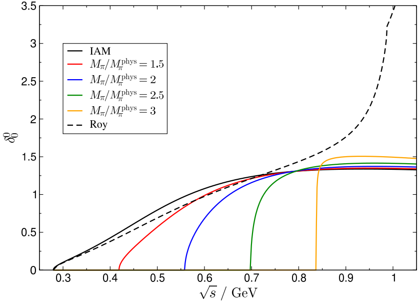
- Example: $I = 2 \Rightarrow$ isovector photons $\Rightarrow 2\pi \sim \rho$

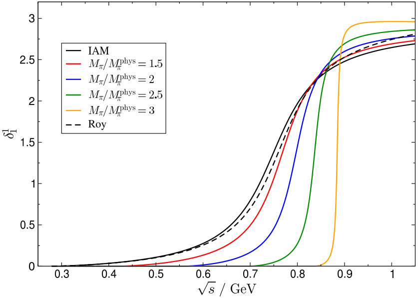
$$a^2(q_1^2, q_2^2) = \alpha_0 \Big[\alpha^2 + \alpha\Big(q_1^2 \mathscr{F}^\rho(q_1^2) + q_2^2 \mathscr{F}^\rho(q_2^2)\Big) + q_1^2 q_2^2 \mathscr{F}^\rho(q_1^2)\mathscr{F}^\rho(q_2^2)\Big]$$

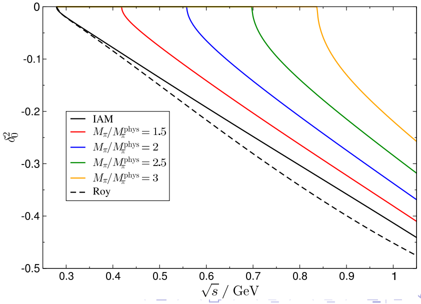
$$\mathscr{F}^\rho(q^2) = \frac{1}{\pi}\int\limits_{4M_\pi^2}^{\infty} \mathrm{d}s \frac{q_{\pi\pi}^3(s)\big(F_\pi^V(s)\big)^* \Omega_1(s)}{s^{3/2}(s-q^2)} \qquad q_{\pi\pi}(s) = \sqrt{\frac{s}{4} - M_\pi^2}$$

$\hookrightarrow$ α_0 and α can be determined from $a^2(q^2, 0)$ alone!

- **1-loop IAM** with low-energy constants from Hanhart, Peláez, Ríos 2008
- Quark-mass dependence of the **phase shifts**



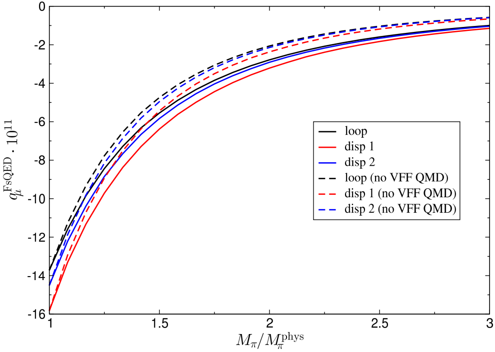




Quark-mass dependence: FsQED

- Check accuracy of IAM

phase shift δ_1^1	loop integrals	DR 1	DR 2
CCL	-13.77 ± 0.01	-15.87 ± 0.01	-14.57 ± 0.01
CCL + ρ', ρ''	-14.65 ± 0.01	-16.90 ± 0.02	-15.53 ± 0.01
IAM	-13.72 ± 0.01	-15.82 ± 0.01	-14.51 ± 0.01



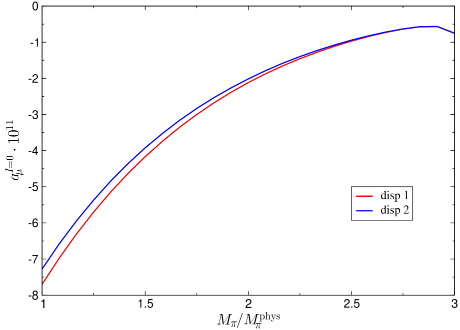
- Quark-mass dependence of $F_\pi^V(s)$ via phase shift in Omnès representation
  Guo, Hanhart, Llanes-Estrada, Meißner 2009
  $\hookrightarrow$ relevant for large M_π
- Quark-mass dependence of a_μ^{FsQED} roughly $\propto M_\pi^{-2}$

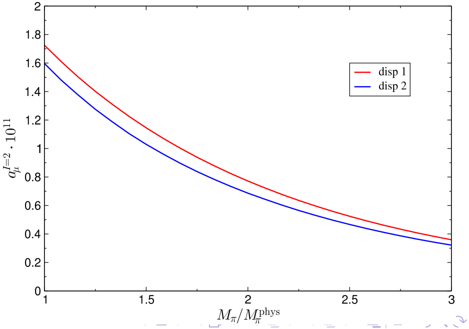
Quark-mass dependence: S-waves

	DR 1	DR 1 + IAM	DR 2	DR 2 + IAM
$I=0$	-7.13 ± 0.03	-7.70 ± 0.04	-6.75 ± 0.06	-7.28 ± 0.06
$I=2$	1.82 ± 0.01	1.73 ± 0.01	1.68 ± 0.01	1.60 ± 0.01

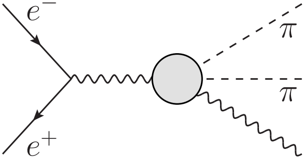
- Choose $\sqrt{s_\text{m}}=2M_K=2\sqrt{(M_K^\text{phys})^2+(M_\pi^2-(M_\pi^\text{phys})^2)/2}$, compare to CCL δ_1^1
- Beyond $3M_\pi$: σ and ρ become **bound states**

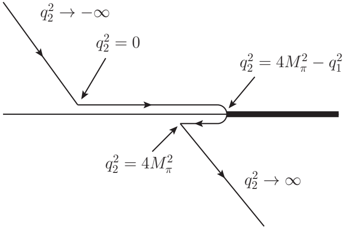
$\hookrightarrow$ highly non-trivial quark-mass dependence





Anomalous thresholds



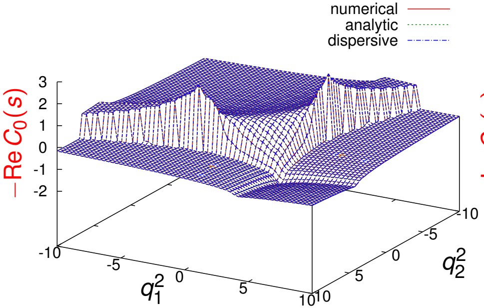


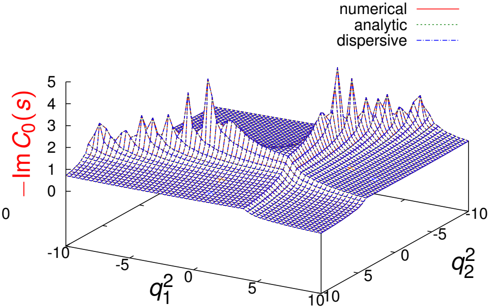
- **Analytic continuation** in q_i^2 in time-like region non-trivial in doubly-virtual case
- Singularities from second sheet move onto first one

  $\hookrightarrow$ need to **deform** the **integration contour**
- Problem already occurs for a simple triangle loop function $C_0(s)$

  $\hookrightarrow$ extra factor $t_\ell(s)/\Omega_\ell(s)$ is well defined in the whole complex plane

  $\hookrightarrow$ remedy in case of $C_0(s)$ can be taken over to full Omnès solution
- Becomes relevant for $e^+e^+ \to e^+e^-\pi\pi$ in time-like kinematics

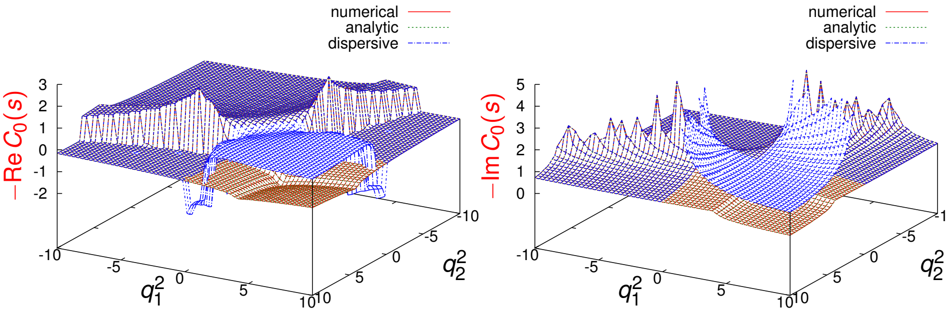




- Comparison for $s = 5$, $M_\pi = 1$

  $\hookrightarrow$ **dispersive reconstruction** of $C_0(s)$ works!

Numerical check of anomalous thresholds



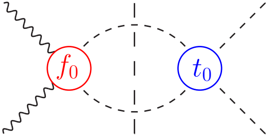
- Ignore anomalous piece

  $\hookrightarrow$ substantial deviations for **large virtualities**!

$\gamma^*\gamma^* \to \pi\pi$

- **Left-hand cut** approximated by **pion pole** + **resonances**
- **Unitarity** for $\gamma^*\gamma^* \to \pi\pi$ system: Watson's theorem

$$\mathrm{disc}\, f_0(s; q_1^2, q_2^2) = 2i\sigma_s f_0(s; q_1^2, q_2^2)\, t_0^*(s)$$



$$t_0(s) = \frac{1}{\sigma_s} e^{i\delta_0(s)} \sin\delta_0(s) \qquad \sigma_s = \sqrt{1 - \frac{4M_\pi^2}{s}}$$

$\hookrightarrow$ solution in terms of **Omnès function**, e.g. for pion pole only

$$f_0(s; q_1^2, q_2^2) = \frac{\Omega_0(s)}{\pi} \int\limits_{4M_\pi^2}^{\infty} \mathrm{d}s' \frac{N_0(s'; q_1^2, q_2^2) \sin\delta_0(s')}{(s'-s)|\Omega_0(s')|} \qquad N_0(s; q_1^2, q_2^2) = \frac{2L}{\sigma_s \sqrt{\lambda(s, q_1^2, q_2^2)}}$$

$$L = \log \frac{s - q_1^2 - q_2^2 + \sigma_s \sqrt{\lambda(s, q_1^2, q_2^2)}}{s - q_1^2 - q_2^2 - \sigma_s \sqrt{\lambda(s, q_1^2, q_2^2)}} \qquad \lambda(x,y,z) = x^2 + y^2 + z^2 - 2(xy + xz + yz)$$

- **Analytic continuation** in q_i^2?

$\gamma^*\gamma^* \to \pi\pi$: analytic continuation

$$L = \log \frac{s-q_1^2-q_2^2+\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}}{s-q_1^2-q_2^2-\sigma_s\sqrt{\lambda(s,q_1^2,q_2^2)}} \xrightarrow{q_2^2\to 0} \pm\log\frac{1+\sigma_s}{1-\sigma_s}$$

- Singularities of the log: **anomalous thresholds**

$$s_\pm = q_1^2+q_2^2-\frac{q_1^2q_2^2}{2M_\pi^2}\pm\frac{1}{2M_\pi^2}\sqrt{q_1^2(q_1^2-4M_\pi^2)q_2^2(q_2^2-4M_\pi^2)}$$

$\hookrightarrow$ usual Omnès derivation breaks down

- Idea: consider first the **scalar loop function**

$$C_0(s) \equiv C_0\big((q_1+q_2)^2;q_1^2,q_2^2\big) = \frac{1}{i\pi^2}\int\frac{\mathrm{d}^4k}{(k^2-M_\pi^2)\big((k+q_1)^2-M_\pi^2\big)\big((k-q_2)^2-M_\pi^2\big)}$$

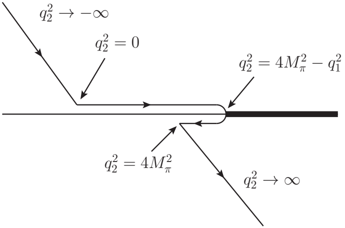
$$\operatorname{disc} C_0(s) = -\frac{2\pi i}{\sqrt{\lambda(s,q_1^2,q_2^2)}}L = -\pi i\sigma_s N_0(s;q_1^2,q_2^2)$$

$\gamma^*\gamma^* \to \pi\pi$: anomalous thresholds

$$s_+ = q_1^2 + q_2^2 - \frac{q_1^2 q_2^2}{2M_\pi^2} + \frac{1}{2M_\pi^2}\sqrt{q_1^2(q_1^2 - 4M_\pi^2)q_2^2(q_2^2 - 4M_\pi^2)}$$

- **Anomalous threshold** usually on the **second sheet**
- Trajectory of $s_+(q_2^2)$ for $0 \leq q_1^2 \leq 4M_\pi^2$

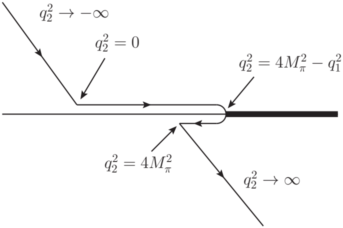
  $\hookrightarrow$ moves through unitarity cut onto first sheet



$$s_+ = q_1^2 + q_2^2 - \frac{q_1^2 q_2^2}{2M_\pi^2} + \frac{1}{2M_\pi^2}\sqrt{q_1^2\left(q_1^2 - 4M_\pi^2\right) q_2^2 \left(q_2^2 - 4M_\pi^2\right)}$$

- **Anomalous threshold** usually on the **second sheet**
- Trajectory of $s_+(q_2^2)$ for $0 \leq q_1^2 \leq 4M_\pi^2$

  $\hookrightarrow$ moves through unitarity cut onto first sheet
- Need to deform the contour



$$C_0(s) = \frac{1}{2\pi i}\int\limits_{4M_\pi^2}^{\infty} ds' \frac{\text{disc}\, C_0(s')}{s'-s}$$

$$+\,\theta\left(q_1^2 + q_2^2 - 4M_\pi^2\right)\frac{1}{2\pi i}\int\limits_0^1 dx \frac{\partial s_x}{\partial x}\frac{\text{disc}_{\text{an}}\, C_0(s_x)}{s_x - s}$$

$$s_x = x4M_\pi^2 + (1-x)s_+ \qquad \text{disc}_{\text{an}}\, C_0(s) = \frac{4\pi^2}{\sqrt{\lambda\left(s, q_1^2, q_2^2\right)}}$$

$\gamma^*\gamma^* \to \pi\pi$: back to the Omnès representation

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi}\int\limits_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|}$$

- Integrand similar to the scalar-loop example

$$\frac{N_0(s;q_1^2,q_2^2)\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\, C_0(s)}{\pi i \sigma_s}\frac{\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\, C_0(s)}{\pi i}\frac{t_0(s)}{\Omega_0(s)}$$

$\hookrightarrow$ additional factor **independent of q_i^2** and **well-defined in the whole s-plane**

$\gamma^*\gamma^* \to \pi\pi$: back to the Omnès representation

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi} \int\limits_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|}$$

- Integrand similar to the scalar-loop example

$$\frac{N_0(s;q_1^2,q_2^2)\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\, C_0(s)}{\pi i \sigma_s}\frac{\sin\delta_0(s)}{|\Omega_0(s)|} = -\frac{\text{disc}\, C_0(s)}{\pi i}\frac{t_0(s)}{\Omega_0(s)}$$

$\hookrightarrow$ additional factor **independent of q_i^2** and **well-defined in the whole s-plane**

Omnès representation for $\gamma^*\gamma^* \to \pi\pi$

$$f_0(s;q_1^2,q_2^2) = \frac{\Omega_0(s)}{\pi} \int\limits_{4M_\pi^2}^{\infty} ds' \frac{N_0(s';q_1^2,q_2^2)\sin\delta_0(s')}{(s'-s)|\Omega_0(s')|}$$

$$+\theta\left(q_1^2+q_2^2-4M_\pi^2\right)\frac{\Omega_0(s)}{2\pi i}\int\limits_0^1 dx \frac{\partial s_x}{\partial x}\frac{\text{disc}_{\text{an}}\, f_0(s_x;q_1^2,q_2^2)}{s_x-s}$$

$$s_x = x4M_\pi^2+(1-x)s_+ \qquad \text{disc}_{\text{an}}\, f_0(s;q_1^2,q_2^2) = -\frac{8\pi}{\sqrt{\lambda(s,q_1^2,q_2^2)}}\frac{t_0(s)}{\Omega_0(s)}$$