Derivation of invariant Gibbs measures for nonlinear Schrödinger equations from many-body quantum states

Benjamin Schlein, University of Zurich

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Joint work with Jürg Fröhlich, Antti Knowles, Vedran Sohinger

I. Hartree theory

Energy: the **Hartree functional** is given by

$$\mathcal{E}_{\mathsf{H}}(\phi) = \int \left[|\nabla \phi(x)|^2 + v(x)|\phi(x)|^2 \right] dx + \frac{1}{2} \int w(x-y)|\phi(x)|^2 |\phi(y)|^2 \, dx dy$$

and acts on $L^2(\mathbb{R}^d)$ (we will consider $d = 1, 2, 3$).

We assume v is **confining** and $w \in L^\infty(\mathbb{R}^d)$ of **positive type**.

Evolution: the **time-dependent Hartree equation** is given by

$$i\partial_t \phi_t = [-\Delta + v(x)]\, \phi_t + (w * |\phi_t|^2)\phi_t$$

Invariant measures: it is interesting to construct the **probability measure**, formally given by

$$\frac{1}{Z} e^{-\left[\mathcal{E}_{\mathsf{H}}(\phi) + \kappa \|\phi\|_2^2\right]} \, d\phi$$

Free invariant measure: let $h = -\Delta + v(x) + \kappa$ and

$$\mathcal{E}_0(\phi) = \langle \phi, h\phi \rangle = \int \left[|\nabla \phi(x)|^2 + v(x)|\phi(x)|^2 + \kappa |\phi(x)|^2 \right] dx$$

Since v confining, h has **pure point spectrum**, i.e.

$$h = \sum_{n \in \mathbb{N}} \lambda_n |u_n\rangle \langle u_n|$$

We will assume v to be s.t.

$$\begin{cases} \mathrm{Tr}\, h^{-1} = \sum_{n \in \mathbb{N}} \lambda_n^{-1} < \infty & \text{for } d = 1 \\ \mathrm{Tr}\, h^{-2} = \sum_{n \in \mathbb{N}} \lambda_n^{-2} < \infty & \text{for } d = 2, 3 \end{cases}$$

Remark 1: the bound $\mathrm{Tr}\, h^{-1} < \infty$ in $d = 2, 3$ cannot hold, because of **lack of decay in momentum**.

Remark 2: for $d = 1, 2$, harmonic oscillator is just not confining enough.

To make sense of $\mu_0 \sim \exp(-\mathcal{E}_0(\phi))d\phi$, we **expand**

$$\phi(x) = \sum_{n\in\mathbb{N}} \frac{\omega_n}{\sqrt{\lambda_n}} u_n(x)$$

Since

$$\mathcal{E}_0(\phi) = \langle \phi, h\phi \rangle = \sum_{n\in\mathbb{N}} |\omega_n|^2$$

we define probability measure μ_0 on set $\mathbb{C}^{\mathbb{N}} = \{\{\omega_n\}_{n\in\mathbb{N}} : \omega_n \in \mathbb{C}\}$, as the **product of Gaussian measures**, each having the density

$$\frac{1}{\pi} e^{-|\omega_n|^2} d\omega_n d\omega_n^*$$

Expected L^2 norm: observe that

$$\mathbb{E}_{\mu_0} \|\phi\|_2^2 = \mathbb{E}_{\mu_0} \sum_{n\in\mathbb{N}} \frac{|\omega_n|^2}{\lambda_n} = \sum_{n\in\mathbb{N}} \frac{1}{\lambda_n} = \mathrm{Tr}\, h^{-1}$$

is finite for $d = 1$, but is **infinite** in $d = 2, 3$.

Expected H^s norm: for $s \in \mathbb{R}$, we have

$$\mathbb{E}_{\mu_0}\|\phi\|^2_{H^s} = \mathbb{E}_{\mu_0}\langle \phi, h^s \phi\rangle = \operatorname{Tr} h^{-1+s}$$

The assumption $\operatorname{Tr} h^{-2} < \infty$ implies that $\mathbb{E}_{\mu_0}\|\phi\|^2_{H^{-1}} < \infty$.

We **conclude**

$$\mu_0(H^{-1}(\mathbb{R}^d)) = 1, \quad \text{while} \quad \mu_0(L^2(\mathbb{R}^d)) = 0$$

for $d = 2, 3$.

Example: if $h = -\Delta + \kappa$ on $\mathbb{T}^d$, we find

$$h e^{ip\cdot x} = (p^2 + \kappa) e^{ip\cdot x} \quad \text{for all } p \in 2\pi\mathbb{Z}^d$$

Hence

$$\mathbb{E}_{\mu_0}\|\phi\|^2_{H^s} = \sum_{p\in 2\pi\mathbb{Z}^d} \frac{1}{(p^2+\kappa)^{1-s}} < \infty$$

iff $s < 1 - d/2$.

Hartree invariant measure: try to define μ_H as **absolutely continuous** probability measure w.r.t. μ_0 with density $Z^{-1}e^{-W(\phi)}$, where

$$W(\phi) = \frac{1}{2}\int w(x-y)|\phi(x)|^2|\phi(y)|^2 dxdy$$

In other words, try to define μ_H such that

$$\mathbb{E}_H f(\phi) = \frac{\int f(\phi)\, e^{-W(\phi)}\, d\mu_0(\phi)}{\int e^{-W(\phi)}\, d\mu_0(\phi)}$$

For $d = 1$, ϕ is typically L^2, and

$$W(\phi) \leq \frac{\|w\|_\infty}{2}\|\phi\|_2^4 < \infty$$

Hence μ_H **well-defined**.

For $d = 2, 3$, on the other hand, typical ϕ is not L^2 and $W = \infty$.

Wick ordering: for $K > 0$ we introduce **cutoff fields**

$$\phi_K(x) = \sum_{n \le K} \frac{\omega_n}{\sqrt{\lambda_n}} u_n(x)$$

We define

$$\rho_K(x) = \mathbb{E}_{\mu_0} |\phi_K(x)|^2 = \sum_{n \le K} \lambda_n^{-1} \, |u_n(x)|^2$$

and the **cutoff renormalized interaction**

$$W_K = \frac{1}{2} \int w(x-y) \left[|\phi_K(x)|^2 - \rho_K(x) \right] \left[|\phi_K(y)|^2 - \rho_K(y) \right] dx dy$$

Lemma: W_K is **Cauchy sequence** in $L^p(\mathbb{C}^{\mathbb{N}}, d\mu_0)$ for all $p < \infty$. We denote by W^r its limit (independent of p).

For $d = 2, 3$, we define μ_H^r through

$$\mathbb{E}_H^r \, f(\phi) = \frac{\int f(\phi) \, e^{-W^r(\phi)} d\mu_0(\phi)}{\int e^{-W^r(\phi)} d\mu_0(\phi)}$$

One can check μ_H^r is **invariant** with respect to the Hartree flow.

II. Quantum systems in mean field regime

N **particle systems**: described on **Hilbert space** $L_s^2(\mathbb{R}^{dN})$.

Mean-field Hamilton operator: has the form

$$H_N = \sum_{j=1}^{N} \left[-\Delta_{x_j} + v(x_j)\right] + \frac{1}{N}\sum_{i<j}^{N} w(x_i - x_j)$$

Coupling constant N^{-1} characterizes **mean field** regime.

Ground state: exhibits **condensation**, $\psi_N \simeq \phi^{\otimes N}$.

Energy of condensate given by

$$\langle \phi^{\otimes N}, H_N\, \phi^{\otimes N}\rangle = N\mathcal{E}_{\mathsf{H}}(\phi)$$

Hence: ground state condensates in **minimizer** of $\mathcal{E}_H$ and

$$\lim_{N\to\infty} \frac{E_N}{N} = \min_{\varphi\in L^2(\mathbb{R}^d):\|\varphi\|_2=1} \mathcal{E}_{\mathsf{H}}(\varphi)$$

Dynamics: governed by the many-body **Schrödinger** equation

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t}$$

Convergence to Hartree: if initial data $\psi_{N,0} \simeq \phi^{\otimes N}$, then

$$\psi_{N,t} \simeq \phi_t^{\otimes N}$$

where ϕ_t solves the time-dependent **Hartree equation**

$$i\partial_t \phi_t = [-\Delta + v]\,\phi_t + (w * |\phi_t|^2)\phi_t$$

with initial data $\phi_{t=0} = \phi$.

Rigorous works: Hepp, Ginibre-Velo, Spohn, Erdős-Yau, Bardos-Golse-Mauser, Fröhlich-Knowles-Schwarz, Rodnianski-S., Knowles-Pickl, Fröhlich-Knowles-Pizzo, Grillakis-Machedon-Margetis, T.Chen-Pavlovic, X.Chen-Holmer, Ammari-Nier, Lewin-Nam-S., . . .

Question: what corresponds to Hartree invariant measures in many-body setting?

Thermal equilibrium: at **temperature** β^{-1}, it is described by Gibbs state

$$\varrho_\beta = \frac{1}{Z_\beta} e^{-\beta H_N}$$

Remark 1: if $\beta > 0$ fixed, ϱ_β still exhibits **condensation**. At one-particle level this leads to **trivial measure** δ_{ϕ_0}.

To recover invariant measure, need to take $\beta = 1/N$.

Remark 2: number of particles at many-body level corresponds to L^2-norm at Hartree level.

To recover invariant measure, need to allow **fluctuations** of number of particles.

III. Fock space and grand canonical ensemble

Fock space: we define

$$\mathcal{F} = \bigoplus_{m \geq 0} L^2(\mathbb{R}^d)^{\otimes_s m} = \bigoplus_{m \geq 0} L^2_s(\mathbb{R}^{md})$$

Creation and annihilation operators: for $f \in L^2(\mathbb{R}^d)$, let

$$(a^*(f)\Psi)^{(m)}(x_1, \dots, x_m) = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} f(x_j) \Psi^{(m-1)}(x_1, \dots, \cancel{x_j}, \dots, x_m)$$

$$(a(f)\Psi)^{(m)}(x_1, \dots, x_m) = \sqrt{m+1} \int dx \, \overline{f(x)} \, \Psi^{(m+1)}(x, x_1, \dots, x_m)$$

They satisfy **canonical commutation relations**

$$[a(f), a^*(g)] = \langle f, g \rangle, \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0$$

We define operator valued **distributions** $a(x), a^*(x)$ such that

$$a^*(f) = \int f(x)\, a^*(x)\, dx, \quad \text{and } a(f) = \int \overline{f(x)}\, a(x)\, dx$$

Number of particles operator: is given by

$$\mathcal{N} = \int a^*(x) a(x)\, dx$$

Hamilton operator: is defined through

$$\mathcal{H}_N = \int a^*(x)\, [-\Delta_x + v(x)]\, a(x) + \frac{1}{2N} \int w(x-y) a^*(x) a^*(y) a(y) a(x)$$

Notice that $[\mathcal{H}_N, \mathcal{N}] = 0$ and

$$\mathcal{H}_N|_{\mathcal{F}_m} = \sum_{j=1}^{m} \left[-\Delta_{x_j} + v(x_j)\right] + \frac{1}{N} \sum_{i<j}^{m} w(x_i - x_j)$$

Grand canonical ensemble: at inverse **temperature** $\beta = N^{-1}$ and **chemical potential** κ, equilibrium is described by

$$\varrho_N = \frac{1}{Z_N} e^{-\frac{1}{N}(\mathcal{H}_N + \kappa \mathcal{N})}, \qquad \text{with} \qquad Z_N = \mathrm{Tr}\, e^{-\frac{1}{N}(\mathcal{H}_N + \kappa \mathcal{N})}$$

Rescaled operators: it is useful to define

$$a_N(x) = \frac{1}{\sqrt{N}} a(x), \qquad a_N^*(x) = \frac{1}{\sqrt{N}} a^*(x)$$

Then

$$[a_N(x), a_N^*(y)] = \frac{1}{N} \delta(x-y), \qquad [a_N(x), a_N(y)] = [a_N^*(x), a_N^*(y)] = 0$$

are **almost commuting**

Expressed in terms of the **rescaled fields**, we find

$$\begin{aligned} \varrho_N = Z_N^{-1} \exp \Big[- \int a_N^*(x)(-\Delta_x + v(x) + \kappa) a_N(x)\, dx \\ + \frac{1}{2} \int w(x-y)\, a_N^*(x)\, a_N^*(y)\, a_N(y)\, a_N(x)\, dx dy \Big] \end{aligned}$$

IV. Non-interacting Gibbs states and Wick ordering

Non-interacting Gibbs state: we **diagonalize**

$$\int a_N^*(x)\left[-\Delta_{x_j}+v(x_j)+\kappa\right]a_N(x)\,dx=\sum_j \lambda_j a_N^*(u_j)a_N(u_j)$$

which leads to

$$\varrho_N^{(0)}=\frac{1}{Z_N^{(0)}}\,e^{-\sum_j \lambda_j a_N^*(u_j)a_N(u_j)}$$

Expectation of rescaled **number of particles**

$$\begin{aligned}\mathbb{E}_N^{(0)}\,a_N^*(u_i)a_N(u_i)&=\frac{\mathrm{Tr}\,a_N^*(u_i)a_N(u_i)\,e^{-\lambda_i a_N^*(u_i)a_N(u_i)}}{\mathrm{Tr}e^{-\lambda_i a_N^*(u_i)a_N(u_i)}}\\&=\frac{1}{N}\frac{\sum_{n\in\mathbb{N}} ne^{-(\lambda_i/N)n}}{\sum_{n\in\mathbb{N}} e^{-(\lambda_i/N)n}}=\frac{1}{N}\frac{1}{e^{\lambda_i/N}-1}\end{aligned}$$

Hence

$$\mathbb{E}_N^{(0)}\sum_i a_N^*(u_i)a_N(u_i)=\frac{1}{N}\sum_{i\in\mathbb{N}}\frac{1}{e^{\lambda_i/N}-1}=\begin{cases}O(1), & \text{for } d=1\\ \to\infty, & \text{for } d=2,3\end{cases}$$

Interaction: expectation of

$$W_N = \frac{1}{2} \int w(x-y) a_N^*(x) a_N^*(y) a_N(y) a_N(x) dx dy$$

is **finite** but, for $d = 2, 3$, it **diverges**, as $N \to \infty$.

Wick ordering: replace W_N with the Wick ordered interaction

$$W_N^r = \frac{1}{2} \int w(x-y) \left[a_N^*(x) a_N(x) - \rho_N(x) \right] \left[a_N^*(y) a_N(y) - \rho_N(y) \right] dx dy$$

with

$$\rho_N(x) = \mathbb{E}_N^{(0)} \, a_N^*(x) a_N(x) = \frac{1}{N} \sum_{j \in \mathbb{N}} \frac{|u_j(x)|^2}{e^{\lambda_j/N} - 1}$$

We write the resulting **grand canonical state**

$$\varrho_N^r = \frac{1}{Z_N^r} e^{-\mathbb{H}_N} = \frac{1}{Z_N^r} e^{-(\mathbb{H}_{N,0} + W_N^r)}$$

with

$$\mathbb{H}_{N,0} = \int a_N^*(x) \left[-\Delta_x + v(x) + \kappa \right] a_N(x) \, dx$$

V. Comparison with invariant measure for Hartree

Correlation functions: for $k \in \mathbb{N}$, define correlation function $\gamma_N^{(k)}$ as non-negative operators on $L^2(\mathbb{R}^{kd})$ with **kernel**

$$
\begin{aligned}
\gamma_N^{(k)}(x_1, \ldots, x_k; y_1, \ldots, y_k) \\
&= \mathbb{E}_N^r \, a_N^*(x_1) \ldots a_N^*(x_k) a_N(y_k) \ldots a_N(y_1) \\
&= \operatorname{Tr} a_N^*(x_1) \ldots a_N^*(x_k) a_N(y_k) \ldots a_N(y_1) \, \varrho_N^r
\end{aligned}
$$

Joint moments: define $\gamma_H^{(k)}$ of **invariant measure** through

$$
\begin{aligned}
\gamma_H^{(k)}(x_1, \ldots, x_k; y_1, \ldots, y_k) \\
&= \mathbb{E}_H^r \, \overline{\phi}(x_1) \ldots \overline{\phi}(x_k) \phi(y_k) \ldots \phi(y_1) \\
&= \frac{\int \overline{\phi}(x_1) \ldots \overline{\phi}(x_k) \phi(y_k) \ldots \phi(y_1) \, e^{-W^r(\phi)} d\mu_0(\phi)}{\int e^{-W^r(\phi)} d\mu_0(\phi)}
\end{aligned}
$$

Conjecture: we expect that, for all fixed $k \in \mathbb{N}$,

$$\lim_{N \to \infty} \left\| \gamma_N^{(k)} - \gamma_H^{(k)} \right\|_{\mathrm{HS}} = 0$$

Conjecture has been proven by **Lewin-Nam-Rougerie** for $d = 1$ (no Wick ordering).

We are mostly interested in the case $d = 2, 3$.

For technical reasons, we show conjecture for **slightly modified** many-body Gibbs states.

Modification: for fixed $\eta > 0$, we consider the **quantum state**

$$\varrho^r_{N,\eta} = \frac{1}{Z_{N,\eta}} e^{-\eta \mathbb{H}_{N,0}} e^{-[(1-2\eta)\mathbb{H}_{N,0} + W^r_N]} e^{-\eta \mathbb{H}_{N,0}}$$

We denote by $\gamma^{(k)}_{\eta,N}$ the correlation functions associated to $\rho^r_{N,\eta}$.

Remark: $\varrho^r_{N,\eta}$ still density matrix of a mixed quantum state.

Theorem [Fröhlich-Knowles-S.-Sohinger]: let $d = 2, 3$,

$$h = -\Delta + v(x) + \kappa$$

with $\operatorname{Tr} h^{-2} < \infty$, $w \in L^\infty(\mathbb{R}^d)$ positive definite. Then, for all fixed $\eta > 0$ and $k \in \mathbb{N}$, we have

$$\lim_{N\to\infty} \left\| \gamma^{(k)}_{N,\eta} - \gamma^{(k)}_H \right\|_{\text{HS}} = 0$$

Remark: for $d = 1$, we recover the result by **Lewin-Nam-Rougerie**; in this case we can choose $\eta = 0$.

Counterterm problem: given rescaled **many-body Hamiltonian** with chemical potential

$$\mathbb{H}_N = \int a_N^*(x)\,[-\Delta_x + v(x) + \kappa]\,a_N(x)\,dx \\ + \frac{1}{2}\int w(x-y)a_N^*(x)a_N(x)a_N^*(y)a_N(y)\,dxdy$$

we rewrite it as

$$\mathbb{H}_N = \int a_N^*(x)\,[-\Delta_x + v(x) + (w*\rho_N)(x) + \kappa]\,a_N(x)\,dx - \langle w*\rho_N, \rho_N\rangle \\ + \frac{1}{2}\int w(x-y)\,[a_N^*(x)a_N(x) - \rho_N(x)]\,[a_N^*(y)a_N(y) - \rho_N(x)]\,dxdy$$

Subtracting constant and **shifting** chemical potential, we obtain

$$\widetilde{\mathbb{H}}_N = \int a_N^*(x)\,[-\Delta_x + v(x) + (w*(\rho_N - \bar{\rho}_N))(x) + \kappa]\,a_N(x)dx \\ + \frac{1}{2}\int w(x-y)\,[a_N^*(x)a_N(x) - \rho_N(x)]\,[a_N^*(y)a_N(y) - \rho_N(y)]\,dxdy$$

with $\bar{\rho}_N = \mathbb{E}^{(0)}_{-\Delta+\kappa}\, a_N^*(x)a_N(x)$ **independent** of x.

The theorem can be applied to $\widetilde{\mathbb{H}}_N$ if we find confining potential

$$\widetilde{v} = v + (w * (\rho_N - \bar{\rho}_N)) \quad \text{s.t.} \qquad \rho_N(x) = \mathbb{E}^{(0)}_{-\Delta+\widetilde{v}+\kappa} a^*_N(x) a_N(x)$$

Theorem [Fröhlich-Knowles-S.-Sohinger]: Let $v \geq 0$ such that $v(x+y) \leq Cv(x)v(y)$ and

$$\mathrm{Tr}\,(-\Delta + v + \kappa)^{-2} < \infty.$$

Then for every $N \in \mathbb{N}$ there exists v_N solving the **counterterm problem**. Furthermore there is a **limiting potential** $\widetilde{v}$ such that

$$\lim_{N\to\infty} \left\| (-\Delta + v_N + \kappa)^{-1} - (-\Delta + \widetilde{v} + \kappa)^{-1} \right\|_{\mathrm{HS}} = 0$$

Hence, after a change of the chemical potential, modified many-body quantum Gibbs state associated with $\mathbb{H}_N$ is such that

$$\lim_{N\to\infty} \left\| \gamma^{(k)}_{N,\eta} - \gamma^{(k)}_{H} \right\|_{\mathrm{HS}} = 0$$

where $\gamma^{(k)}_H$ are moments of **inv. measure** with $h = -\Delta + \widetilde{v} + \kappa$.

VI. Some ideas from the proof

Duhamel expansion: start from

$$e^{-(1-2\eta)\mathbb{H}_N} = e^{-(1-2\eta)\mathbb{H}_{N,0}} + \int_0^{1-2\eta} dt\, e^{-(1-2\eta-t)\,\mathbb{H}_{N,0}}\, W_N^r e^{-t\mathbb{H}_N}$$

Iterating, we find

$$e^{-(1-2\eta)\mathbb{H}_N} = e^{-(1-2\eta)\mathbb{H}_{N,0}}$$

$$+ \sum_{m=1}^{n-1} \int_0^{1-2\eta} dt_1 \dots \int_0^{t_{m-1}} dt_m\, e^{-(1-2\eta-t_1)H_{N,0}}\, W_N^r \dots W_N^r\, e^{-t_m\mathbb{H}_{N,0}}$$

$$+ \int_0^{1-2\eta} dt_1 \dots \int_0^{t_{n-1}} dt_n\, e^{-(1-2\eta-t_1)\mathbb{H}_{N,0}}\, W_N^r \dots W_N^r e^{-t_n\mathbb{H}_N}$$

Hence

$$e^{-\eta\mathbb{H}_{N,0}} e^{-(1-2\eta)\mathbb{H}_N} e^{-\eta\mathbb{H}_{N,0}} = e^{-\mathbb{H}_{N,0}}$$

$$+ \sum_{m=1}^{n-1} \int_\eta^{1-\eta} dt_1 \dots \int_\eta^{t_{m-1}} dt_m\, e^{-(1-t_1)\mathbb{H}_{N,0}}\, W_N^r \dots W_N^r\, e^{-t_m\mathbb{H}_{N,0}}$$

$$+ \int_\eta^{1-\eta} dt_1 \dots \int_\eta^{t_{n-1}} dt_n\, e^{-(1-t_1)\mathbb{H}_{N,0}}\, W_N^r \dots W_N^r\, e^{-t_n\mathbb{H}_N} e^{-\eta\mathbb{H}_{N,0}}$$

Evolved fields operator: remark

$$e^{t\mathbb{H}_{0,N}} a_N^*(f) e^{-t\mathbb{H}_{0,N}} = \sum_j \langle u_j, f\rangle\, e^{t\lambda_j a_N^*(u_j) a_N(u_j)} a_N^*(u_j) e^{-t\lambda_j a_N^*(u_j) a_N(u_j)}$$
$$= \sum_j \langle u_j, f\rangle\, e^{t\lambda_j/N}\, a_N^*(u_j) = a_N^*(e^{-th/N} f)$$

Fully expanded terms: need to compute **free expectations**!

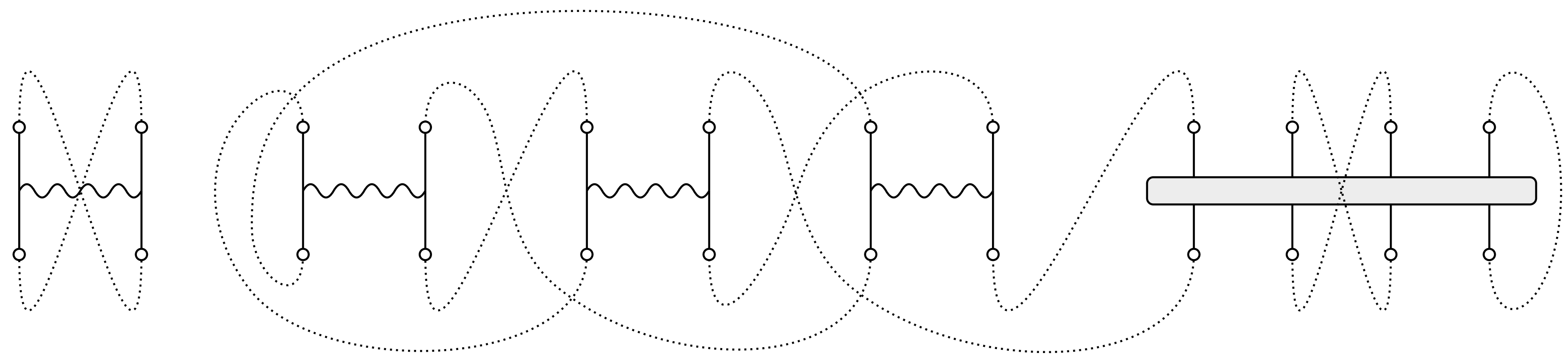
Wick theorem: we have

$$\mathbb{E}_{N,\kappa}^{(0)}\, a_N^{\sharp_1}(f_1)\dots a_N^{\sharp_{2m}}(f_{2m})$$
$$= \sum_\pi \mathbb{E}_{N,\kappa}^{(0)} \left[a_N^{\sharp_{i_1}}(f_{i_1}) a_N^{\sharp_{\ell_1}}(f_{\ell_1}) \right] \dots \mathbb{E}_{N,\kappa}^{(0)} \left[a_N^{\sharp_{i_m}}(f_{i_m}) a_N^{\sharp_{\ell_m}}(f_{\ell_m}) \right]$$

Non-vanishing expectations: are only

$$\mathbb{E}_{N,\kappa}^{(0)}\, [a_N^*(x) a_N(y)] = \frac{1}{N} \frac{1}{e^{h/N}-1}(x;y)$$
$$\mathbb{E}_{N,\kappa}^{(0)}\, [a_N(x) a_N^*(y)] = \frac{1}{N} \frac{1}{e^{h/N}-1}(x;y) + \frac{1}{N}\,\delta(x-y)$$

Diagrammatic expansion: pairings encoded in diagrams



Wick ordering implies **no pairing** between fields with same x.

Bound: using diagrammatic representation and **assumption**

$$\operatorname{Tr} h^{-2} < \infty,$$

we conclude that contribution of each pairing is bounded by a constant, **uniformly** in N.

Convergence: as $N \to \infty$, contribution of each pairing tends to corresponding term in expansion of **Hartree inv. measure**.

Error term: use **Cauchy-Schwarz** to get rid of interacting term.

All interactions W_N^r are controlled through the free state.

Here we need **modification**, to avoid that interacting exponential carries full time.

Final obstacle: number of pairing $\sim (2n)!$ Time integral $\sim 1/n!$

The series **does not converge**!

What saves us is **Borel resummation**.

Theorem [Sokal]: Let $A(z)$ and $(A_N(z))_{N\in\mathbb{N}}$ be **analytic** on ball

$$\mathcal{C}_R = \left\{ z \in \mathbb{C} : (\mathrm{Re}\, z - R)^2 + \mathrm{Im}^2 z \le R^2 \right\}$$

for some $R > 0$. For $n \in \mathbb{N}$ suppose

$$A(z) = \sum_{m=0}^{n-1} a_m z^m + R_n(z), \qquad A_N(z) = \sum_{m=0}^{n-1} a_{m,N} z^m + R_{n,N}(z)$$

with

$$|a_m| + \sup_N |a_{m,N}| \le C^m m!, \qquad |R_m(z)| + \sup_N |R_{m,N}(z)| \le C^m |z|^m\, m!$$

for all $m \in \mathbb{N}$, $z \in \mathcal{C}_R$.

Suppose moreover that, for all $m \in \mathbb{N}$: $\quad \lim_{N\to\infty} |a_{m,N} - a_m| = 0$.

Then $A_N(z) \to A(z)$ for all $z \in \mathcal{C}_R$.