Worldline calculation of the three-gluon-vertex

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(N. Ahmadiniaz and C. Schubert, arXiv:1206.1843 [hep-ph])

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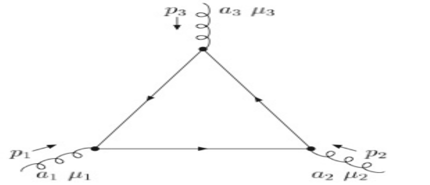

Figure 1: Three-gluon vertex.

- Three gluon-vertex is a basic object of interest in non-abelian gauge theory
- It is an input for the Schwinger-Dyson equation

The general (off-shell) three-gluon vertex, proposed by Ball and Chiu (1980)

$$
\begin{aligned}
\Gamma_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) &= A(p_1^2,p_2^2;p_3^2)g_{\mu_1\mu_2}(p_1-p_2)_{\mu_3} + B(p_1^2,p_2^2;p_3^2)g_{\mu_1\mu_2}(p_1+p_2)_{\mu_3} \\
&- C(p_1^2,p_2^2;p_3^2)[(p_1p_2)g_{\mu_1\mu_2} - p_{1\mu_2}p_{2\mu_1}](p_1-p_2)_{\mu_3} \\
&+ \frac{1}{3}S(p_1^2,p_2^2,p_3^2)(p_{1\mu_3}p_{2\mu_1}p_{3\mu_2} + p_{1\mu_2}p_{2\mu_3}p_{3\mu_1}) + F(p_1^2,p_2^2;p_3^2) \\
&\times [(p_1p_2)g_{\mu_1\mu_2} - p_{1\mu_2}p_{2\mu_1}][p_{1\mu_3}(p_2p_3) - p_{2\mu_3}(p_1p_3)] \\
&+ H(p_1^2,p_2^2,p_3^2)\Big(-g_{\mu_1\mu_2}[p_{1\mu_3}(p_2p_3) - p_{2\mu_3}(p_1p_3)] \\
&+ \frac{1}{3}(p_{1\mu_3}p_{2\mu_1}p_{3\mu_2} - p_{1\mu_2}p_{2\mu_3}p_{3\mu_1})\Big) \\
&+ [\text{cyclic permutations of } (p_1,\mu_1),(p_2,\mu_2),(p_3,\mu_3)] \qquad (1)
\end{aligned}
$$

where

- A, C, F $\longrightarrow$ symmetric functions in the first two arguments
- B $\longrightarrow$ antisymmetric function in the first two arguments
- H $\longrightarrow$ totally symmetric function with respect to interchange of any pair of arguments
- S $\longrightarrow$ totally antisymmetric function with respect to interchange of any pair of arguments

- *J. S. Ball and T-W Chiu, Phys. Rev. D 22, 2542 (1980); 22, 2550 (1980)*

  One loop calculation of the three-gluon amplitude for the gluon loop.

- *J. M. Cornwall and J. Papavassiliou, Phys. Rev. D 40, 3474 (1989)*

  Non-perturbative study of the three-gluon vertex from the pinch technique.

- *A. I. Davydychev, P. Osland and L. Saks, JHEP 08 (2001) 050*

  Massive quark contribution to the three-gluon vertex in arbitrary dimension.

- *M. Binger and S. J. Brodsky Phys. Rev. D 74, 054016 (2006)*

  The gauge-invariant three-gluon vertex obtained from the pinch technique, the results are given in arbitrary space-time dimension.

  SUSY related : $\mathbf{3\Gamma_0 + 2\Gamma_{\frac{1}{2}} + \Gamma_1 = 0}$

Worldline Green's function

Z. Bern and D. A. Kosower (Phys. Rev. Lett. 66, 1669, 1991),
M. J. Strassler (Nucl. Phys. B 385, 145, 1992)

$$\Gamma^{a_1 a_2 a_3}_{\mathrm{1PI,scalar}}[k_1, \varepsilon_1; k_2, \varepsilon_2; k_3, \varepsilon_3] = (-ig)^3 \mathrm{tr}(T^{a_1} T^{a_2} T^{a_3}) \int_0^\infty \frac{dT}{T} \mathrm{e}^{-m^2 T}$$

$$\times \quad \int \mathcal{D}x(\tau) \int_0^T d\tau_1 \varepsilon_1 \cdot \dot{x}_1 \mathrm{e}^{ik_1 \cdot x_1} \int_0^{\tau_1} d\tau_2 \varepsilon_2 \cdot \dot{x}_2 \mathrm{e}^{ik_2 \cdot x_2} \varepsilon_3 \cdot \dot{x}_3 \mathrm{e}^{ik_3 \cdot x_3} \ \mathrm{e}^{-\int_0^T \frac{\dot{x}^2}{4}} \quad (2)$$

$$\int d\tau \varepsilon \cdot \dot{x} \mathrm{e}^{\mathrm{ik \cdot x}} \Rightarrow \text{gluon/photon vertex operator}$$

- T = total proper time of the loop particle
- τ = proper time which parametrizes the loop
- m = mass of the loop particle
- T^a = generator of the gauge group
- $\int \mathcal{D}(x)$ is over closed trajectory in Minkovski space-time with periodicity T, $x(T) = x(0)$
- ε =gluon polarization
- k =gluon momentum

This path integral is Gaussian so we just need the Green's function which is

$$< x^{\mu_1}(\tau_1) x^{\mu_2}(\tau_2) >= -G_B(\tau_1, \tau_2) g^{\mu_1 \mu_2}$$

$$\begin{aligned} G_B(\tau_1, \tau_2) &= |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T} \\ \dot{G}_B(\tau_1, \tau_2) &= \mathrm{sign}(\tau_1 - \tau_2) - 2\frac{(\tau_1 - \tau_2)}{\mathrm{T}} \\ \ddot{G}_B(\tau_1, \tau_2) &= 2\delta(\tau_1 - \tau_2) - \frac{2}{T} \end{aligned} \tag{3}$$

Scalar loop calculation

After integration by part and removing all $\ddot{G}_B(\tau_i, \tau_j)$ we have

$$\Gamma_{\text{scalar}} = \frac{g^3}{(4\pi)^{D/2}}(\Gamma^3_{\text{scalar}} + \Gamma^2_{\text{scalar}} + \Gamma^{\text{bt}}_{\text{scalar}}) \tag{4}$$

$$\begin{aligned}
\Gamma^3_{\mathrm{scalar}} &= -\mathrm{tr}(T^{a_1}[T^{a_2}, T^{a_3}]) \int_0^\infty \frac{dT}{T^{\frac{D}{2}}} e^{-m^2 T} \\
&\times \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2\, Q_3^3|_{\tau_3=0}\, e^{(G_{B12} k_1\cdot k_2 + G_{B13} k_1\cdot k_3 + G_{B23} k_2\cdot k_3)} \\
\Gamma^2_{\mathrm{scalar}} &= \Gamma^3_{\mathrm{scalar}}(Q_3^3 \to Q_3^2) \\
\Gamma^{\mathrm{bt}}_{\mathrm{scalar}} &= -\mathrm{tr}(T^{a_1}[T^{a_2}, T^{a_3}]) \int_0^\infty \frac{dT}{T^{\frac{D}{2}}} e^{-m^2 T} \int_0^T d\tau_1\, \dot{G}_{B12} \dot{G}_{B21} \\
&\times \Big[\varepsilon_3 \cdot f_1 \cdot \varepsilon_2\, e^{G_{B12} k_1\cdot(k_2+k_3)} + \varepsilon_1 \cdot f_2 \cdot \varepsilon_3\, e^{G_{B12} k_2\cdot(k_1+k_3)} \\
&+ \varepsilon_2 \cdot f_3 \cdot \varepsilon_1\, e^{G_{B12} k_3\cdot(k_1+k_2)}\Big]
\end{aligned} \tag{5}$$

$$\begin{aligned}
Q_3^3 &= \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}\mathrm{tr}\,(f_1 f_2 f_3) \\
Q_3^2 &= \frac{1}{2}\dot{G}_{B12}\dot{G}_{B21}\mathrm{tr}\,(f_1 f_2)\dot{G}_{B3k}\varepsilon_3 \cdot k_k + \frac{1}{2}\dot{G}_{B13}\dot{G}_{B31}\mathrm{tr}\,(f_1 f_3)\dot{G}_{B2j}\varepsilon_2 \cdot k_j \\
&+ \frac{1}{2}\dot{G}_{B23}\dot{G}_{B32}\mathrm{tr}\,(f_2 f_3)\dot{G}_{B1i}\varepsilon_1 \cdot k_i \\
f_i^{\mu\nu} &= k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu
\end{aligned} \tag{6}$$

so the final result for the the scalar loop calculation is :

$$\Gamma_s = \frac{g^3}{(4\pi)^{\frac{D}{2}}} \mathrm{tr}(T^{a_1}[T^{a_2}, T^{a_3}])(\gamma_{\rm s}^3 + \gamma_{\rm s}^2 + \gamma_{\rm s}^{\rm bt}) \tag{7}$$

with

$$\begin{aligned}
\gamma_0^3 &= \Gamma\big(3 - \frac{D}{2}\big)\mathrm{tr}\,(f_1 f_2 f_3) I^D_{3,B}(k_1^2, k_2^2, k_3^2) \\
\gamma_0^2 &= \frac{1}{2}\Gamma\big(3 - \frac{D}{2}\big)\Big[\mathrm{tr}\,(f_1 f_2)\Big(\varepsilon_3 \cdot k_1 I^D_{2,B}(k_1^2, k_2^2, k_3^2) - \varepsilon_3 \cdot k_2 I^D_{2,B}(k_2^2, k_1^2, k_3^2)\Big) \\
&\qquad + \mathrm{tr}\,(f_2 f_3)\Big(\varepsilon_1 \cdot k_2 I^D_{2,B}(k_2^2, k_3^2, k_1^2) - \varepsilon_1 \cdot k_3 I^D_{2,B}(k_3^2, k_2^2, k_1^2)\Big) \\
&\qquad + \mathrm{tr}\,(f_3 f_1)\Big(\varepsilon_2 \cdot k_3 I^D_{2,B}(k_3^2, k_1^2, k_2^2) - \varepsilon_2 \cdot k_1 I^D_{2,B}(k_1^2, k_3^2, k_2^2)\Big)\Big] \\
\gamma_0^{\rm bt} &= -\Gamma\big(2 - \frac{D}{2}\big)\Big[\varepsilon_3 \cdot f_1 \cdot \varepsilon_2 I^D_{{\rm bt},B}(k_1^2) + \varepsilon_1 \cdot f_2 \cdot \varepsilon_3 I^D_{{\rm bt},B}(k_2^2) + \varepsilon_2 \cdot f_3 \cdot \varepsilon_1 I^D_{{\rm bt},B}(k_3^2)\Big]
\end{aligned} \tag{8}$$

Rewriting the integrals in term of the standard *Feynman/Schwinger* parameter $\alpha_{1,2,3}$:

$$\begin{aligned}
I^D_{3,B}(k_1^2,k_2^2,k_3^2) &= \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_1-\alpha_2-\alpha_3) \\
&\times \frac{(1-2\alpha_1)(1-2\alpha_2)(1-2\alpha_3)}{\left(m^2+\alpha_1\alpha_2 k_1^2+\alpha_2\alpha_3 k_2^2+\alpha_1\alpha_3 k_3^2\right)^{3-\frac{D}{2}}} \\
I^D_{2,B}(k_1^2,k_2^2,k_3^2) &= \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_1-\alpha_2-\alpha_3) \\
&\times \frac{(1-2\alpha_2)^2(1-2\alpha_1)}{\left(m^2+\alpha_1\alpha_2 k_1^2+\alpha_2\alpha_3 k_2^2+\alpha_1\alpha_3 k_3^2\right)^{3-\frac{D}{2}}} \\
I^D_{bt,B}(k^2) &= \int_0^1 d\alpha \frac{(1-2\alpha)^2}{\left(m^2+\alpha(1-\alpha)k^2\right)^{2-\frac{D}{2}}} \qquad (9)
\end{aligned}$$

where

$$\alpha_1+\alpha_2+\alpha_3=1$$

Off-shell generalization of the Bern-Koswer replacement rules: to get the spinor and gluon results from the scalar one:

$$\begin{array}{ll} \gamma_0 \to \gamma_{1/2} & I^D_{3,2,\mathrm{bt},B} \to I^D_{3,2,\mathrm{bt},B} - I^D_{3,2,\mathrm{bt},F} \\ \gamma_0 \to \gamma_1 & I^D_{3,2,\mathrm{bt},B} \to I^D_{3,2,\mathrm{bt},B} - 4 I^D_{3,2,\mathrm{bt},F} \end{array} \tag{10}$$

$$\begin{aligned} I^D_{2,F}(k_1^2,k_2^2,k_3^2) &= \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_1-\alpha_2-\alpha_3) \\ &\times \frac{1-2\alpha_1}{\left(m^2+\alpha_1\alpha_2 k_1^2+\alpha_2\alpha_3 k_2^2+\alpha_1\alpha_3 k_3^2\right)^{3-\frac{D}{2}}} \\ I^D_{3,F}(k_1^2,k_2^2,k_3^2) &= \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_1-\alpha_2-\alpha_3) \\ &\times \frac{1}{\left(m^2+\alpha_1\alpha_2 k_1^2+\alpha_2\alpha_3 k_2^2+\alpha_1\alpha_3 k_3^2\right)^{3-\frac{D}{2}}} \\ I^D_{bt,F}(k^2) &= \int_0^1 d\alpha \frac{1}{\left(m^2+\alpha(1-\alpha)k^2\right)^{2-\frac{D}{2}}} \end{aligned} \tag{11}$$

Now from worldline SUSY, not space-time SUSY, again get Binger-Brodsky relation

$$\mathbf{3\Gamma_0 + 2\Gamma_{1/2} + \Gamma_1 = 0} \tag{12}$$

Comparison with the effective action

Compare with the QCD effective action

$$\Gamma_{\mathrm{scalar}}[F] = \int_0^\infty \frac{dT}{T} \frac{\mathrm{e}^{-m^2 T}}{(4\pi T)^{\frac{D}{2}}} \mathrm{tr} \int dx_0 \sum_{n=2}^{\infty} \frac{(-T)^n}{n!} \mathrm{O}_n[F] \tag{13}$$

This expansion is known to order $O(T^6)$, here for our comparison we need O_2 and O_3 which are:

$$\begin{aligned} O_2 &= -\frac{1}{6} F_{\kappa\lambda} F^{\kappa\lambda} \\ O_3 &= -\frac{2}{15} i F_\kappa{}^\lambda F_\lambda{}^\mu F_\mu{}^\kappa - \frac{1}{20} D_\kappa F_{\lambda\mu} D^\kappa F^{\lambda\mu} \end{aligned} \tag{14}$$

where

$$\begin{aligned} F_{\mu\nu} &= f_{\mu\nu} + ig[A_\mu, A_\nu] \\ f_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ D_\mu &= \partial_\mu + igA_\mu \end{aligned} \tag{15}$$

so we have can recognize our Q_3^3 and Q_3^2

$$\begin{aligned} & Q_3^3 \leftrightarrow \gamma^3 \leftrightarrow F_\kappa{}^\lambda F_\lambda{}^\mu F_\mu{}^\kappa = f_\kappa^\lambda f_\lambda^\mu f_\mu^\kappa + \text{higher point terms} \\ & Q_3^2 \leftrightarrow \gamma^2 \leftrightarrow (\partial + ig\, \underbrace{A)F(\partial}\, + igA)F \\ & \gamma_3^{\mathrm{bt}} \leftrightarrow (f + ig\, \underbrace{[A, A])(f}\, + ig[A, A]) \end{aligned} \tag{16}$$

Comparison with the Ball-Chiu vertex

How does this relate to the Ball-Chiu decomposition?

For the Scalar loop case:

$$\begin{aligned}
H(p_1^2, p_2^2, p_3^2) &= -\frac{d_0 g^2}{2(4\pi)^{D/2}}\Gamma(3-\frac{D}{2}) I^D_{3,B}(p_1^2, p_2^2, p_3^2) \\
A(p_1^2, p_2^2; p_3^2) &= \frac{d_0 g^2}{4(4\pi)^{D/2}}\Gamma(2-\frac{D}{2})\left[I^D_{\mathrm{bt,B}}(p_1^2) + I^D_{\mathrm{bt,B}}(p_2^2)\right] \\
B(p_1^2, p_2^2; p_3^2) &= \frac{d_0 g^2}{4(4\pi)^{D/2}}\frac{1}{2}\Gamma(2-\frac{D}{2})\left[I^D_{\mathrm{bt,B}}(p_1^2) - I^D_{\mathrm{bt,B}}(p_2^2)\right] \\
F(p_1^2, p_2^2; p_3^2) &= \frac{d_0 g^2}{2(4\pi)^{D/2}}\Gamma(3-\frac{D}{2})\frac{I^D_{2,B}(p_1^2, p_2^2, p_3^2) - I^D_{2,B}(p_2^2, p_1^2, p_3^2)}{p_1^2 - p_2^2} \\
C(p_1^2, p_2^2; p_3^2) &= \frac{d_0 g^2}{2(4\pi)^{D/2}}\Gamma(2-\frac{D}{2})\frac{I^D_{\mathrm{bt,B}}(p_1^2) - I^D_{\mathrm{bt,B}}(p_2^2)}{p_1^2 - p_2^2} \\
S(p_1^2, p_2^2; p_3^2) &= 0
\end{aligned} \tag{17}$$

Replacement rules → spinor and gluon loop cases similar

Just for comparison with previous calculations let me show you “H” function:
from Davydychev, Osland and Saks JHEP 08 (2001) 050

$$
\begin{aligned}
H \quad = \quad & -c\frac{g^2\eta}{(4\pi^2)^{D/2}}\frac{2}{(D-2)(D-1)\mathcal{K}^2} \times \\
& \times\Big\{(D-1)^2\mathcal{K}^{-1}(k_1k_2)(k_1k_3)(k_2k_3) \times \\
& \big\{(k_1k_2)(k_1k_3)(k_2k_3)\varphi_3 + (k_1k_2)(k_1k_3)\kappa_{2,1} + (k_1k_2)(k_2k_3)\kappa_{2,2} \\
& +(k_1k_3)(k_2k_3)\kappa_{2,3}\big\} - 3(D-1)(k_1k_2)(k_1k_3)(k_2k_3) \times \\
& \big\{(\mathcal{Q}+4m^2)\varphi_3 + \kappa_{2,1} + \kappa_{2,2} + \kappa_{2,3}\big\} + (D-1)(D-2)\mathcal{K}^2\varphi_3 \\
& -2(D-2)^2\mathcal{K}m^2\tilde{\kappa} + (D-2)(k_1^2-4m^2)\big\{k_1^2(k_2k_3) + (k_1k_2)(k_1k_3)\big\}\kappa_{2,1} \\
& +(D-2)(k_2^2-4m^2)\big\{k_2^2(k_1k_3) + (k_1k_2)(k_2k_3)\big\}\kappa_{2,2} \\
& +(D-2)(k_3^2-4m^2)\big\{k_3^2(k_1k_2) + (k_1k_3)(k_2k_3)\big\}\kappa_{2,1}\Big\} \qquad (18)
\end{aligned}
$$

where

$$
\begin{aligned}
\eta &= \frac{\Gamma^2(D/2-1)}{\Gamma(D-3)}\Gamma(3-D/2) \\
\mathcal{Q} &= \frac{1}{2}(k_1^2+k_2^2+k_3^2) \\
\mathcal{K} &= k_1^2 k_2^2 - (k_1 k_2)^2
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
J_3(1,1,1) &= i\pi^{D/2}\eta\varphi_3 \\
J_3(0,1,1) &= i\pi^{D/2}\eta\kappa_{2,1} \\
J_3(1,0,1) &= i\pi^{D/2}\eta\kappa_{2,2} \\
J_3(1,1,0) &= i\pi^{D/2}\eta\kappa_{2,3} \\
J_3(0,0,1) &= i\pi^{D/2}\eta m^2\tilde{\kappa}
\end{aligned}
\tag{20}
$$

where three-point integrals with equal masses define as

$$
J_3(\nu_1,\nu_2,\nu_3) = \int \frac{d^n q}{[(k_2-q)^2-m^2]^{\nu_1}\,[(k_1+q)^2-m^2]^{\nu_2}\,[q^2-m^2]^{\nu_3}} \tag{21}
$$

Conclusions

- Significantly more efficient calculation than previous ones.
- More compact representation of the vertex.
- The comparison with the effective action shows that S remains zero also beyond one loop.
- Now: on to the four-gluon vertex!

Thank you for your attention