# Bulk-boundary correspondence in (3+1)d topological phases

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- Introduction
  Topological order
  bulk-boundary correspondence in (2+1) dimensions

- Topological order in (3+1) dimensions

- Closing

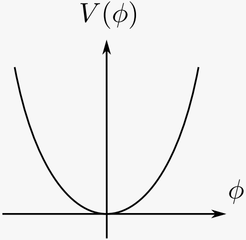
# Symmetry breaking phases

Many phases of matter can be described by local order parameters associated to spontaneous symmetry breaking (SSB)

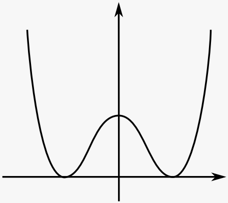
Example: magnnet

Characterized by non-zero order parameter $\langle \phi \rangle$

Theoretical framework: Landau-Ginzburg theory




$\langle \phi \rangle = 0$

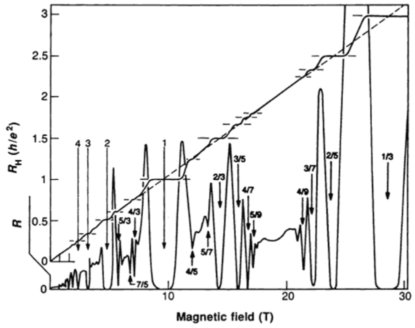


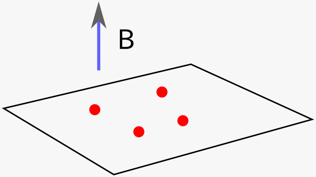
$\langle \phi \rangle \neq 0$

# Topological phases

There are, however, phases of matter which cannot be described by any local order parameter

Example: quantum Hall effect



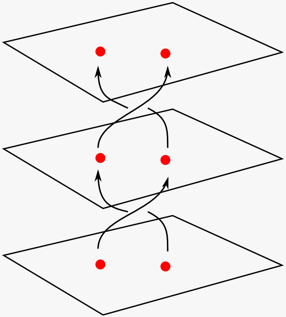




Different plateaus = different gapped quantum phases

# Topological phases

- Symmetry-breaking phases support bosonic excitations:
Nambu-Goldstone bosons.

- Topologically ordered phases support excitations realizing exotic exchage statistics: "anyons"

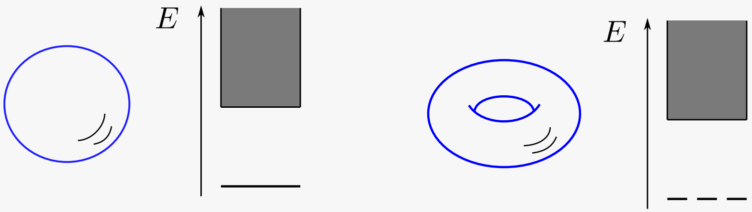


- Theoretical framework:
topolgical quantum field theory
Chern-Simons theory

# Topological ground state degeneracy

How do we characterized topological order?
Idea: response of the system to topology of spacetime

Topological orders: ground-state degeneracy depending
on the topology of the space
(Topological degeneracy) [Wen '90]

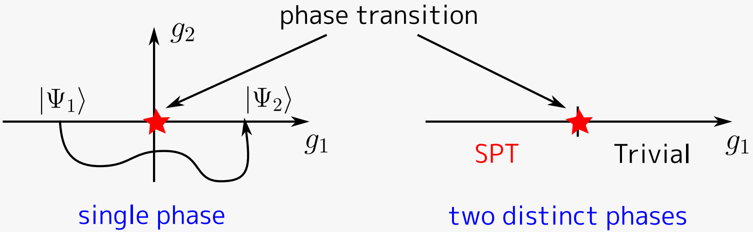



Non-symmetry related degeneracy

# Symmetry protected topological phases (SPT)

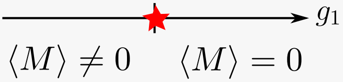
- Not a topologically-ordered phase w/o symmetry
  "deformable" to a trivial phase (state w/o entanglement)

- But sharply distinct from trivial state once symmetries are enforced:




- E.g. topological insulator, topolgical superconductor

- Need to go beyond
  "symmetry breaking" paradigm.



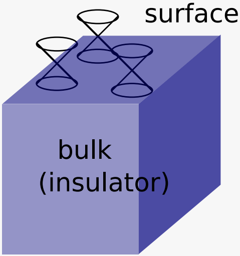
# (3+1) dimensional SPT: topological insulator

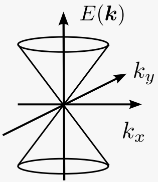
- d=3 dimensions,  Time-reversal symmetry

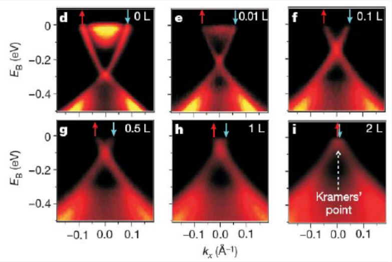
- Topological distinction by a topological invariant

- Anomalous surface state with in the bulk band gap:
  odd number of Dirac cones

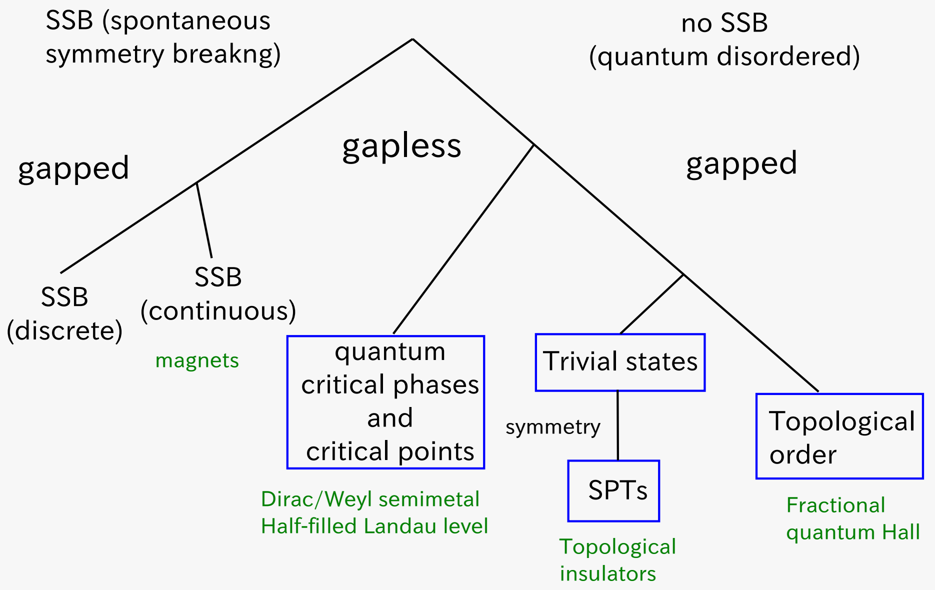
- Gapless and completely immnue to disorder as far as TRS is preserved



# Phases of condensed matter (at T=0)



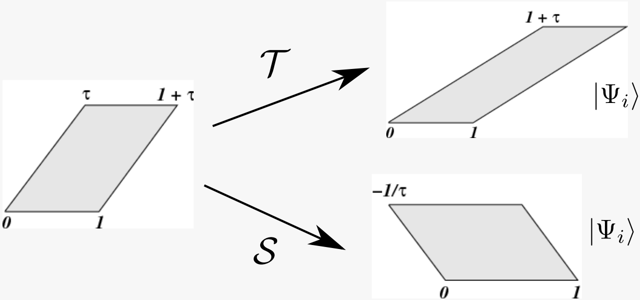
# Characterization of Topological order

- Gapped excitations: "anyons"

- Ground state degeneracy depending on topology of space $|\Psi_i\rangle$

- Non-trivial braiding statistics and spin of gapped quasiparticle excitations

Encoded in modular S- and T- matrices on spatial torus [Wen 92]




$|\Psi_i\rangle \to \sum_{ij} \mathcal{T}_{ij} |\Psi_j\rangle$

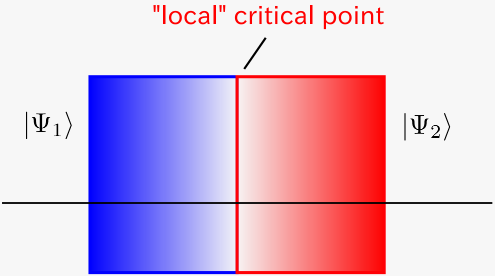
$|\Psi_i\rangle \to \sum_{ij} \mathcal{S}_{ij} |\Psi_j\rangle$

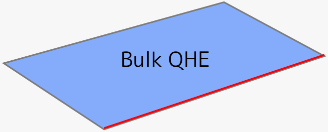
# Bulk and boundary correspondence

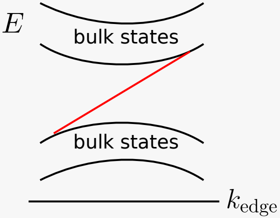
- In the presence of a boundary, topologically non-trivial state is accompanied by a gapless state localized at the boundary.

- "Anomaly inflow"

Laughlin (81), Callan-Harvey (85)..

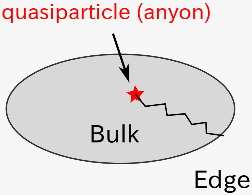


# Extracting bulk data from boundary

- Bulk quasiparticles <---> Twisted boundary conditions at edge




- Bulk ground state degeneracy
  <---> Possible b.c.'s

- Bulk wfn $|\Psi_i\rangle$
  <---> boundary partition func. $Z_i$

- Bulk modular S and T matrices of GS wfn on spatial torus
  <---> Boundary S and T matrices
  of partition functions on spacetime torus [Cappelli 96]

$$Z_i \to \mathcal{T}_{ij} Z_j \qquad Z_i \to \mathcal{S}_{ij} Z_j$$

- Temporal boundary: wave functional [Bos-Nair (90)]

Is there a bulk-boundary correspondence in (3+1)d?

- SPT setting: yes

- This talk: topological orderd phases in (3+1) dimensions

- [Xiao Chen, Apoorv Tiwari, SR, (2015)]
  Carried out the calculations of S and T matrices in
  (2+1)d surface of a (3+1)d bulk topological phases.

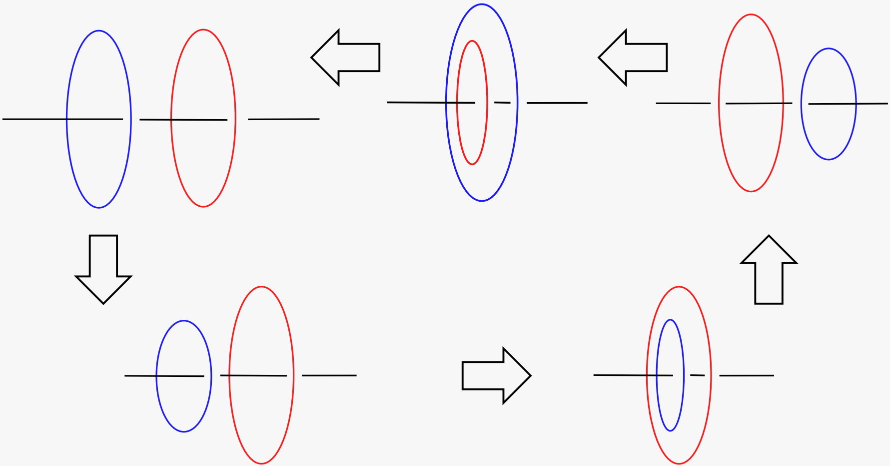
  Proposed a field theory model realzing 3-loop braiding statistics
  S and T matrices computed from boundary

  Established Bulk-boundary correspondence by
  extracting S and T matrices from boundary theory

# What to expect in (3+1)d ?

- Non trivial braiding statistics of loops in (3+1)d
  "Three-loop" braiding statistics

[Wang-Levin (14) Jiang-Mesaros-Ran (14), Wang-Wen (14)]



# (3+1)d BF topological theory

- Action: $$S_{bulk} = \int_{\mathcal{M}} \left[ \frac{\mathrm{K}}{2\pi} b \wedge da - a \wedge J_{qp} - b \wedge J_{qv} \right]$$

$b$: two-form field
$a$: one-form gauge field
$J_{qp}$: quasi-particle current (3-form, Wilson loop)
$J_{qv}$: quasi-vortex current (2-form, surface operator)
$\mathrm{K}$: "level" (integer); parameter of the theory.

- Application:
  - BCS superconductors
    [Balachandran et al 93, Hansson et al 04, Sergej Moroz's talk]
  - discrete gauge theories [e.g. Banks-Seiberg 11]
  - topological insulators [Chan et al 13-15]

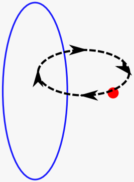
# (3+1)d BF topological theory

- Action:

$$S_{bulk} = \int_{\mathcal{M}} \left[ \frac{\mathrm{K}}{2\pi} b \wedge da - a \wedge J_{qp} - b \wedge J_{qv} \right]$$

- Non-trivial particle-string statistics

$$\int \mathcal{D}[a,b] e^{iS_{bulk}} = e^{iS_{eff}} \qquad S_{eff} = \frac{2\pi}{\mathrm{K}} Link(J_{qv}, J_{qp})$$



- K^3 ground state degeneracy on T^3

# Surface theory

- Surface theory
$$S_{\partial\mathcal{M}} = \int_{\partial\mathcal{M}} dtdxdy \left[ \frac{\mathrm{K}}{2\pi}\epsilon_{ij}\partial_i\zeta_j\partial_t\varphi - V(\varphi,\zeta) \right]$$
$$\mathcal{M} = \Sigma \times S^1$$

- Goal: Establish bulk-boundary correspondence

- Twisted b.c.:

$$\int_{\partial\Sigma} dxdy\, \epsilon_{ij}\partial_i\zeta_j = \frac{2\pi M_0}{\mathrm{K}}$$ (from bulk quasiparticle)

$$\varphi(t,x+2\pi R_1,y) = \varphi(t,x,y) + 2\pi M_1/\mathrm{K}$$
$$\varphi(t,x,y+2\pi R_2) = \varphi(t,x,y) + 2\pi M_2/\mathrm{K}$$ (from bulk quasivortex)

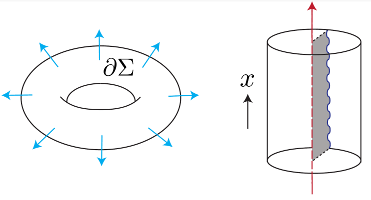
# Surface theory

- Twisted b.c. is related to bulk quasi particles/vortices

$$M_\mu = \mathrm{K} N_\mu + n_\mu$$

$$\frac{\mathrm{K}}{4\pi}\int_\Sigma d^3x\,\varepsilon^{0ijk}\partial_i b_{jk} = \int_\Sigma d^3x\, j^0_{qp} = n_0$$

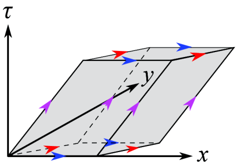
$$L_1 \times \frac{\mathrm{K}}{2\pi}\int dydz\,\varepsilon^{01ij}\partial_i a_j = n_2 \times L_1$$



(a) (b)

# Large coord. transformations on T^3

- Surface theory put on T^3 with flat background metric $g$




- 5 modular parameters

$$R_1/R_0,\ R_2/R_0,\ \alpha,\ \beta,\ \gamma$$

$$\begin{aligned} ds^2 &= g_{\mu\nu} d\theta^\mu d\theta^\nu \\ &= R_0^2 (d\theta^0)^2 + R_1^2 (d\theta^1 - \alpha d\theta^0)^2 + R_2^2 (d\theta^2 - \beta d\theta^1 - \gamma d\theta^0)^2 \end{aligned}$$

- Symmetry (large diffeo): SL(3,Z) $\quad g_{\mu\nu} \xrightarrow{L} (LgL^T)_{\mu\nu} = L_\mu{}^\rho L_\nu{}^\sigma g_{\rho\sigma}$

- SL(3,Z): generated by two generators $\quad L = U_1^{n_1} U_2^{n_2} U_1^{n_3} \cdots$

$$U_1 = U_1' M, \quad U_1' = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad U_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\alpha \to \alpha - 1, \quad \gamma \to \gamma + \beta$$

## Surface theory: results

- Surface theories with twisted b.c.:

$$\int dxdy\, \epsilon_{ij}\partial_i\zeta_j = \frac{2\pi M_0}{\mathrm{K}}$$

$$\varphi(t, x + 2\pi R_1, y) = \varphi(t, x, y) + 2\pi M_1/\mathrm{K}$$
$$\varphi(t, x, y + 2\pi R_2) = \varphi(t, x, y) + 2\pi M_2/\mathrm{K}$$

- Calculated the partition functions on flat T^3 with twisted b.c:

$$Z^{n_0 n_1 n_2} \qquad M_\mu = \mathrm{K}N_\mu + n_\mu$$

- Extracted the modular S and T matrices:

$$\mathcal{S}_{n_i,n_i'} = \frac{1}{\mathrm{K}}\delta_{n_1,n_2'} e^{-\frac{2\pi i}{\mathrm{K}}(n_0' n_2 - n_0 n_1')},$$
$$\mathcal{T}_{n_i,n_i'} = \delta_{n_0,n_0'}\delta_{n_1,n_1'}\delta_{n_2,n_2'} e^{\frac{2\pi i}{\mathrm{K}} n_0 n_1}$$

- The result agrees with the bulk calculations [E.g. Moradi-Wen (14)]
  Established the bulk-boundary correspondence.

# Coupled BF theories

- Let's now move on to more complicated theory

- Unique topological order with Zk gauge symmetry, as seen from the 4th group cohomology:

$$H^4[\mathbb{Z}_\mathrm{K}, U(1)] = 0$$

[Dijkgraaf -Witten (90)
Chen-Gu-Liu-Wen (11-13) in SPT context]

- K^2 distinct topological orders with Zk x Zk gauge symmetry since

$$H^4[\mathbb{Z}_\mathrm{K} \times \mathbb{Z}_\mathrm{K}, U(1)] = \mathbb{Z}_\mathrm{K} \times \mathbb{Z}_\mathrm{K}$$

Many different ways to gauge the system
Distinguished by 3-loop braiding statistics [Wang-Levin (14)]

# Coupled BF theories

- Motivating cubic theory (I,J=1,2):

$$\begin{aligned} S_{bulk} = \int_{\mathcal{M}} \Big[ & \frac{\mathrm{K}}{2\pi} \delta_{IJ} b^I \wedge da^J \\ & + \frac{\mathrm{p}_1}{4\pi^2} a^1 \wedge a^2 \wedge da^2 + \frac{\mathrm{p}_2}{4\pi^2} a^2 \wedge a^1 \wedge da^1 \\ & - \delta_{IJ} b^I \wedge J^J_{qv} - \delta_{IJ} a^I \wedge J^J_{qp} \Big], \end{aligned} \tag{169}$$

[Kapustin-Thorngren (14) Wang-Gu-Wen (15) Ye-Gu (15) Wang-Wen-Yau (16)]

Coupling: p1 and p2

- Gauge invariance:

$$\begin{aligned} & b^1 \to b'^1 = b^1 + d\zeta^1 - \frac{\mathrm{p}_2}{2\pi \mathrm{K}} \left( a^2 \wedge d\varphi^1 + d\varphi^2 \wedge a^1 \right), \\ & b^2 \to b'^2 = b^2 + d\zeta^2 - \frac{\mathrm{p}_1}{2\pi \mathrm{K}} \left( a^1 \wedge d\varphi^2 + d\varphi^1 \wedge a^2 \right), \\ & a^I \to a'^I = a^I + d\varphi^I. \end{aligned} \tag{173}$$

# Coupled BF theories

- Quadratic theory:

$$\begin{aligned}S'_{bulk} = \frac{\mathrm{K}}{2\pi}\int \delta_{IJ} b^I \wedge da^J - \int \delta_{IJ} a^J \wedge J^I_{qp} \\ - \int \left[b^1 + \frac{\mathrm{p}_2}{2\pi \mathrm{K}} a^1 \wedge a^2\right] \wedge J^1_{qv} \\ - \int \left[b^2 + \frac{\mathrm{p}_1}{2\pi \mathrm{K}} a^2 \wedge a^1\right] \wedge J^2_{qv}.\end{aligned}$$

- 3-loop braiding statistics

$$\begin{aligned}S_{eff} = -\frac{2\pi}{\mathrm{K}}\int (d^{-1}J^I_{qv}) \wedge J^I_{qp} \\ + \left(\frac{2\pi}{\mathrm{K}}\right)^3 \mathrm{p}_1 \int (d^{-1}J^1_{qv}) \wedge (d^{-1}J^2_{qv}) \wedge J^2_{qv} \\ + \left(\frac{2\pi}{\mathrm{K}}\right)^3 \mathrm{p}_2 \int (d^{-1}J^2_{qv}) \wedge (d^{-1}J^1_{qv}) \wedge J^1_{qv}\end{aligned}$$

# Coupled BF theories

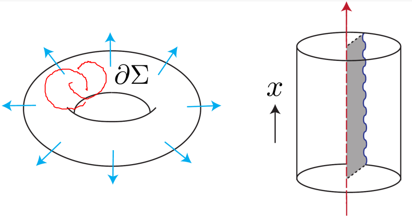
- Surface theories with twisted b.c.:

Twisted b.c. is related to bulk quasi particles/vortices

$$\frac{\mathrm{K}}{2\pi}\int_\Sigma db^1 = -\frac{\mathrm{p}_1}{\mathrm{K}^2}\int_\Sigma (d^{-1}J^2_{qv})\wedge J^2_{qv} + \frac{\mathrm{p}_2}{\mathrm{K}^2}\int_\Sigma (d^{-1}J^2_{qv})\wedge J^1_{qv} + \int_\Sigma J^1_{qp}$$

Hopf linking of vortex lines twists b.c.

$$L_1 \times \frac{\mathrm{K}}{2\pi}\int dydz\, \varepsilon^{01ij}\partial_i a_j = n_2 \times L_1$$




(a) (b)

# Coupled BF theories: results

- Surface partition function:

$$Z^{n_0 n_1 n_2}_{r_0 r_1 r_2}$$

$$M_\mu = \mathrm{K}N_\mu + n_\mu, \quad n_\mu = 0,1,\ldots,\mathrm{K}-1,$$
$$Q_\mu = \mathrm{K}R_\mu + r_\mu, \quad r_\mu = 0,1,\ldots,\mathrm{K}-1,$$

- Coupling generates "twist"

$$M_0 \to M_0 + p_1(Q \wedge M)$$
$$Q_0 \to Q_0 + p_2(M \wedge Q)$$

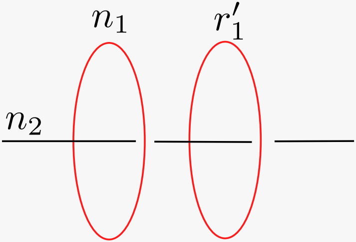
- Extracted the modular S and T matrices (when $p_1=p_2=K$):

$$\begin{aligned}
\mathcal{S}_{n_\mu,n'_\mu,r_\mu,r'_\mu} &= \frac{1}{\mathrm{K}^2}\delta_{n_1,n'_2}\delta_{r_1,r'_2}e^{-\frac{2\pi i}{\mathrm{K}}(\tilde{n}'_0 n_2 - \tilde{n}_0 n'_1 + \tilde{r}'_0 r_2 - \tilde{r}_0 r'_1)} \\
&\quad \times e^{-\frac{2\pi i}{\mathrm{K}^2}\left[(n_1+r_1)(n_2 r'_1 + n'_1 r_2) - 2n_2 n'_1 r_1 - 2n_1 r_2 r'_1\right]} \\
\mathcal{T}_{n_\mu,n'_\mu,r_\mu,r'_\mu} &= \delta_{n_\mu,n'_\mu}\delta_{r_\mu,r'_\mu} \\
&\quad \times e^{\frac{2\pi i}{\mathrm{K}}(\tilde{n}_0 n_1 + \tilde{r}_0 r_1) + \frac{2\pi i}{\mathrm{K}^2}(r_1 n_2 - r_2 n_1)(n_1 - r_1)}.
\end{aligned}$$

- The result is consistent with the bulk calculations
  [E.g.Wang-Levin (14),Jiang-Mesaros-Ran (14), Wang-Wen (14)]

- Interpretation of S-matrix:

$$e^{\frac{2\pi i}{\mathrm{K}^2} n_1 n_2 r_1'}$$




- Dimensional reduction to (2+1)d S-matrix (K=2):

$$(n_2, r_2) = (0,0) \qquad \mathbf{K} = 2\sigma_x \oplus 2\sigma_x$$

$$(n_2, r_2) = (1,1) \qquad \mathbf{K} = 2\sigma_z \oplus 2\sigma_z$$

$$(n_2, r_2) = (1,0)$$

$$\mathbf{K} = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

## Summary

- We have carried out the calculations of S and T matrices in (2+1)d surface of (3+1)d BF theories.

- Proposed a field theory model realzing 3-loop braiding statistics S and T matrices computed from boundary

- Bulk-boundary correspondence was established by extracting S and T matrices from boundary theory

- Other cases -- 4 loop braiding statistics etc.