Universality of charge transport in interacting fermionic systems

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"Condensed matter and critical phenomena"
Laboratori Nazionali di Frascati

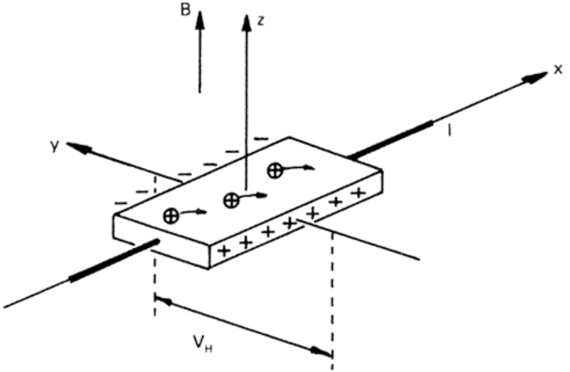
Outline

- Motivations: understand charge trasport in interacting $2d$ systems.
- Results:
  1. Integer quantum Hall effect for interacting fermionic systems
  2. Hall transitions in the Haldane-Hubbard model.
- Sketch of the proof.
- Conclusions.

Introduction

Integer quantum Hall effect

- $2d$ condensed matter systems display remarkable transport properties.
- Paradigmatic example: Integer quantum Hall effect (IQHE).
- Setting. Thin samples of suitable insulators, at low temperatures, exposed to strong magnetic field B and weak electric field E.



Integer quantum Hall effect

- $2d$ condensed matter systems display remarkable transport properties.
- Paradigmatic example: Integer quantum Hall effect (IQHE).
- J = current generated by weak field E. Linear response: $J_i = \sigma_{ij} E_j$.

$$\sigma_{11} = \sigma_{22} = 0\,, \qquad \sigma_{12} = -\sigma_{21} \in \frac{e^2}{h} \cdot \mathbb{Z}\,.$$

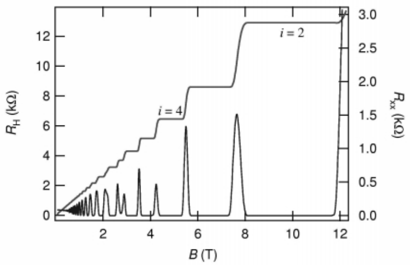


Figure: The IQHE. (von Klitzing, Nobel prize 1985.)

Theory: noninteracting particles

Thouless - Kohmoto - Nightingale - Den Nijs '82, Avron - Seiler - Simon '83, '94, Bellissard - van Elst - Schulz-Baldes '94, Aizenman - Graf '98...

- $H = H_0 + \lambda W$ = one-particle Schrödinger operator on $\ell^2(\mathbb{Z}^2)$.
  H_0 = magnetic lattice Laplacian, W = random local potential.

  $$H_0(x;y) = e^{i\phi_{xy}}\delta_{|x-y|,1}\ , \qquad W(x;y) = w_x\delta_{|x-y|,0}\ ,$$

  with w_x = i.i.d. random variables.

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- Let $P_\mu = \chi(H \leq \mu)$ = Fermi projector. If $\mathbf{E}|P_\mu(x; y)| \leq Ce^{-c|x-y|}$:

  $$\sigma_{12} = \frac{ie^2}{\hbar} \mathrm{Tr}\, P_\mu[[X_1, P_\mu], [X_2, P_\mu]] \in \frac{e^2}{h} \cdot \mathbb{Z}$$

  with $\mathrm{Tr}\,\cdot = \lim_{|\Lambda|\to\infty} |\Lambda|^{-1} \mathrm{tr}\,\cdot\; \chi(x \in \Lambda)$ = trace per unit volume.

- P_μ decays exp. if $\mu \in$ spectral gap, or $\mu \in$ mobility gap (strong disorder).

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- P_μ decays exp. if $\mu \in$ spectral gap, or $\mu \in$ mobility gap (strong disorder).
- If no disorder: $\sigma_{12} =$ Chern number of Bloch bundle.

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- Today.
  1. Universality of σ_{ij} for weakly interacting fermionic systems.
  2. Hall transitions in the Haldane-Hubbard model (gapless limit).

IQHE for interacting systems

Fermions on the lattice

- $\Lambda_L = 2d$ Bravais lattice, periodic b.c. (e.g. square, honeycomb lattice).

- Fock space fermionic operators: $a^{\pm}_{x,\alpha}$, with $\alpha = 1, \ldots, N$ "color" index (e.g. spin, sublattice).

- Fock space Hamiltonian: $\mathcal{H} = \mathcal{H}^{(0)} + U\mathcal{V}$, where

$$\begin{aligned} \mathcal{H}^{(0)} &= \sum_{x,y} \sum_{\alpha,\alpha'} a^{+}_{x,\alpha} H^{(0)}_{\alpha\alpha'}(x,y) a^{-}_{y,\alpha'} \,, \qquad \big(H^{(0)}(x,y) \equiv H^{(0)}(x-y)\big) \\ \mathcal{V} &= \sum_{x,y} \sum_{\alpha,\alpha'} n_{x,\alpha} v_{\alpha\alpha'}(x-y) n_{y,\alpha'} \,, \qquad (n_{x,\alpha} = a^{+}_{x,\alpha} a^{-}_{x,\alpha}) \end{aligned}$$

  $H^{(0)}(x-y) =$ short-range hopping, $v(x-y) =$ short-range interaction.

- For $k \in \mathbb{T}^2$, Bloch Hamiltonian: $\hat{H}^{(0)}(k) = \sum_z e^{ik\cdot z} H^{(0)}(z)$.
  Assumption: the spectrum of $\hat{H}^{(0)}(k)$ is gapped.

Conductivity

- Finite temperature, finite volume Gibbs state:

$$\langle \cdot \rangle_{\beta,L} = \frac{\mathrm{Tr}\ \cdot e^{-\beta(\mathcal{H}-\mu\mathcal{N})}}{\mathcal{Z}_{\beta,L}}\ .$$

- Conductivity defined via Kubo formula ($e^2 = \hbar = 1$):

$$\sigma_{ij} := \lim_{\eta\to 0^+} \frac{i}{\eta}\Big(\int_{-\infty}^{0} dt\, e^{\eta t}\, \langle\!\langle \big[e^{i\mathcal{H}t}\mathcal{J}_i e^{-i\mathcal{H}t}, \mathcal{J}_j \big] \rangle\!\rangle_\infty - \langle\!\langle \big[\mathcal{J}_i, \mathcal{X}_j \big] \rangle\!\rangle_\infty \Big)$$

where $\mathcal{X} = \sum_{x,\alpha} x\, n_{x,\alpha}$ = 2nd quantization of position operator and

$$\mathcal{J} := i\big[\mathcal{H},\, \mathcal{X}\big] = \text{current operator}\ , \quad \langle\!\langle \cdot \rangle\!\rangle_\infty = \lim_{\beta,L\to\infty} L^{-2} \langle \cdot \rangle_{\beta,L}.$$

- Kubo formula: linear response at $t = 0$, after introducing a weak external field $e^{\eta t} E \cdot \mathcal{X}$ at $t = -\infty$ (Bru-Pedra '14: derivation for finite times)

Stability of IQHE

Theorem (Giuliani, Mastropietro, P. - Comm. Math. Phys. '16.)

Let $\mu \notin \sigma(\hat{H}^{(0)}(k))$. *There is* $U_0 > 0$ *s.t. for* $U \in (-U_0, U_0)$ *and* $\beta, L \to \infty$:

$$\sigma_{ij} = \sigma_{ij} \,|_{U=0}$$

In particular, $\sigma_{ii} = 0$ *and* $\sigma_{12} = -\sigma_{21} \in (e^2/h) \cdot \mathbb{Z}$.

- The conductivity matrix is equal to the noninteracting one.

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- Strategy.
  1. Construction of Euclidean correlations (cluster expansion).
  2. Wick rotation to imaginary times ($\rightsquigarrow$ Euclidean conductivity).
  3. Universality of Euclidean conductivity matrix. Inspired by:

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  Coleman-Hill '85: "no corrections beyond 1-loop to the topological mass in QED_{2+1}."
- Here, $U_0 \equiv U_0(\text{gap})$. What about Hall transitions (where $U \gg$ gap)?

The Haldane-Hubbard model

Graphene

- First realisation of a 2d crystal (Geim-Novoselov, Nobel prize 2010).
- Simplest model: Laplacian on the honeycomb lattice.

$$\mathcal{H}_{\rm G}^{(0)} = t_1 \sum_{x,\sigma} \left[a^+_{x,A,\sigma} a^-_{x,B,\sigma} + a^+_{x,A,\sigma} a^-_{x-\ell_1,B,\sigma} + a^+_{x,A,\sigma} a^-_{x-\ell_2,B,\sigma} + h.c. \right]$$

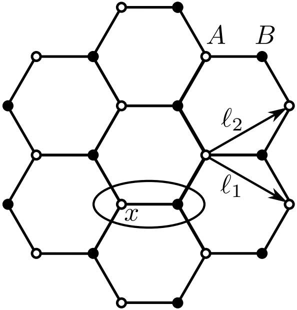
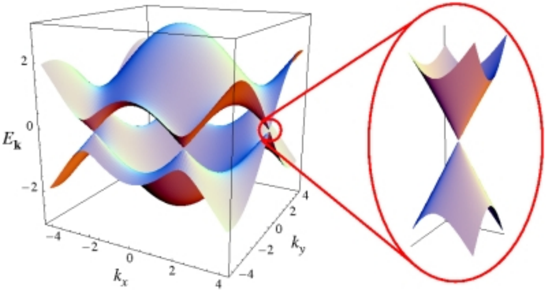


Figure: Dimer $\rightsquigarrow (a^\pm_{x,A,\sigma},\, a^\pm_{x,B,\sigma})$.

Dirac cones

- The spectrum is gapless:



- Fermi level: $\mu = 0$ corresponds to undoped graphene (half-filling).
- Low energy excitations: $2d$ massless Dirac fermions (with $v \ll c$).
- "Relativistic" charge carriers, remarkable transport properties.

Universal longitudinal conductivity

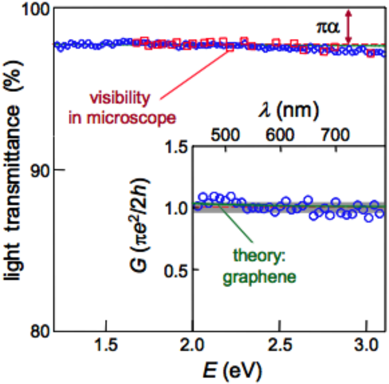


Figure: Graphene's longitudinal conductivity [Nair et al., Science '08].

Same value predicted by $2d$ massless Dirac fermions!

The Haldane model

- Haldane '88. Graphene + nnn hopping + staggered potential.

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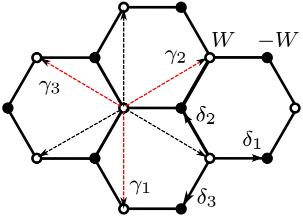
- Black: $t_2 e^{i\phi}$. Red: $t_2 e^{-i\phi}$
- Zero net magnetic flux.



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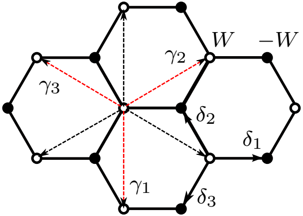

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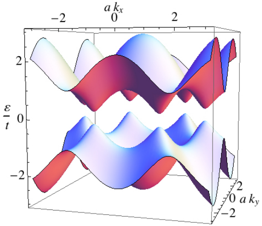
- Gapped system. Gaps:

$$\Delta_\pm = |m_\pm| \,, \quad m_\pm = W \pm 3\sqrt{3} t_2 \sin\phi.$$

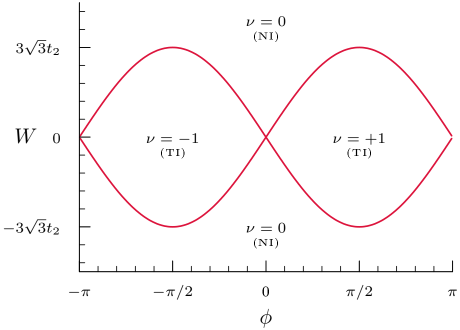
= "mass" of Dirac fermions.



Phase diagram

- IQHE without net external magnetic flux:

$$\sigma_{12} = \frac{2e^2}{h}\nu\,, \qquad \nu = \frac{1}{2}[\mathrm{sgn}(m_-) - \mathrm{sgn}(m_+)]$$



- Simplest model of topological insulator.
  Building brick for more complex systems (*e.g.* Kane-Mele model).

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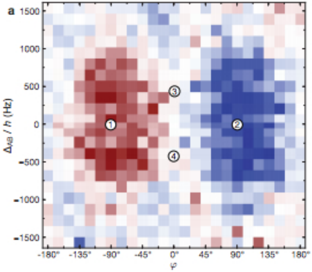

Figure: Experimental realization (Esslinger group, Nature '14)

- What is the effect of many-body interactions on the phase diagram?

Phase transitions in the Haldane-Hubbard model

Theorem (Giuliani, Jauslin, Mastropietro, P. - arXiv 2016)

There exists $U_0 > 0$ and a function ("renormalized mass")

$$m_{R,\omega} = m_\omega + F_\omega(m_\pm; U) \quad \textit{where} \quad F_\omega = O(U), \quad F_\omega\big|_{m_\pm=0} = 0\,, \quad \omega = \pm$$

such that, for $U \in (-U_0, U_0)$, choosing $\mu = \mu(m_\pm; U)$:

$$\Big[\lim_{m_{R,\omega}\to 0^+} - \lim_{m_{R,\omega}\to 0^-}\Big]\sigma_{12} = \frac{2e^2}{h}\omega$$

$$\sigma_{ii}^{cr} := \lim_{\eta\to 0^+}\lim_{m_{R,\omega}\to 0}\sigma_{ii}(\eta) = \frac{e^2}{h}\frac{\pi}{4}\,.$$

- U_0 is now uniform in the gap.
- $m_{R,\pm} = 0$: renormalized transition curves.

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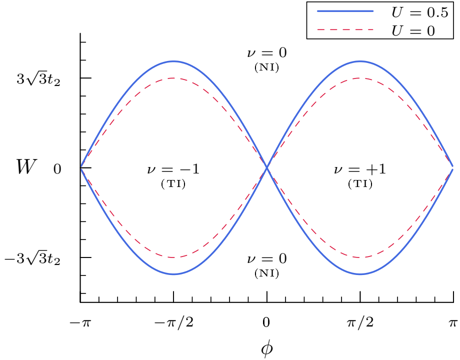
- If $m_{R,+} = m_{R,-} \to 0$, $\sigma_{ii}^{\text{cr}} = (e^2/h)(\pi/2)$. Same as interacting graphene:

  Giuliani, Mastropietro, P. - Phys. Rev. B '11, Comm. Math. Phys. '12.

Renormalized transition curves



- Away from the blue curve the correlations decay exponentially fast. On the blue curve the decay is algebraic.
- Method: constructive fermionic RG, combined with Ward identities. [Brydges-Battle-Federbush, Gawedzki-Kupiainen, Benfatto-Gallavotti-Mastropietro, Feldman-Knörrer-Salmhofer-Trubowitz, Magnen-Rivasseau-Sénéor ...]

Sketch of the proofs

Wick rotation &
universality for the critical Haldane-Hubbard model

Wick rotation

- Let us define the Euclidean conductivity matrix as:

$$\overline{\sigma}_{ij} := -\lim_{\eta\to 0^+}\frac{1}{\eta}\Big[\widehat{K}_{ij}(\eta)-\widehat{K}_{ij}(0)\Big]$$

where, setting $\mathcal{J}(-it) := e^{\mathcal{H}t}\mathcal{J}e^{-\mathcal{H}t}$ = imaginary time evolution of $\mathcal{J}$:

$$\widehat{K}_{ij}(\eta) = \lim_{\beta,L\to\infty}\frac{1}{L^2}\int_{-\beta/2}^{\beta/2} dt\, e^{-i\eta t}\big\langle \mathbf{T}\,\mathcal{J}_i(-it)\,;\mathcal{J}_j\big\rangle_{\beta,L}$$

($\mathbf{T}$ = time ordering).

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- Euclidean correlations can be studied via cluster expansion and RG methods. For weak interactions:

$$\Big|\frac{1}{L^2}\big\langle \mathbf{T}\,\mathcal{J}_i(-it)\,;\mathcal{J}_j(-is)\big\rangle_{\beta,L}\Big| \le \frac{C_M}{1+|t-s|^M} \qquad (\beta, L\to\infty)$$

$\forall M>0$ if $\mu\notin\sigma(\hat{H}^{(0)}(k))$ or $M=2$ for conical intersections.

Wick rotation

- We would like to show that, for $U \in (-U_0, U_0)$:

$$
\begin{aligned}
\overline{\sigma}_{ij} &:= -\lim_{\eta\to 0^+} \frac{1}{\eta}\int_{-\infty}^{\infty} dt\,(e^{-i\eta t}-1)\langle\!\langle \mathbf{T}\, e^{t\mathcal{H}}\mathcal{J}_i e^{-t\mathcal{H}}\,;\mathcal{J}_j\rangle\!\rangle_\infty \\
&= \lim_{\eta\to 0^+}\frac{i}{\eta}\Big(\int_{-\infty}^{0} dt\, e^{\eta t}\,\langle\!\langle\big[e^{i\mathcal{H}t}\mathcal{J}_i e^{-i\mathcal{H}t}\,,\,\mathcal{J}_j\big]\rangle\!\rangle_\infty - \langle\!\langle\big[\mathcal{J}_i\,,\,\mathcal{X}_j\big]\rangle\!\rangle_\infty\Big) \\
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- Proof based on complex deformation, for $\eta > 0$. We'll show:

$$\frac{1}{\eta}\int_{-\infty}^{\infty} dt\, e^{-i\eta t}\langle\!\langle \mathbf{T}\, e^{t\mathcal{H}}\mathcal{J}_i e^{-t\mathcal{H}}\,;\mathcal{J}_j\rangle\!\rangle_\infty = \frac{i}{\eta}\int_{-\infty}^{0} dt\, e^{\eta t}\,\langle\!\langle\big[e^{i\mathcal{H}t}\mathcal{J}_i e^{-i\mathcal{H}t}\,,\,\mathcal{J}_j\big]\rangle\!\rangle_\infty$$

(part of the statement is that the r.h.s. exists.)

Wick rotation

- Fix T, $\eta \in \mathbb{R}^+$. We have:

$$\int_{-T}^{T} dt\, e^{-i\eta t} \langle\!\langle \mathbf{T}\, \mathcal{J}_i(-it)\, ; \mathcal{J}_j \rangle\!\rangle_\infty$$

$$= \int_{-T}^{0} dt\, e^{-i\eta t} \langle\!\langle \mathcal{J}_j \mathcal{J}_i(-it) \rangle\!\rangle_\infty + \int_{0}^{T} dt\, e^{-i\eta t} \langle\!\langle \mathcal{J}_i(-it) \mathcal{J}_j \rangle\!\rangle_\infty \equiv I_1 + I_2\ .$$

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- Consider I_2. Fix $\varepsilon > 0$. Claim: up to $O(\varepsilon)$,

$$I_2 = \Big[-i \int_{-T+\varepsilon}^{0} e^{\eta(t - i\varepsilon)} \langle\!\langle \mathcal{J}_i(t - i\varepsilon) \mathcal{J}_j \rangle\!\rangle_\infty + \text{boundary term} \Big]$$

Wick rotation

- Fix T, $\eta \in \mathbb{R}^+$. We have:

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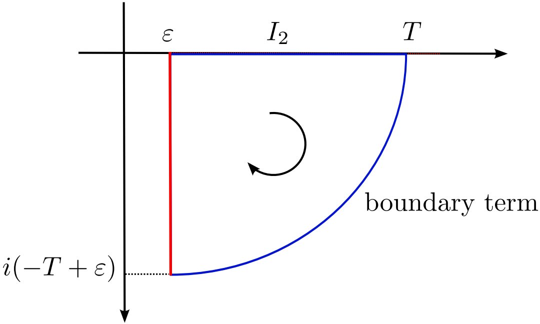

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- Bound for correlations at complex times: for $\mathrm{Re}\, z \geq 0$,

$$\begin{aligned} \Big| \langle \mathcal{J}_j(-iz)\mathcal{J}_i(0)\rangle_{\beta,L} \Big| &\leq \Big| \langle \mathcal{J}_j(-i\mathrm{Re}\, z)\mathcal{J}_j(0)\rangle_{\beta,L} \Big|^{1/2} \Big| \langle \mathcal{J}_i(-i\mathrm{Re}\, z)\mathcal{J}_i(0)\rangle_{\beta,L} \Big|^{1/2} \\ &\leq \frac{L^2 C_M}{1 + |\mathrm{Re}\, z|^M} \qquad (M \geq 2) \end{aligned}$$

Wick rotation

$$\left|\frac{1}{L^2}\langle \mathcal{J}_j(-iz)\mathcal{J}_i(0)\rangle_{\beta,L}\right| \leq \frac{C_M}{1+|\mathrm{Re}\, z|^M} \qquad (M \geq 2, \quad \mathrm{Re}\, z \geq 0) \qquad (*)$$

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  3. existence of the β, $L \to \infty$ limit on positive real axis.

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Wick rotation

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- Repeating the same analysis for I_1:

$$I_1 + I_2 = -i \lim_{\varepsilon\to 0}\int_{-\infty}^{0} dt\, e^{\eta t} \langle\!\langle [\mathcal{J}_i(t\mp i\varepsilon),\, \mathcal{J}_j] \rangle\!\rangle_\infty = -i\int_{-\infty}^{0} dt\, e^{\eta t}\langle\!\langle [\mathcal{J}_i(t),\, \mathcal{J}_j]\rangle\!\rangle_\infty$$

by Lieb-Robinson bounds, $(*)$ and dominated convergence. ■

Universality in the Haldane-Hubbard model

- Noninteracting theory. Euclidean conductivity:

$$\begin{aligned}\overline{\sigma}_{ij} &= -\lim_{\eta\to 0^+}\frac{1}{\eta}\int_{-\infty}^{\infty} dt\,\big(e^{-i\eta t}-1\big)\langle\!\langle \mathbf{T}\,\mathcal{J}_i(-it)\,;\mathcal{J}_j\rangle\!\rangle_\infty \\ &\equiv -\lim_{\eta\to 0^+}\frac{1}{\eta}[\widehat{K}_{ij}(\eta)-\widehat{K}_{ij}(0)]\,.\end{aligned}$$

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- The state is quasi-free. By Wick rule:

$$\begin{aligned} \widehat{K}_{ij}(\eta) &= (-1)\int_{\mathbb{R}\times\mathbb{T}^2} \mathrm{tr}\,\Gamma_i(k)\,g(\mathbf{k}+\eta)\,\Gamma_j(k)\,g(\mathbf{k}) \\ g(\mathbf{k})^{-1} &= -\begin{pmatrix} ik_0-m(k) & t_1\Omega^*(k) \\ t_1\Omega(k) & ik_0+m(k)\end{pmatrix}\,, \qquad \mathbf{k}=(k_0,k)\,, \end{aligned}$$

where, for k' small:

$$\Omega(k'+k_F^\omega)\simeq\frac{3}{2}\big(ik_1'+\omega k_2'\big) \qquad m(k'+k_F^\omega)\simeq m_\omega \qquad \Gamma_i\simeq \text{Pauli matrices}$$

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- In presence of interactions, this is just the zero-th order.

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$$\equiv -\lim_{\eta\to 0^+}\frac{1}{\eta}[\widehat{K}_{ij}(\eta)-\widehat{K}_{ij}(0)]\,.$$

- The state is not quasi-free. By RG methods:

$$\widehat{K}_{ij}(\eta) = (-1)\int_{\mathbb{R}\times\mathbb{T}^2}\operatorname{tr}\Gamma_{i,R}(k)\,g_R(\mathbf{k}+\eta)\,\Gamma_{j,R}(k)\,g_R(\mathbf{k})+\text{h.o.t.}$$
$$g_R(\mathbf{k})^{-1} \simeq -\begin{pmatrix} iZ_1k_0-m_R(k) & v_R\Omega^*(k)\\ v_R\Omega(k) & iZ_2k_0+m_R(k)\end{pmatrix}$$

where, for k' small:

$$m_R(k'+k_F^\omega)\simeq m_{R,\omega} \qquad \Gamma_{i,R}\simeq \text{“renormalized” Pauli matrices}$$
$$(Z_i,\,v_R,\,m_{R,\omega}) = (1,\,t_1,\,m_\omega)+\text{convergent series in } U$$

Universality in the Haldane-Hubbard model

$$\overline{\sigma}_{ii} = -\lim_{\eta \to 0^+} \frac{1}{\eta}[\widehat{K}_{ii}(\eta) - \widehat{K}_{ii}(0)]$$

- Crucial remark: $\widehat{K}_{ii}(\eta) = \widehat{K}_{ii}(-\eta)$. Differentiability $\Rightarrow \overline{\sigma}_{ii} = 0$.

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- If $m_{R,\omega} = 0$ then $\widehat{K}_{ii}(\eta)$ is not differentiable. Convenient rewriting:

$$\widehat{K}_{ii}(\eta) = \int_{\mathbb{R}\times B_\varepsilon(k_F^\omega)} \mathrm{tr}\,\Gamma_{i,R}(k)\, g_R(\mathbf{k}+\eta)\,\Gamma_{i,R}(k)\, g_R(\mathbf{k}) + \widetilde{K}_{ii}(\eta) \equiv I_1 + I_2$$

I_2 is differentiable (interaction RG-irrelevant) $\Rightarrow \overline{\sigma}_{ii}$ only depends on I_1.

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- Term I_1 still depends on $Z_{i,R}$, v_R, $m_{R,\omega}$, $\Gamma_{i,R}$. Ward identity:

$$\Gamma_{i,R}(k_F^\omega) = -\partial_i g_R(\mathbf{k}_F^\omega)^{-1}\ , \qquad \mathbf{k}_F^\omega = (0, k_F^\omega)\ ,$$

  following from $U(1)$ gauge invariance. All parameters cancel out!

$$\boxed{\overline{\sigma}_{ii}^{\mathrm{cr}} = \frac{1}{8}} \qquad (e^2 = \hbar = 1) \qquad \blacksquare$$

Conclusions

- We discussed the transport properties of interacting fermionic systems, on two-dimensional lattices.
- We proved:
  - the stability of the IQHE for general interacting gapped systems
  - the universality of the conductivity matrix of the critical Haldane-Hubbard model.

  Tools: LR bounds, determinant bounds, rigorous RG, Ward identities.
- Open questions:
  - Spin transport in time-reversal invariant $2d$ insulators (e.g., interacting Kane-Mele model)?
  - Interacting bulk-edge correspondence?
  - Effect of long-range interactions (e.g. Coulomb)?
  - ...

Thank you!

Universality of $\overline{\sigma}_{ij}$: Schwinger-Dyson equation

- For U in the analyticity domain, the Schwinger-Dyson equation holds:

$$\begin{aligned} \widehat{K}^{(U)}_{i,j}(\mathbf{p}) &= \widehat{K}^{(0)}_{i,j}(\mathbf{p}) \\ &+ \int_0^U dU' \int d\mathbf{q}\, \hat{v}(p) \widehat{K}^{(U')}_{i,j,0,0}(\mathbf{p}, -\mathbf{p}, \mathbf{q}) \\ &+ 2\int_0^U dU' \hat{v}(0) \widehat{K}^{(U')}_{i,j,0}(\mathbf{p}, -\mathbf{p}) \widehat{K}^{(U')}_{0} \\ &+ 2\int_0^U dU' \hat{v}(p) \widehat{K}^{(U')}_{i,0}(\mathbf{p}) \widehat{K}^{(U')}_{j,0}(-\mathbf{p}) \end{aligned}$$

with $\mathbf{p} = (\eta, p) \in \mathbb{R}^3$ and

$$\widehat{K}^{(U')}_{i,j,0,0}(\mathbf{p}, -\mathbf{p}, \mathbf{q}) = \langle \mathbf{T}\, J_{i,\mathbf{p}}\, ; J_{j,-\mathbf{p}}\, ; n_{\mathbf{q}}\, ; n_{-\mathbf{q}} \rangle\ .$$

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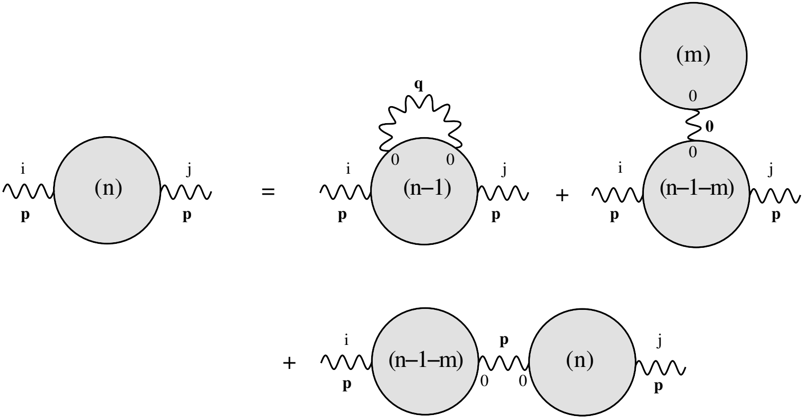

Universality of $\overline{\sigma}_{ij}$: Ward identities

- Continuity equation (recall $O(-it) = e^{t\mathcal{H}} O e^{-t\mathcal{H}}$):

$$\partial_t n_p(-it) := \left[\mathcal{H}, n_p(-it)\right] = p \cdot J_p(-it)$$

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- It implies relations among correlations: Ward identities. E.g.:

$$\eta \widehat{K}_{0,0}(\mathbf{p}) + p_i \widehat{K}_{i,0}(\mathbf{p}) = 0 \Rightarrow \widehat{K}_{j,0}(\mathbf{p}) = -\eta \frac{\partial}{\partial p_j} \widehat{K}_{0,0}(\mathbf{p}) - p_i \frac{\partial}{\partial p_j} \widehat{K}_{i,0}(\mathbf{p})$$
$$= O(\mathbf{p}) \; .$$

Similarly, $\widehat{K}_{i,j,0,0}(\mathbf{p}, -\mathbf{p}, \mathbf{q}) = O(\mathbf{p}^2)$, $\widehat{K}_{i,j,0}(\mathbf{p}, -\mathbf{p}) = O(\mathbf{p}^2)$. Therefore,

$$\widehat{K}^{(U)}_{i,j}(\mathbf{p}) - \widehat{K}^{(0)}_{i,j}(\mathbf{p}) = O(\mathbf{p}^2) \; .$$

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Similarly, $\widehat{K}_{i,j,0,0}(\mathbf{p}, -\mathbf{p}, \mathbf{q}) = O(\mathbf{p}^2)$, $\widehat{K}_{i,j,0}(\mathbf{p}, -\mathbf{p}) = O(\mathbf{p}^2)$. Therefore,

$$\widehat{K}^{(U)}_{i,j}(\mathbf{p}) - \widehat{K}^{(0)}_{i,j}(\mathbf{p}) = O(\mathbf{p}^2) \; .$$

- Since $\overline{\sigma}^{(U)}_{ij} = \lim_{\eta \to 0^+} (-1/\eta) \big[\widehat{K}^{(U)}_{i,j}(\eta, 0, 0) - \widehat{K}^{(U)}_{i,j}(\mathbf{0})\big] \equiv -\partial_\eta \widehat{K}^{(U)}_{i,j}(\mathbf{0})$,

$$\overline{\sigma}^{(U)}_{ij} = \overline{\sigma}^{(0)}_{ij} \; .$$

■

Universality of critical conductivity

- Let $0 < |m_{R,\omega}| \ll |m_{R,-\omega}|$. Let $\varepsilon > 0$ small. Convenient rewriting:

$$\overline{\sigma}_{12} = \int_{\mathbb{R}\times B_\varepsilon(k_F^\omega)} \mathrm{tr}\,\Gamma_{1,R}(k)\,\partial_{k_0} g_R(\mathbf{k})\,\Gamma_{2,R}(k)\,g_R(\mathbf{k}) + \widetilde{\sigma}_{12} \equiv I_1 + I_2$$

I_1 : Integrand $\sim [|\mathbf{k}-\mathbf{k}_F^\omega|^2 + m_{R,\omega}^2]^{-\frac{3}{2}} \Rightarrow$ integral not continuous in $m_{R,\omega}$

I_2 : $\widetilde{\sigma}_{12}$ is continuous in $m_{R,\omega}$ (interaction irrelevant in RG sense).

$\Rightarrow$ $\Delta_\omega = \big[\lim_{m_{R,\omega}\to 0^+} - \lim_{m_{R,\omega}\to 0^-}\big]\overline{\sigma}_{12}$ only determined by term I_1.

- Still, term I_1 depends on $Z_{i,R}$, v_R, $m_{\omega,R}$, $\Gamma_{i,R}$. Crucial ingredient:

$$\Gamma_{i,R}(k_F^\omega) = -\partial_i g_R(k_F^\omega)^{-1} \qquad \text{(Ward identity)}$$

following from $U(1)$ gauge invariance. All parameters cancel out!

$$\boxed{\Delta_\omega = \frac{\omega}{\pi}} \qquad (e^2 = \hbar = 1) \qquad \blacksquare$$