High-Energy Gravitational Scattering: from ACV to EOB

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50 Years of the Veneziano Model:
From Dual Models to Strings, M-theory and Beyond
Florence, Italy, 11-15 May 2018

ACV = Amati-Ciafaloni-Veneziano

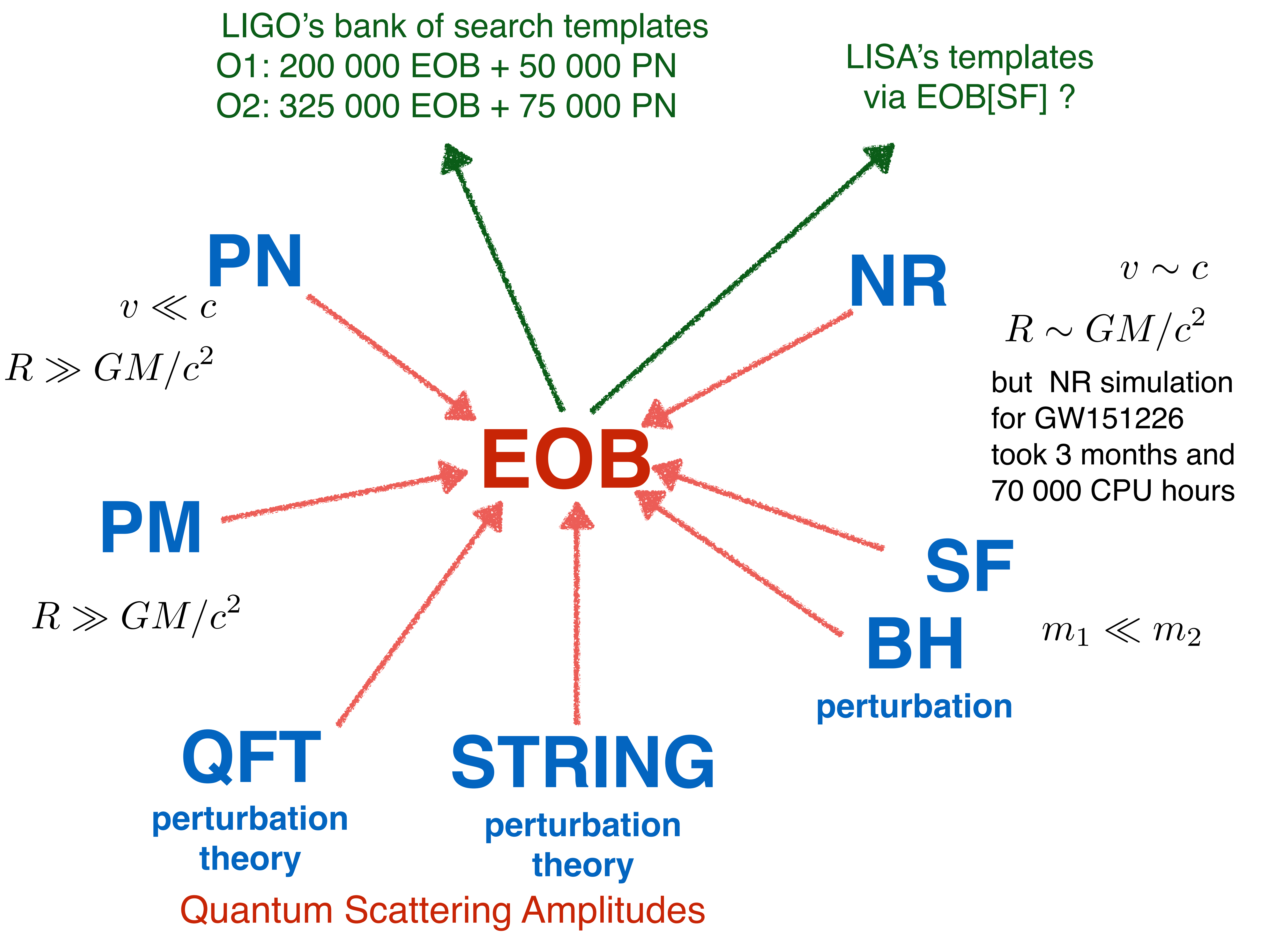
D. Amati, M. Ciafaloni and G. Veneziano,``Superstring Collisions at Planckian Energies,”Phys.\ Lett.\ B {\bf 197}, 81 (1987).

D. Amati, M. Ciafaloni and G. Veneziano, ``Higher Order Gravitational Deflection and Soft Bremsstrahlung in Planckian Energy Superstring Collisions," Nucl.\ Phys.\ B {\bf 347}, 550 (1990).

D. Amati, M. Ciafaloni and G. Veneziano,``Towards an S-matrix description of gravitational collapse,"JHEP {\bf 0802}, 049 (2008)

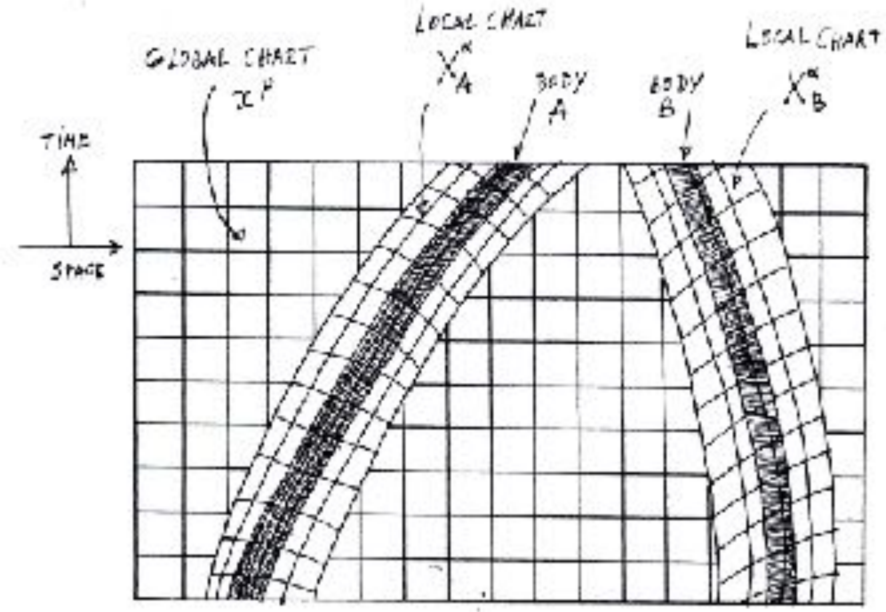
EOB= Effective-One-Body

A. Buonanno and T. Damour,
``Effective one-body approach to
general relativistic two-body dynamics,"
Phys.\ Rev.\ D \ bf{59}, 084006 (1999)


LIGO's bank of search templates
O1: 200 000 EOB + 50 000 PN
O2: 325 000 EOB + 75 000 PN
LISA's templates
via EOB[SF] ?
PN
$v \ll c$
$R \gg GM/c^2$
NR
$v \sim c$
$R \sim GM/c^2$
but NR simulation
for GW151226
took 3 months and
70 000 CPU hours
EOB
PM
$R \gg GM/c^2$
SF
BH
perturbation
$m_1 \ll m_2$
QFT
perturbation
theory
STRING
perturbation
theory
Quantum Scattering Amplitudes

Computing the Motion of NSs or BHs

Multi-chart approach to motion
of strong-self-gravity bodies,
and matched asymptotic expansions
[EIH '38], Manasse '63, Demianski-Grishchuk '74, D'Eath'75, Kates '80, Damour '82



Skeletonization : $T_{\mu\nu} \longrightarrow$ point-masses (Mathisson '31)

delta-functions in GR : Infeld '54, Infeld-Plebanski '60

Like in Dirac 1938, the use of point-particles (delta functions) introduces UV divergences (linked to self-field effects).

It has been shown (« Effacing property » Damour '83) that possible internal-structure dependence in strong self-gravity objects (NSs, BHs) only arise at 5PN= 5-loop level

Below 5PN (5-loop) point-masses in GR are renormalizable (Damour '83, Goldberger-Rothstein '06)

One still needs a regularization method that respects gravity's gauge invariance:
dimensional regularization D=4+ epsilon ('t Hooft-Veltman '72)

It has been explicitly shown that S_eff was UV finite (in ADM gauge, and in dim.reg.) at 3 loops (Damour-Jaranowski-Schäfer '01) and 4 loops (Damour-Jaranowski-Schäfer '14, Jaranowski-Schäfer '15)

There appear IR divergences at 4PN (4 loop) linked to non-locality (Blanchet-Damour '88).

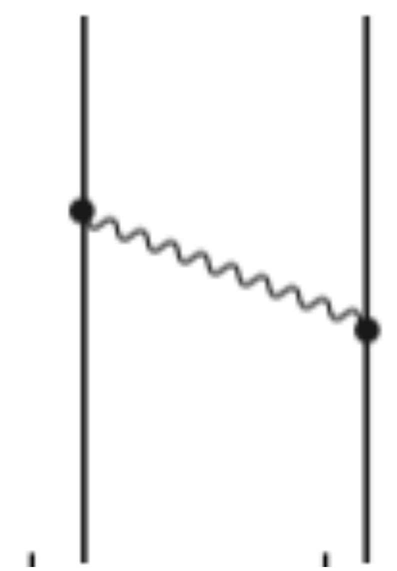
Fokker Action in Electrodynamics (1929)

$$S_{\rm tot}[x_a^\mu, A_\mu] = -\sum_a \int m_a ds_a + \sum_a \int e_a dx_a^\mu A_\mu(x_a) - \int d^D x \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + S_{\rm gf}$$

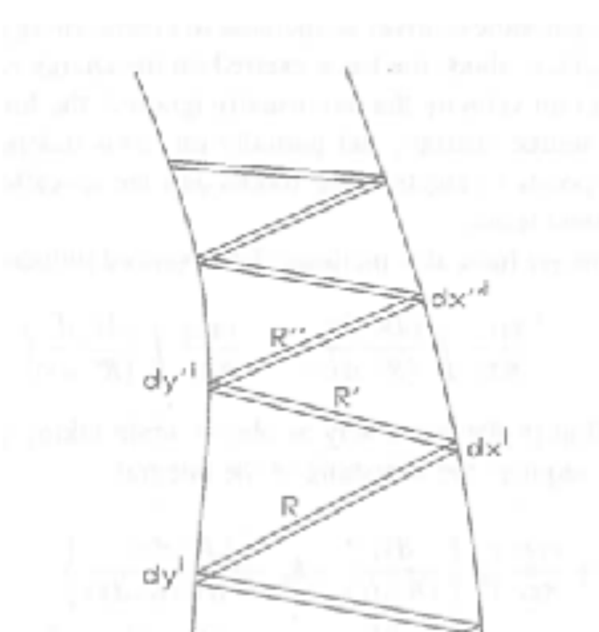
« Integrate out » the field A_mu in the total (particle+field) action

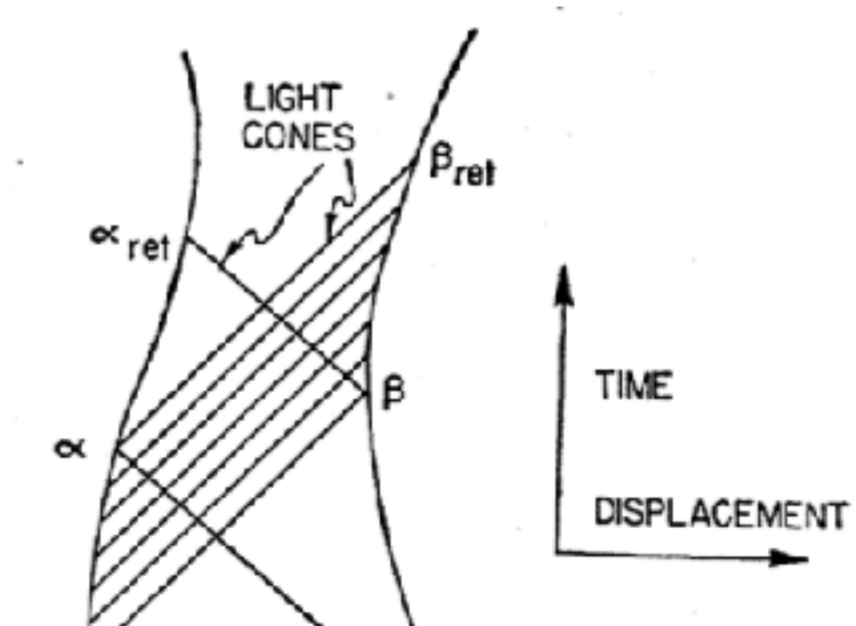
$$\boxed{\begin{aligned} S_{\rm eff}^{\rm class}[x_a(s_a)] = &-\sum_a m_a \int \mathrm{d}s_a \\ &+ \frac{1}{2}\sum_{a,b} e_a e_b \iint \mathrm{d}x_a^\mu \, \mathrm{d}x_{b\mu} \, \delta\left((x_a - x_b)^2\right) . \end{aligned}}$$

One-photon-exchange diagram



The effective action S_eff(x_a) was heavily used in the (second) Wheeler-Feynman paper (1949) together with similar diagrams to those used by Fokker





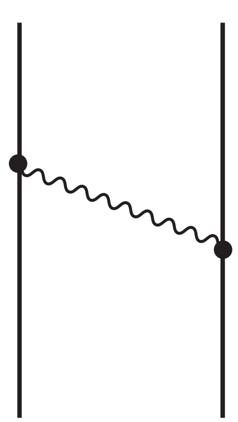
Fokker-type Action in Gravity and its Diagrammatic Expansion

either $S^{\rm class}_{\rm eff}[x_a(s_a)] = [S_{\rm pm} + S_{\rm EH} + S_{\rm gf}]_{g_{\mu\nu}(x) \to g^{\rm gf}_{\mu\nu}[x_a(s_a)]}$ Damour-Esposito-Farese ‘96

or $e^{\frac{i}{\hbar} S^{\rm quant}_{\rm eff}} = \int D g_{\mu\nu}\, e^{\frac{i}{\hbar}(S_{\rm pm} + S_{\rm EH} + S_{\rm gf})}$. Goldberger-Rothstein ’06

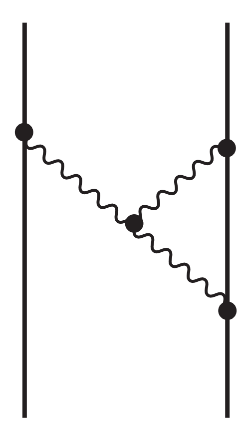
Needs gauge-fixed* action and time-symmetric Green function G.
*E.g. Arnowitt-Deser-Misner Hamiltonian formalism or harmonic coordinates.
Perturbatively solving (in dimension D=4 - eps) Einstein’s equations
to get the equations of motion and the action for the conservative dynamics

$$g = \eta + h$$
$$S(h,T) = \int \left(\frac{1}{2} h \Box h + \partial\partial hhh + \ldots + (h + hh + \ldots) T \right)$$
$$\Box h = -T + \ldots \to h = GT + \ldots$$
$$S_{\rm red}(T) = \frac{1}{2} TGT + V_3(GT, GT, GT) + \ldots$$



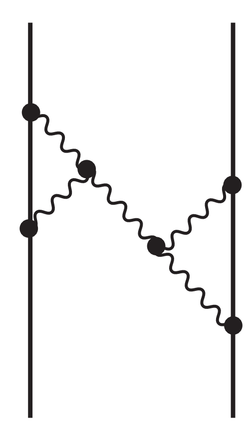
O(G)= Newtonian
+ (v/c)^n corrections

$+$



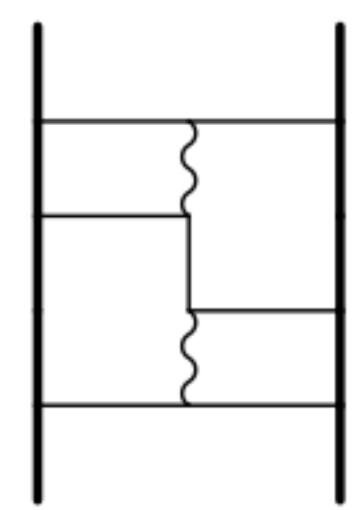
O(G^2)=1PN
=1 loop

$+$



O(G^3)=2PN
=2 loop

$+ \cdots$

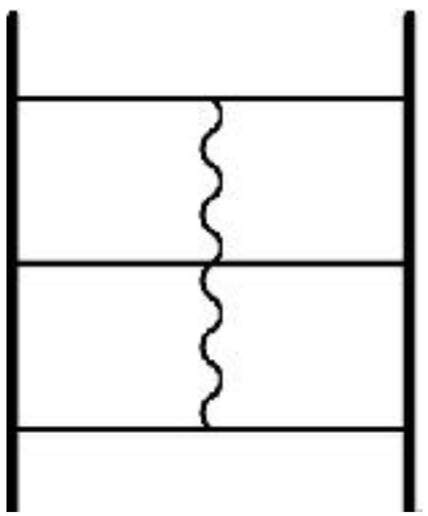


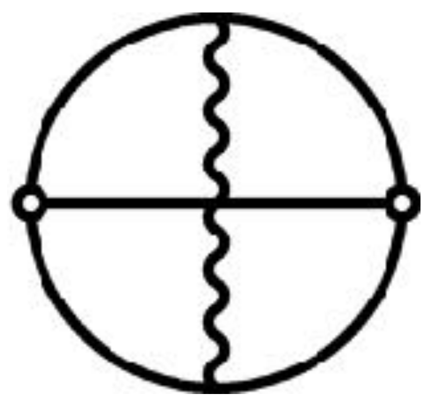
O(G^5)=4PN
=4 loop

PN computation of the Fokker-type Gravity Action

Slow Motion (PN) expansion: in powers of 1/c^2: 1PN= (v/c)^2; 2PN= (v/c)^4, etc nPN=(v/c)^(2n)

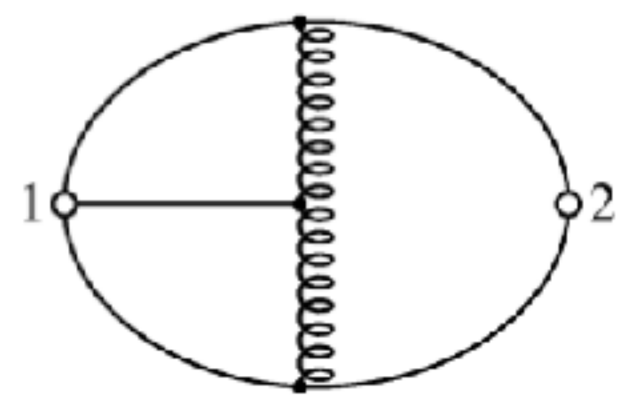
$$\boxed{\Box^{-1} = (\Delta - \frac{1}{c^2}\partial_t^2)^{-1} = \Delta^{-1} + \frac{1}{c^2}\partial_t^2\Delta^{-2} + \ldots}$$



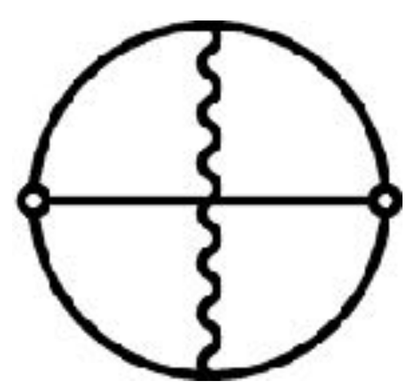


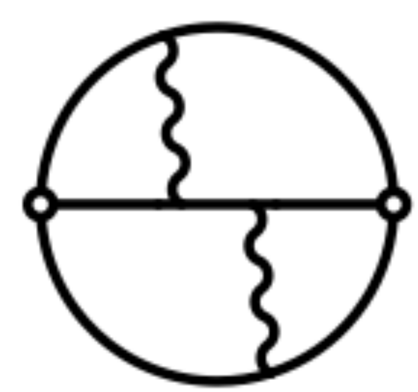
Use PN-expanded Green function for explicit computations.
This transforms spacetime diagrams (between two worldlines) into (massless) two-point space diagrams (in three dimensions)

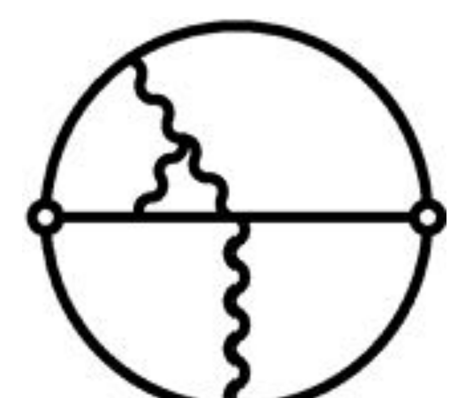
E.g. at 3PN, a 3-loop space diagram
~ G^4 m_1^3 m_2^2
(Damour-Jaranowski-Schaefer 2001)



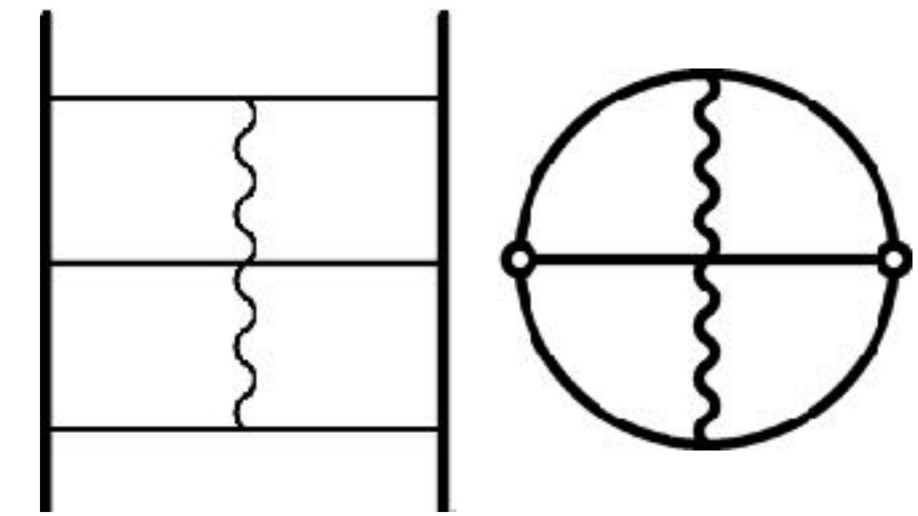
E.g. at 4PN, some 4-loop space diagrams
~ G^5 m_1^3 m_2^3
among 515 4PN-level diagrams,
(Damour-Jaranowski-Schaefer '14, Bernard et al '16, Foffa et al '17)

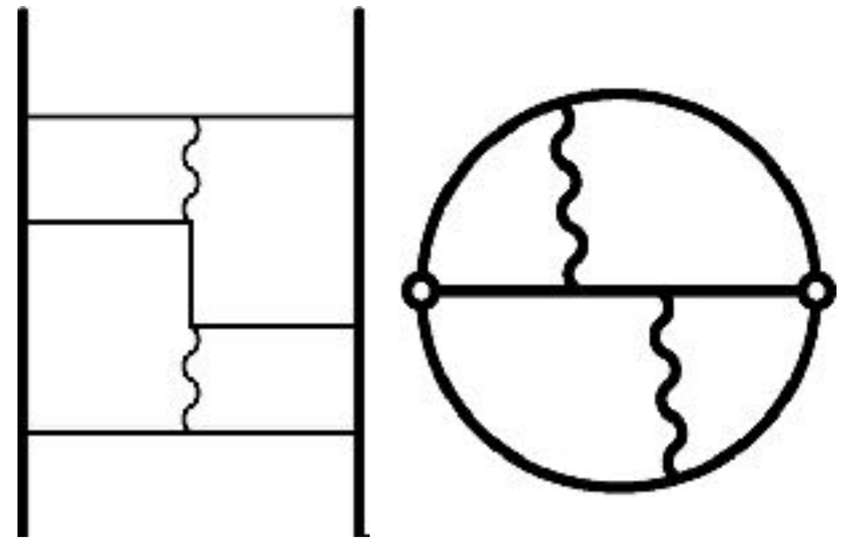


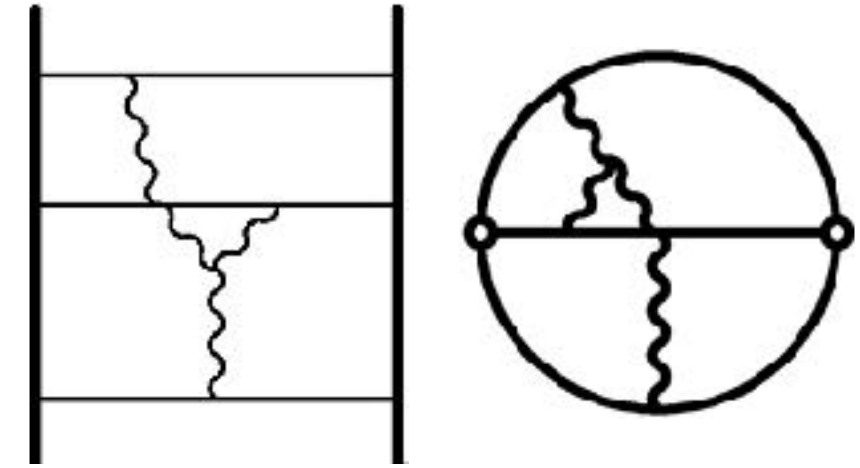




Four-loop static contribution to the gravitational interaction potential of two point masses (Foffa-Mastrolia-Sturani-Sturm'16, Damour-Jaranowski '17)



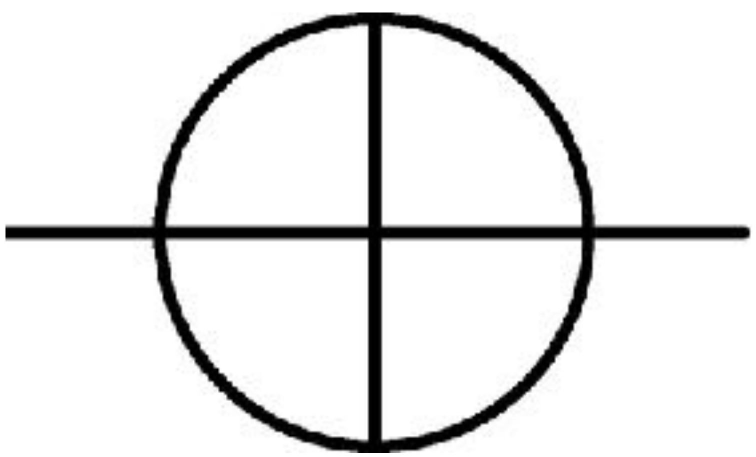




$$L_{33} = (32 - 2\pi^2)\frac{G^5 m_1^3 m_2^3}{c^8 r_{12}^5}.$$

$$L_{49} = (64 - 6\pi^2)\frac{G^5 m_1^3 m_2^3}{c^8 r_{12}^5}.$$

$$L_{50} = \left(-\frac{248}{3} + 8\pi^2\right)\frac{G^5 m_1^3 m_2^3}{c^8 r_{12}^5}.$$

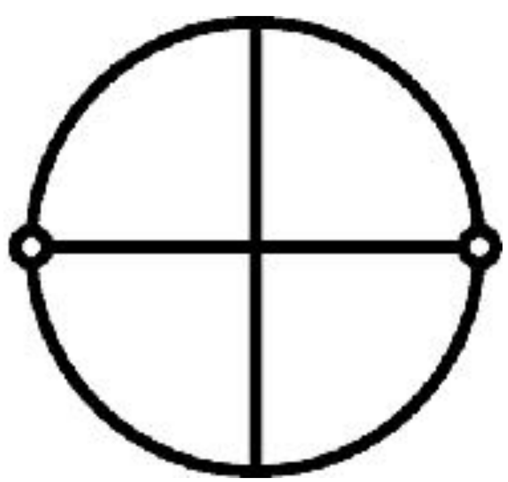


$$\mathcal{M}_{3,6}(\boldsymbol{p}) = \widehat{\mathcal{M}}_{3,6}|\boldsymbol{p}|^{4\varepsilon-4},$$

(num.) Lee-Mingulov'15, Damour-Jaranowski '17

where (with γ denoting Euler's constant)

$$\widehat{\mathcal{M}}_{3,6} = (4\pi)^{-4-2\varepsilon}\frac{e^{2\gamma\varepsilon}}{2}\left[\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} + \frac{\pi^2}{12} - 8 + O(\varepsilon)\right].$$



All integrals computable by the x-space generalized Riesz integral (Jaranowski-Schaefer '00)

$$\int \mathrm{d}^3x\, r_1^a r_2^b (r_1 + r_2 + r_{12})^c = 2\pi R(a,b,c)\, r_{12}^{a+b+c+3},$$

Post-Newtonian Equations of Motion [2-body, wo spins]

- 1PN (including v^2/c^2) [Lorentz-Droste '17], Einstein-Infeld-Hoffmann '38
- 2PN (inc. v^4/c^4) Ohta-Okamura-Kimura-Hiida '74, Damour-Deruelle '81
  Damour '82, Schäfer '85, Kopeikin '85
- 2.5 PN (inc. v^5/c^5) Damour-Deruelle '81, Damour '82, Schäfer '85,
  Kopeikin '85
- 3 PN (inc. v^6/c^6) Jaranowski-Schäfer '98, Blanchet-Faye '00,
  Damour-Jaranowski-Schäfer '01, Itoh-Futamase '03,
  Blanchet-Damour-Esposito-Farèse' 04, Foffa-Sturani '11
- 3.5 PN (inc. v^7/c^7) Iyer-Will '93, Jaranowski-Schäfer '97, Pati-Will '02,
  Königsdörffer-Faye-Schäfer '03, Nissanke-Blanchet '05, Itoh '09
- 4PN (inc. v^8/c^8) Jaranowski-Schäfer '13, Foffa-Sturani '13,'16
  Bini-Damour '13, Damour-Jaranowski-Schäfer '14, Bernard et al'16

New feature : non-locality in time

Inclusion of spin-dependent effects: Barker-O' Connell'75, Faye-Blanchet-Buonanno'06, Damour-Jaranowski-Schaefer'08, Porto-Rothstein '06, Levi '10, Steinhoff-Hergt-Schaefer '10, Steinhoff'11, Levi-Steinhoff'15-18

2-body Taylor-expanded N + 1PN + 2PN Hamiltonian

$$H_{\rm N}(\mathbf{x}_a,\mathbf{p}_a)=\frac{\mathbf{p}_1^2}{2m_1}-\frac{1}{2}\frac{Gm_1m_2}{r_{12}}+(1\leftrightarrow 2)$$

$$c^2H_{\rm 1PN}(\mathbf{x}_a,\mathbf{p}_a)=-\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(-12\frac{\mathbf{p}_1^2}{m_1^2}+14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$+\frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}}+(1\leftrightarrow 2),$$

$$c^4H_{\rm 2PN}(\mathbf{x}_a,\mathbf{p}_a)=\frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4}-\frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}-\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}+5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.$$
$$\left.-6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}-\frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)$$
$$+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2}+19\frac{\mathbf{p}_2^2}{m_2^2}\right)-\frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2)+6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2}+(1\leftrightarrow 2),$$

2-body Taylor-expanded 3PN Hamiltonian [DJS 01]

$$
\begin{aligned}
c^6 H_{3\mathrm{PN}}(\mathbf{x}_a,\mathbf{p}_a) = {} & -\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7} + \frac{1}{32}\frac{G m_1 m_2}{r_{12}}\left(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6} + 4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4 m_2^2} + 6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4 m_2^2}\right.\\
& - 10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2 + \mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4 m_2^2} + 24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4 m_2^2}\\
& + 2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3 m_2^3} + \frac{(7\mathbf{p}_1^2\mathbf{p}_2^2 - 10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2^3}\\
& + \frac{(\mathbf{p}_1^2\mathbf{p}_2^2 - 2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2^3} + 15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3 m_2^3}\\
& \left. - 18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3 m_2^3} + 5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3 m_2^3}\right) + \frac{G^2 m_1 m_2}{r_{12}^2}\left(\frac{1}{16}(m_1 - 27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}\right.\\
& - \frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2} - \frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2 + 371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} + \frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3} + \frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}\\
& - \frac{1}{8}m_1\frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2) + 11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3 m_2} - \frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2}\\
& + \frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2 m_2^2} + \frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\\
& \left. - \frac{1}{48}(220m_1 + 193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2}\right) + \frac{G^3 m_1 m_2}{r_{12}^3}\left(-\frac{1}{48}\left(425m_1^2 + \left(473 - \frac{3}{4}\pi^2\right)m_1 m_2 + 150m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.\\
& + \frac{1}{16}\left(77(m_1^2 + m_2^2) + \left(143 - \frac{1}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \frac{1}{16}\left(20m_1^2 - \left(43 + \frac{3}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}\\
& \left. + \frac{1}{16}\left(21(m_1^2 + m_2^2) + \left(119 + \frac{3}{4}\pi^2\right)m_1 m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2}\right)\\
& + \frac{1}{8}\frac{G^4 m_1 m_2^3}{r_{12}^4}\left(\left(\frac{227}{3} - \frac{21}{4}\pi^2\right)m_1 + m_2\right) + (1 \leftrightarrow 2).
\end{aligned}
$$

2-body Taylor-expanded 4PN Hamiltonian [DJS, 2014]

$$
\begin{aligned}
c^8 H^{\rm local}_{\rm 4PN}(\mathbf{x}_a,\mathbf{p}_a) =& \frac{7(\mathbf{p}_1^2)^5}{256 m_1^9} + \frac{G m_1 m_2}{r_{12}} H_{48}(\mathbf{x}_a,\mathbf{p}_a) + \frac{G^2 m_1 m_2}{r_{12}^2} m_1 H_{46}(\mathbf{x}_a,\mathbf{p}_a) \\
&+ \frac{G^3 m_1 m_2}{r_{12}^3}\left(m_1^2 H_{441}(\mathbf{x}_a,\mathbf{p}_a) + m_1 m_2 H_{442}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
&+ \frac{G^4 m_1 m_2}{r_{12}^4}\left(m_1^3 H_{421}(\mathbf{x}_a,\mathbf{p}_a) + m_1^2 m_2 H_{422}(\mathbf{x}_a,\mathbf{p}_a)\right) \\
&+ \frac{G^5 m_1 m_2}{r_{12}^5} H_{40}(\mathbf{x}_a,\mathbf{p}_a) + (1 \leftrightarrow 2),
\end{aligned}
\tag{A3}
$$

$$
\begin{aligned}
H_{48}(\mathbf{x}_a,\mathbf{p}_a) =& \frac{45(\mathbf{p}_1^2)^4}{128 m_1^8} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{64 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^3}{64 m_1^6 m_2^2} - \frac{9(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{16 m_1^6 m_2^2} \\
&- \frac{3(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{32 m_1^6 m_2^2} + \frac{15(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{21(\mathbf{p}_1^2)^3\mathbf{p}_2^2}{64 m_1^6 m_2^2} - \frac{35(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{256 m_1^5 m_2^3} \\
&+ \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{128 m_1^5 m_2^3} + \frac{33(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1^2)^2}{256 m_1^5 m_2^3} - \frac{85(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} \\
&- \frac{45(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2^3} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2^3} + \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} \\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{64 m_1^5 m_2^3} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{64 m_1^5 m_2^3} + \frac{55(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} \\
&- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{25(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{23(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} \\
&+ \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{128 m_1^5 m_2^3} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{256 m_1^5 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{64 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4(\mathbf{p}_1^2)^2}{64 m_1^4 m_2^4} \\
&- \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^4 m_2^4} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{64 m_1^4 m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{64 m_1^4 m_2^4} \\
&- \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} - \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{4 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{16 m_1^4 m_2^4} + \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{16 m_1^4 m_2^4} \\
&- \frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{32 m_1^4 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_2^2)^2}{64 m_1^4 m_2^4} - \frac{3(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{32 m_1^4 m_2^4} - \frac{7(\mathbf{p}_1^2)^2(\mathbf{p}_2^2)^2}{128 m_1^4 m_2^4},
\end{aligned}
\tag{A4a}
$$

$$
\begin{aligned}
H_{46}(\mathbf{x}_a,\mathbf{p}_a) =& \frac{369(\mathbf{n}_{12}\cdot\mathbf{p}_1)^6}{160 m_1^6} - \frac{889(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_1^2}{192 m_1^6} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1^2)^2}{16 m_1^6} - \frac{63(\mathbf{p}_1^2)^3}{64 m_1^6} - \frac{549(\mathbf{n}_{12}\cdot\mathbf{p}_1)^5(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{128 m_1^5 m_2} \\
&+ \frac{67(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{16 m_1^5 m_2} - \frac{167(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1^2)^2}{128 m_1^5 m_2} + \frac{1547(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} - \frac{851(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{128 m_1^5 m_2} \\
&+ \frac{1099(\mathbf{p}_1^2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{256 m_1^5 m_2} + \frac{3263(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2} + \frac{1067(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{480 m_1^4 m_2^2} - \frac{4567(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1^2)^2}{3840 m_1^4 m_2^2} \\
&- \frac{3571(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{320 m_1^4 m_2^2} + \frac{3073(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{480 m_1^4 m_2^2} + \frac{4349(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{1280 m_1^4 m_2^2} \\
&- \frac{3461\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{3840 m_1^4 m_2^2} + \frac{1673(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4\mathbf{p}_2^2}{1920 m_1^4 m_2^2} - \frac{1999(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} + \frac{2081(\mathbf{p}_1^2)^2\mathbf{p}_2^2}{3840 m_1^4 m_2^2} - \frac{13(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{8 m_1^3 m_2^3} \\
&+ \frac{191(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3\mathbf{p}_1^2}{192 m_1^3 m_2^3} - \frac{19(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3} - \frac{5(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2^3} \\
&+ \frac{11(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{192 m_1^3 m_2^3} + \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^3}{96 m_1^3 m_2^3} + \frac{233(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_2^2}{96 m_1^3 m_2^3} - \frac{47(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2\mathbf{p}_2^2}{32 m_1^3 m_2^3} \\
&+ \frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{185\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{384 m_1^3 m_2^3} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4}{4 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_2)^4\mathbf{p}_1^2}{4 m_1^2 m_2^4} \\
&- \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3(\mathbf{p}_1\cdot\mathbf{p}_2)}{2 m_1^2 m_2^4} + \frac{21(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{16 m_1^2 m_2^4} + \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{6 m_1^2 m_2^4} + \frac{49(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2\mathbf{p}_2^2}{48 m_1^2 m_2^4} \\
&- \frac{133(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)\mathbf{p}_2^2}{24 m_1^2 m_2^4} - \frac{77(\mathbf{p}_1\cdot\mathbf{p}_2)^2\mathbf{p}_2^2}{96 m_1^2 m_2^4} + \frac{197(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_2^2)^2}{96 m_1^2 m_2^4} - \frac{173\mathbf{p}_1^2(\mathbf{p}_2^2)^2}{48 m_1^2 m_2^4} + \frac{13(\mathbf{p}_2^2)^3}{8 m_2^6},
\end{aligned}
\tag{A4b}
$$

$$
\begin{aligned}
H_{441}(\mathbf{x}_a,\mathbf{p}_a) =& \frac{3027(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{384 m_1^4} - \frac{22993(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{960 m_1^4} - \frac{6695(\mathbf{p}_1^2)^2}{1152 m_1^4} - \frac{3191(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{640 m_1^3 m_2} \\
&+ \frac{28561(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{1920 m_1^3 m_2} + \frac{8777(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{384 m_1^3 m_2} - \frac{752969\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{28800 m_1^3 m_2} \\
&- \frac{16481(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{960 m_1^2 m_2^2} + \frac{94433(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{4800 m_1^2 m_2^2} - \frac{103957(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{2400 m_1^2 m_2^2} \\
&+ \frac{791(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{400 m_1^2 m_2^2} + \frac{26627(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_2^2}{1600 m_1^2 m_2^2} - \frac{118261\mathbf{p}_1^2\mathbf{p}_2^2}{4800 m_1^2 m_2^2} + \frac{105(\mathbf{p}_2^2)^2}{32 m_2^4},
\end{aligned}
\tag{A4c}
$$

$$
\begin{aligned}
H_{442}(\mathbf{x}_a,\mathbf{p}_a) =& \left(\frac{2749\pi^2}{8192} - \frac{211189}{19200}\right)\frac{(\mathbf{p}_1^2)^2}{m_1^4} + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2\mathbf{p}_1^2}{m_1^4} + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^4} \\
&+ \left(\frac{10631\pi^2}{8192} - \frac{1918349}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{13723\pi^2}{16384} - \frac{2492417}{57600}\right)\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2 m_2^2} \\
&+ \left(\frac{1411429}{19200} - \frac{1059\pi^2}{512}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2\mathbf{p}_1^2}{m_1^2 m_2^2} + \left(\frac{248991}{6400} - \frac{6153\pi^2}{2048}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^2 m_2^2} \\
&- \left(\frac{30383}{960} + \frac{36405\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2 m_2^2} + \left(\frac{1243717}{14400} - \frac{40483\pi^2}{16384}\right)\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2} \\
&+ \left(\frac{2369}{60} + \frac{35655\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3 m_2} + \left(\frac{43101\pi^2}{16384} - \frac{391711}{6400}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)\mathbf{p}_1^2}{m_1^3 m_2} \\
&+ \left(\frac{56955\pi^2}{16384} - \frac{1646983}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3 m_2},
\end{aligned}
\tag{A4d}
$$

$$
H_{421}(\mathbf{x}_a,\mathbf{p}_a) = \frac{64861\mathbf{p}_1^2}{4800 m_1^2} - \frac{91(\mathbf{p}_1\cdot\mathbf{p}_2)}{8 m_1 m_2} + \frac{105\mathbf{p}_2^2}{32 m_2^2} - \frac{9841(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{1600 m_1^2} - \frac{7(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{2 m_1 m_2},
\tag{A4e}
$$

$$
\begin{aligned}
H_{422}(\mathbf{x}_a,\mathbf{p}_a) =& \left(\frac{1937033}{57600} - \frac{199177\pi^2}{49152}\right)\frac{\mathbf{p}_1^2}{m_1^2} + \left(\frac{176033\pi^2}{24576} - \frac{2864917}{57600}\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1 m_2} + \left(\frac{282361}{19200} - \frac{21837\pi^2}{8192}\right)\frac{\mathbf{p}_2^2}{m_2^2} \\
&+ \left(\frac{698723}{19200} + \frac{21745\pi^2}{16384}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2} - \left(\frac{63641\pi^2}{24576} - \frac{2712013}{19200}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1 m_2} \\
&+ \left(\frac{3200179}{57600} - \frac{28691\pi^2}{24576}\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_2^2},
\end{aligned}
\tag{A4f}
$$

$$
H_{40}(\mathbf{x}_a,\mathbf{p}_a) = -\frac{m_1^4}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169799}{2400}\right)m_1^3 m_2 + \left(\frac{44825\pi^2}{6144} - \frac{609427}{7200}\right)m_1^2 m_2^2.
\tag{A4g}
$$

$$
\begin{aligned}
H^{\rm nonloc}_{\rm 4PN}(t) =& -\frac{1}{5}\frac{G^2 M}{c^8} I^{(3)}_{ij}(t) \\
&\times \mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{dv}{|v|} I^{(3)}_{ij}(t+v).
\end{aligned}
$$

Analytical GW Templates for BBH Coalescences ?

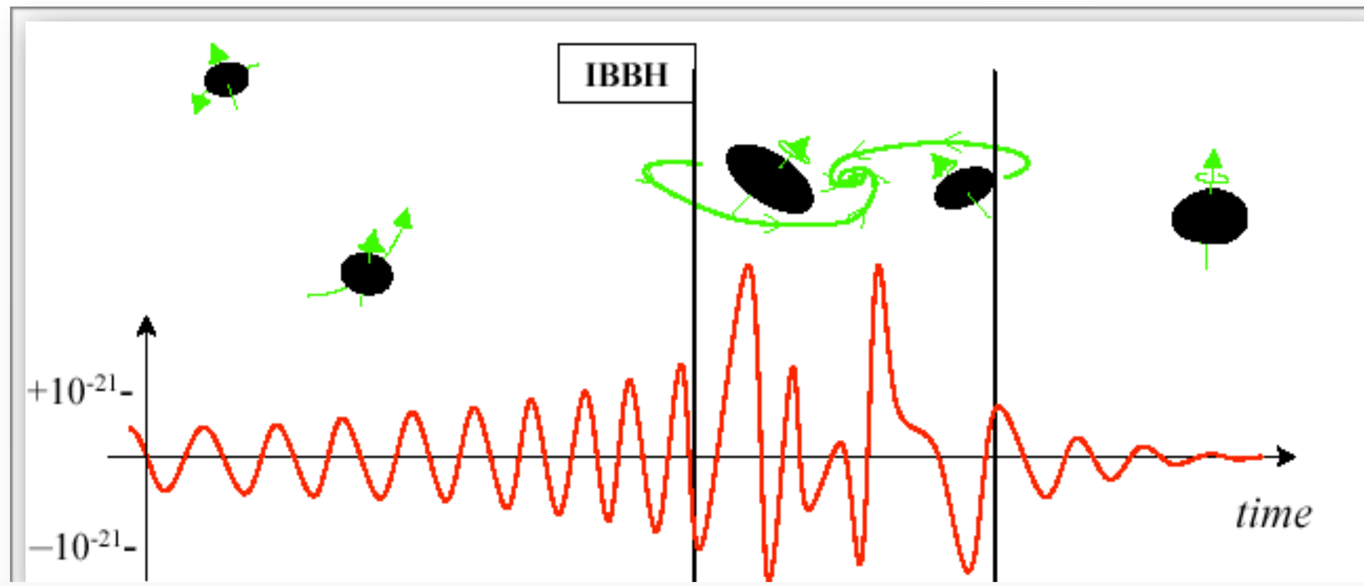
PN corrections to Einstein's quadrupole frequency « chirping »
from PN-improved balance equation dE(f)/dt = - F(f)

$$\frac{d\phi}{d\ln f} = \frac{\omega^2}{d\omega/dt} = Q_\omega^N \widehat{Q}_\omega \qquad \frac{v}{c} = \left(\frac{\pi G(m_1+m_2)f}{c^3}\right)^{\frac{1}{3}}$$

$$Q_\omega^N = \frac{5\,c^5}{48\,\nu\,v^5}\,;\ \widehat{Q}_\omega = 1 + c_2\left(\frac{v}{c}\right)^2 + c_3\left(\frac{v}{c}\right)^3 + \cdots \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

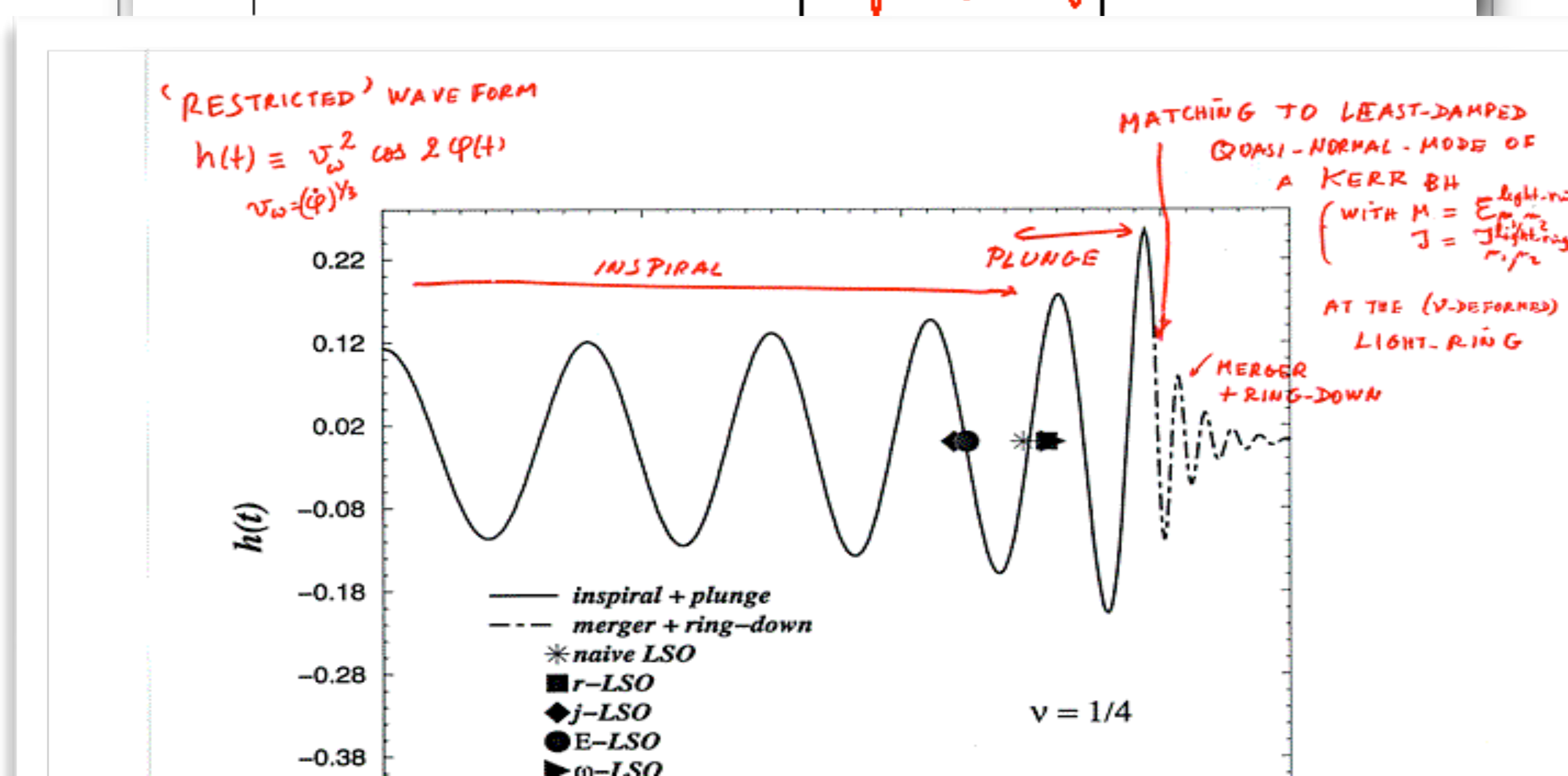
Cutler et al. '93:
« slow convergence of PN »

Brady-Creighton-Thorne'98:
« inability of current computational techniques to evolve a BBH through its last ~10 orbits of inspiral » and to compute the merger



Damour-Iyer-Sathyaprakash'98:
use resummation methods for E and F

Buonanno-Damour '99-00:
novel, resummed approach:
Effective-One-Body
analytical formalism

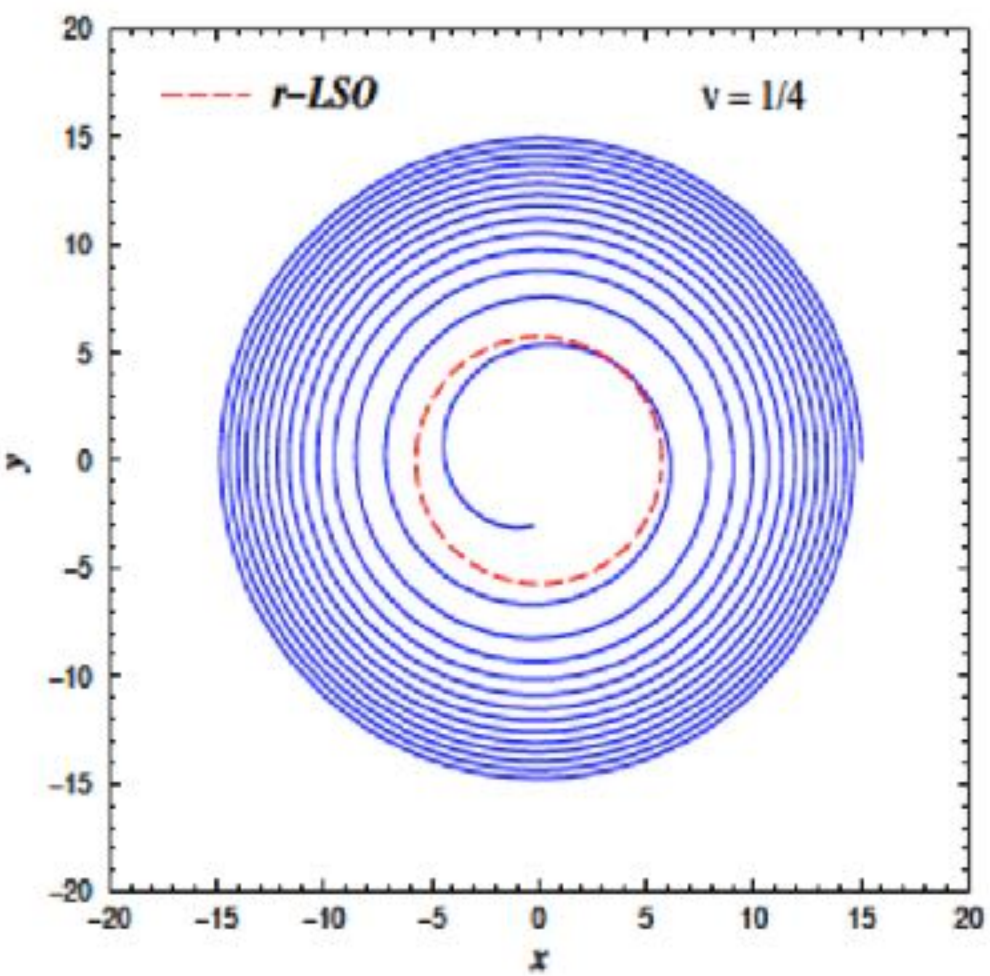


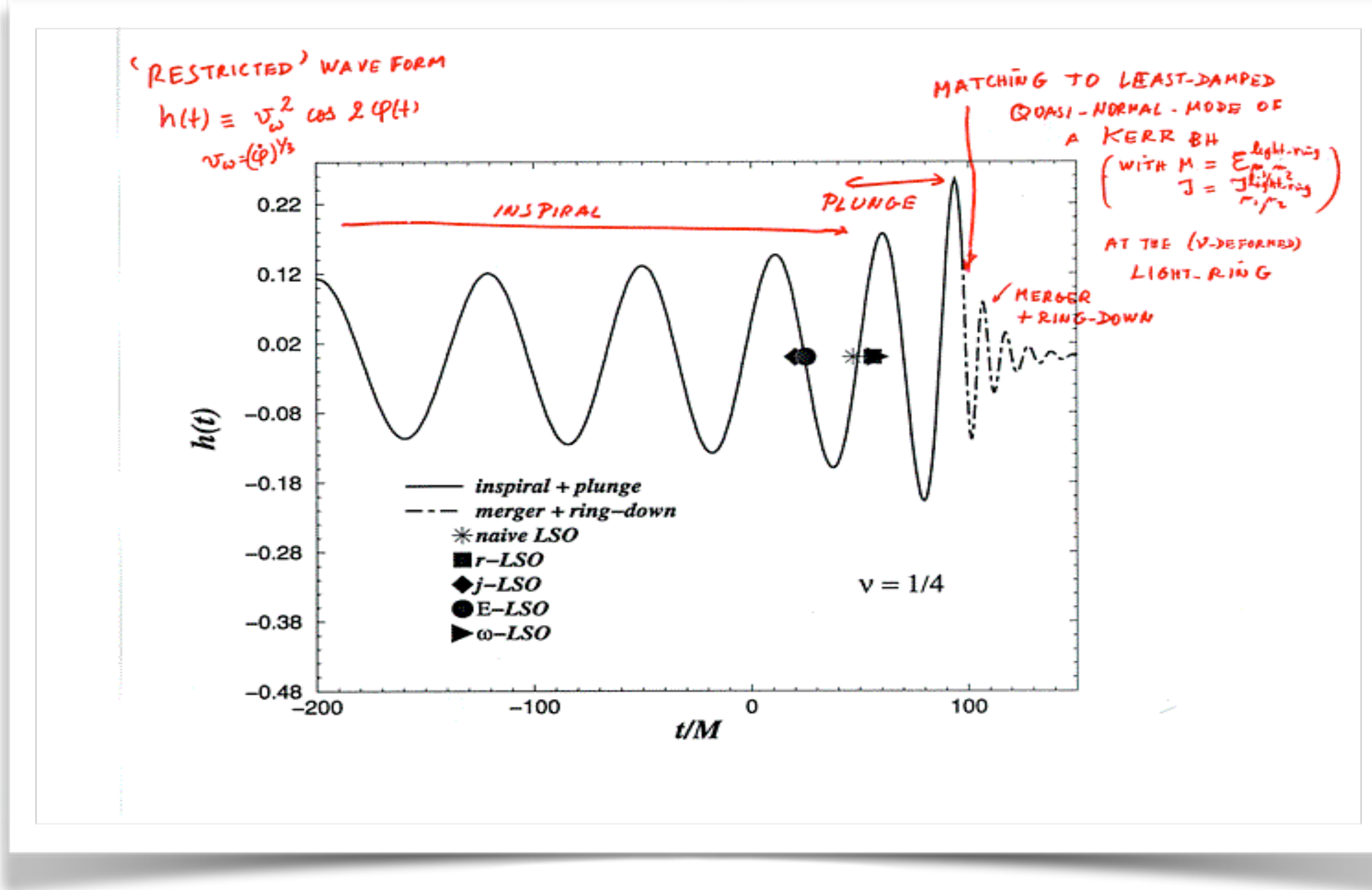
Effective One Body (EOB) Method)

Buonanno-Damour 1999, 2000; Damour-Jaranowski-Schaefer 2000; Damour 2001; Damour-Nagar 2007; Damour-Iyer-Nagar 2009

Resummation of both the Hamiltonian, the waveform and radiation-reaction
—> description of the coalescence + addition of ringdown (Vishveshwara 70, Davis-Ruffini-Tiomno 72)

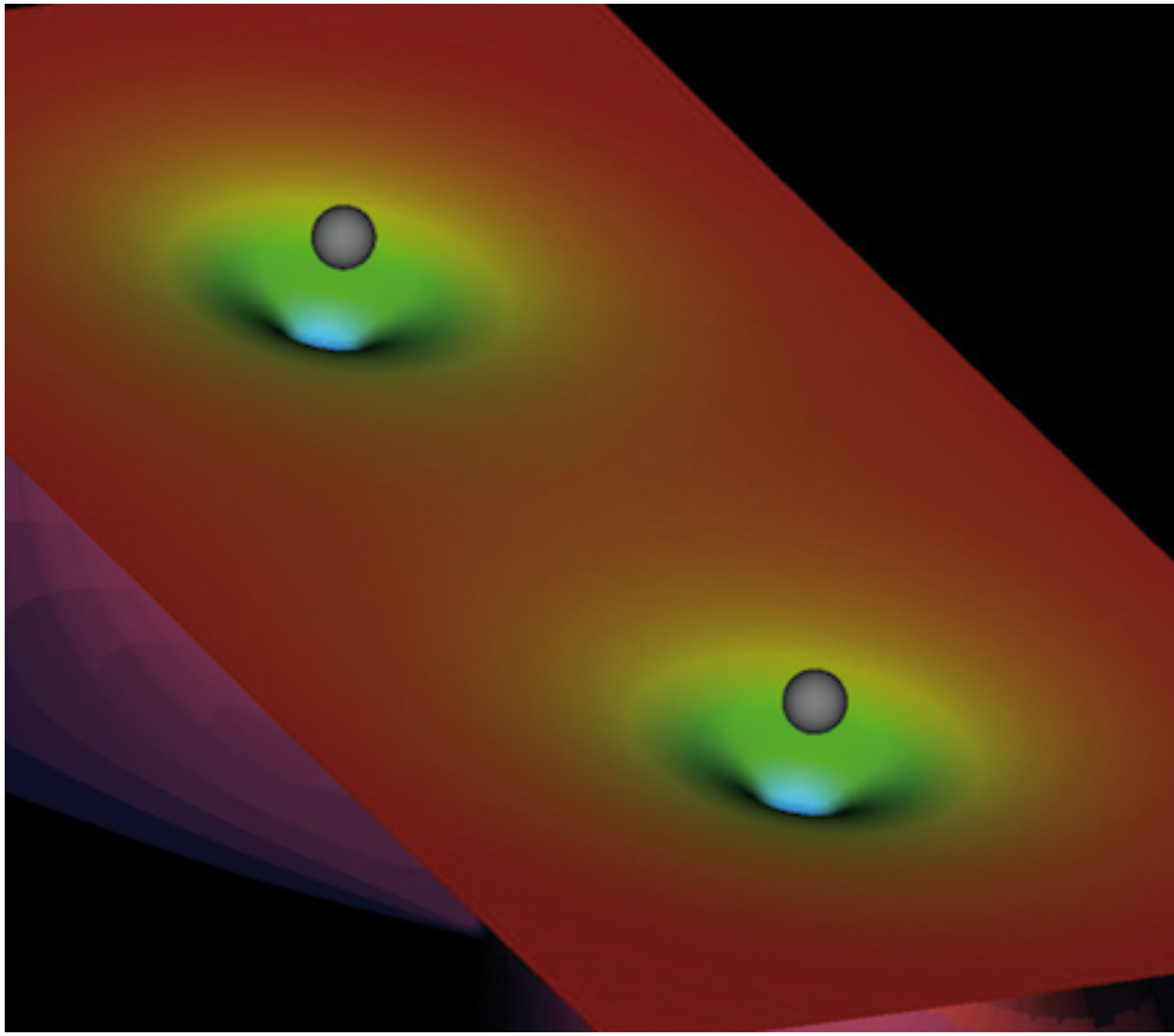
Buonanno-Damour 2000

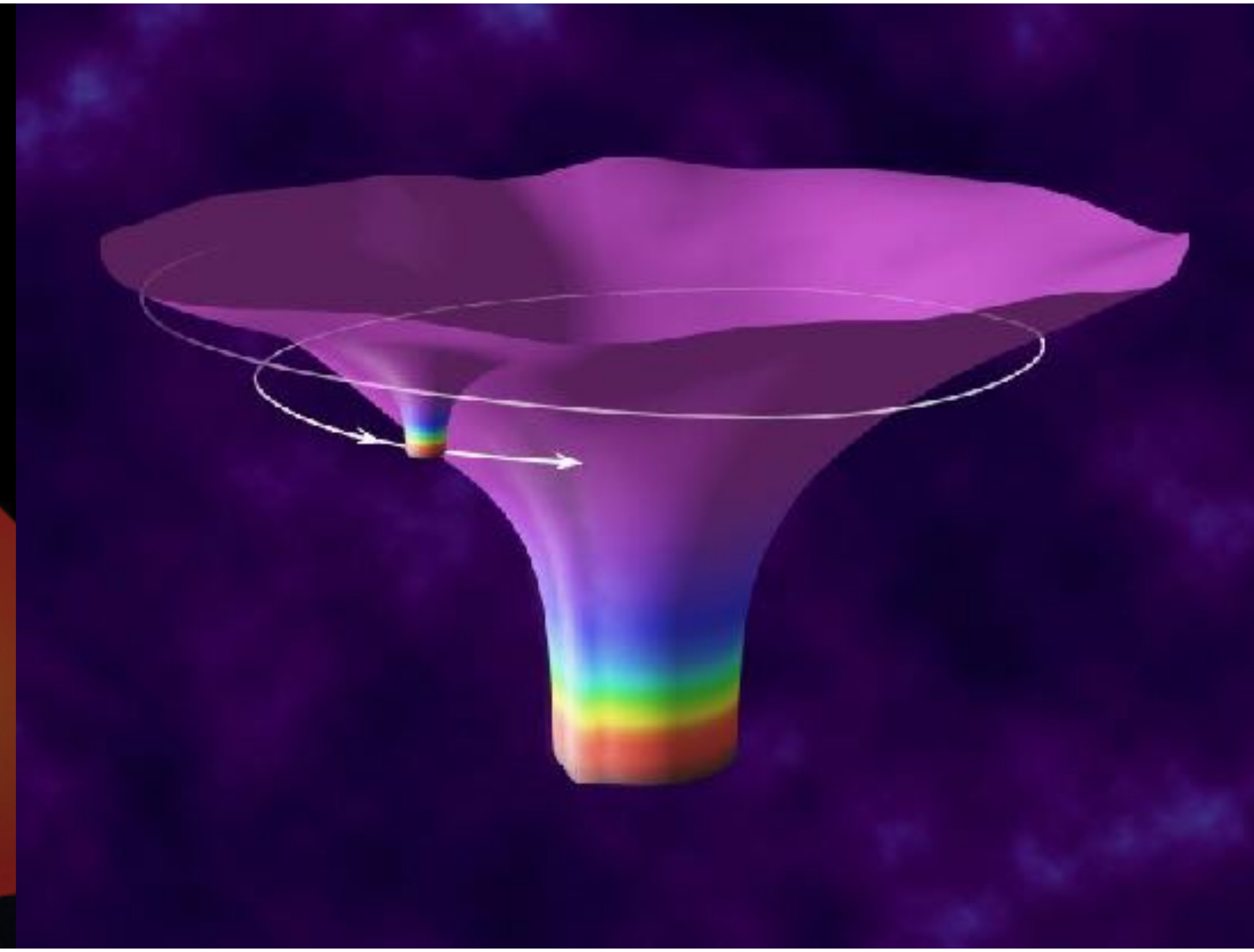




Predictions as early as 2000 :
continued transition, non adiabaticity, first complete waveform, final spin (OK within 10%), final mass

EOB: resumming the dynamics of a two-body system (m_1,m_2,S_1,S_2) in terms of the dynamics of a particle of mass mu and spin S* moving in some effective metric g(M,S)





Effective metric for non-spinning bodies: a nu-deformation of Schwarzschild

$$M = m_1 + m_2 \qquad \mu = \frac{m_1 m_2}{m_1 + m_2} \qquad \nu = \frac{\mu}{M} = \frac{m_1 m_2}{(m_1 + m_2)^2}$$

$$\boxed{ds^2_{\text{eff}} = -A(r;\nu)\,dt^2 + B(r;\nu)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right)}$$

TWO-BODY/EOB "CORRESPONDENCE":

THINK QUANTUM-MECHANICALLY (J.A. WHEELER)

Real 2-body system
(in the c.o.m. frame)
(m_1, m_2)

1:1 map

An effective particle
in some effective metric

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$g_{\mu\nu}^{\mathrm{eff}}$$

$$\mathcal{E} = f(E)$$

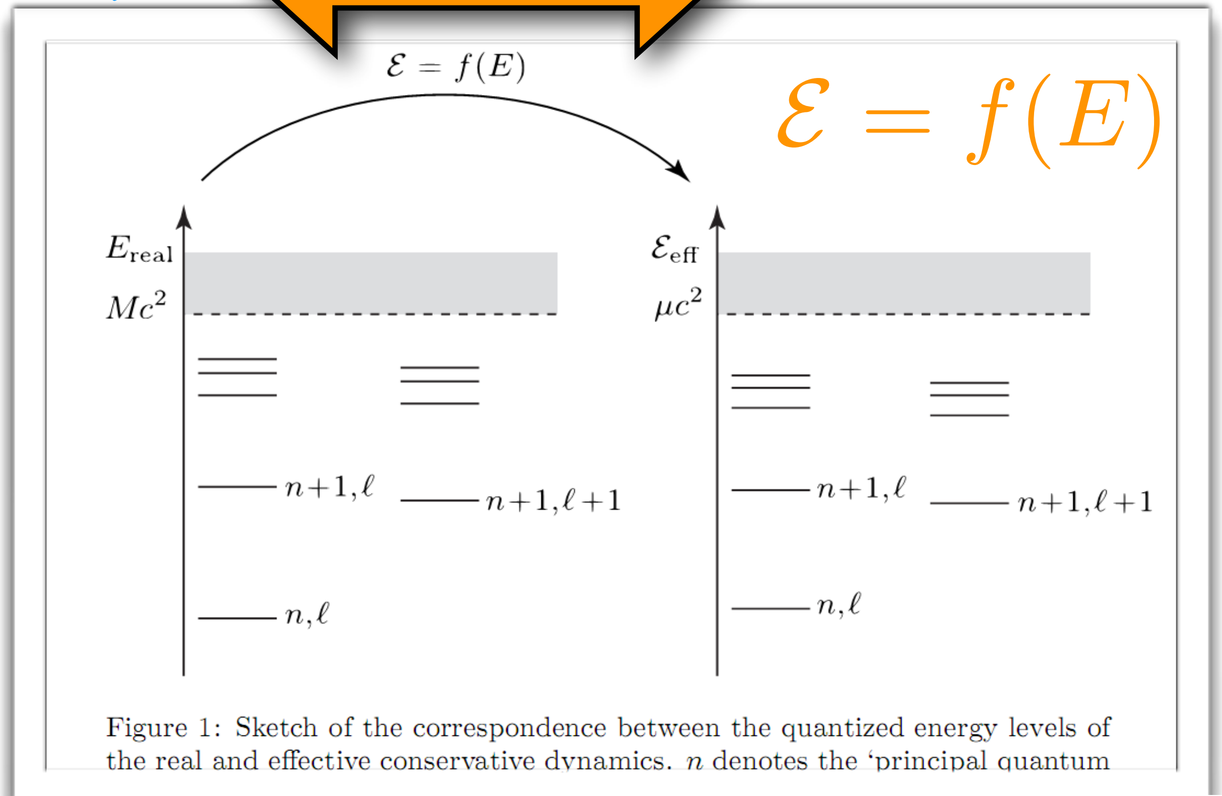


Figure 1: Sketch of the correspondence between the quantized energy levels of the real and effective conservative dynamics. n denotes the 'principal quantum

$$\mu^2 + g_{\rm eff}^{\mu\nu} \frac{\partial S_{\rm eff}}{\partial x^\mu} \frac{\partial S_{\rm eff}}{\partial x^\nu} + \mathcal{O}(p^4) = 0$$

Bohr-Sommerfeld's
Quantization Conditions
(action-angle variables &
Delaunay Hamiltonian)

$$\begin{aligned} J &= \ell\hbar = \frac{1}{2\pi}\oint p_\varphi d\varphi \\ N &= n\hbar = I_r + J \\ I_r &= \frac{1}{2\pi}\oint p_r dr \end{aligned}$$

$$\boxed{\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)}}$$

$$H^{\rm classical}(q,p) \longrightarrow H^{\rm classical}(I_a) \longrightarrow \boxed{E^{\rm quantum}(I_a = n_a h) = f^{-1}[\mathcal{E}_{\rm eff}^{\rm quantum}(I_a^{\rm eff} = n_a h)]}$$

Resumed (non-spinning) 4PN EOB interaction potentials

$$M = m_1 + m_2\,; \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}\,, \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2} = \frac{\mu}{M} \qquad u \equiv \frac{GM}{R\,c^2}$$

$$ds^2_{\rm eff} = -A(r;\nu)\,dt^2 + B(r;\nu)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) \qquad \bar{D} \equiv (A\,B)^{-1}$$

DJS gauge: describing the main effects via the radial potential A(r)

$$A^{\rm PN}(u;\nu) = 1 - 2u + 2\nu u^3 + \nu a_4 u^4 + \left(\nu a_5^c + \nu^2 a_5' + \frac{64}{5}\nu \ln u\right)u^5$$

$$A^{\rm EOB}(u) = {\rm Pade}^1_4[A^{PN}(u)]$$

$$a_4 = \frac{94}{3} - \frac{41\pi^2}{32}\,;\; a_5^c = \frac{2275\pi^2}{512} + \ldots\,;\; a_5' = \ldots$$

$$\bar{D}(u) = 1 + 6\nu u^2 + (52\nu - 6\nu^2)u^3 + \left(\left(-\frac{533}{45} - \frac{23761\pi^2}{1536} + \frac{1184}{15}\gamma_{\rm E} - \frac{6496}{15}\ln 2 + \frac{2916}{5}\ln 3\right)\nu + \left(\frac{123\pi^2}{16} - 260\right)\nu^2 + \frac{592}{15}\nu \ln u\right)u^4,$$

$$\hat{Q}(\mathbf{r}', \mathbf{p}') = \left(2(4-3\nu)\nu u^2 + \left(\left(-\frac{5308}{15} + \frac{496256}{45}\ln 2 - \frac{33048}{5}\ln 3\right)\nu - 83\nu^2 + 10\nu^3\right)u^3\right)(\mathbf{n}'\cdot\mathbf{p}')^4 + \left(\left(-\frac{827}{3} - \frac{2358912}{25}\ln 2 + \frac{1399437}{50}\ln 3 + \frac{390625}{18}\ln 5\right)\nu - \frac{27}{5}\nu^2 + 6\nu^3\right)u^2(\mathbf{n}'\cdot\mathbf{p}')^6 + \mathcal{O}[\nu u(\mathbf{n}'\cdot\mathbf{p}')^8].$$

Spinning EOB effective Hamiltonian

$$H_{\rm eff} = H_{\rm orb} + H_{\rm so} \quad \rightarrow H_{\rm EOB} = Mc^2\sqrt{1+2\nu\left(\frac{H_{\rm eff}}{\mu c^2}-1\right)}$$

$$\hat{H}^{\rm eff}_{\rm orb} = \sqrt{A\left(1 + B_p \boldsymbol{p}^2 + B_{np}(\boldsymbol{n}\cdot\boldsymbol{p})^2 - \frac{1}{1+\frac{(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2}{r^2}}\frac{(r^2+2r+(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2)}{\mathcal{R}^4+\Delta(\boldsymbol{n}\cdot\boldsymbol{\chi}_0)^2}((\boldsymbol{n}\times\boldsymbol{p})\cdot\boldsymbol{\chi}_0)^2 + Q_4\right)}.$$

$$H_{\rm so} = G_S \boldsymbol{L}\cdot\boldsymbol{S} + G_{S^*}\boldsymbol{L}\cdot\boldsymbol{S}^*,$$

$$\mathbf{S} = \mathbf{S_1} + \mathbf{S_2}\,;\; \mathbf{S}_* = \frac{m_2}{m_1}\mathbf{S_1} + \frac{m_1}{m_2}\mathbf{S_2}\,,$$

Gyrogravitomagnetic ratios (when neglecting spin^2 effects)

$$r^3 G_S^{\rm PN} = 2 - \frac{5}{8}\nu u - \frac{27}{8}\nu p_r^2 + \nu\left(-\frac{51}{4}u^2 - \frac{21}{2}up_r^2 + \frac{5}{8}p_r^4\right) + \nu^2\left(-\frac{1}{8}u^2 + \frac{23}{8}up_r^2 + \frac{35}{8}p_r^4\right)$$

$$r^3 G_{S_*}^{\rm PN} = \frac{3}{2} - \frac{9}{8}u - \frac{15}{8}p_r^2 + \nu\left(-\frac{3}{4}u - \frac{9}{4}p_r^2\right) - \frac{27}{16}u^2 + \frac{69}{16}up_r^2 + \frac{35}{16}p_r^4 + \nu\left(-\frac{39}{4}u^2 - \frac{9}{4}up_r^2 + \frac{5}{2}p_r^4\right)$$
$$+\nu^2\left(-\frac{3}{16}u^2 + \frac{57}{16}up_r^2 + \frac{45}{16}p_r^4\right)$$

EOB

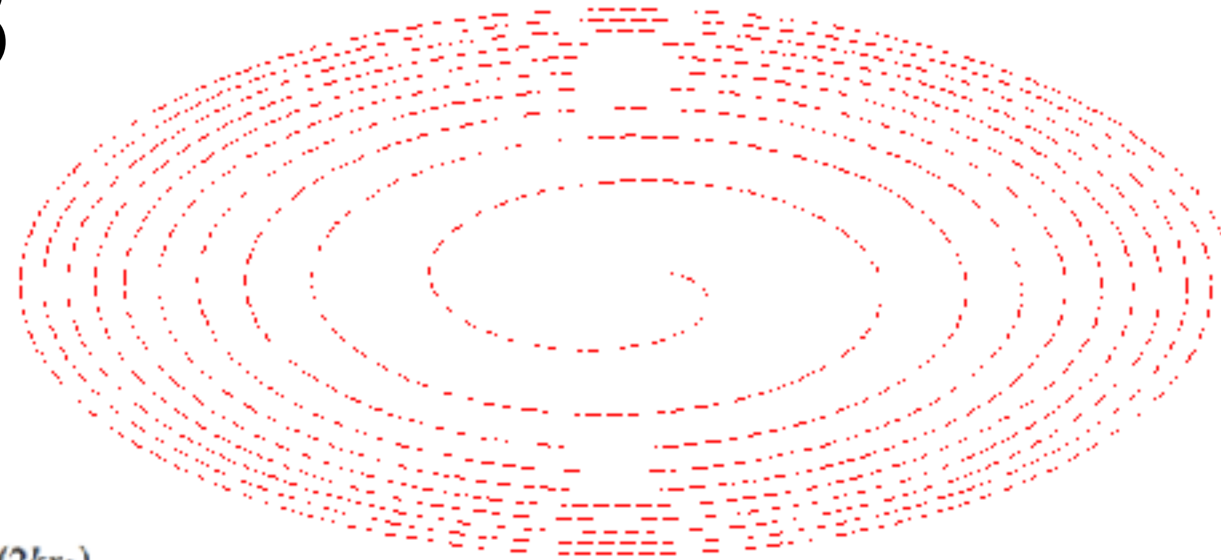
$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial p_{r_*}},$$

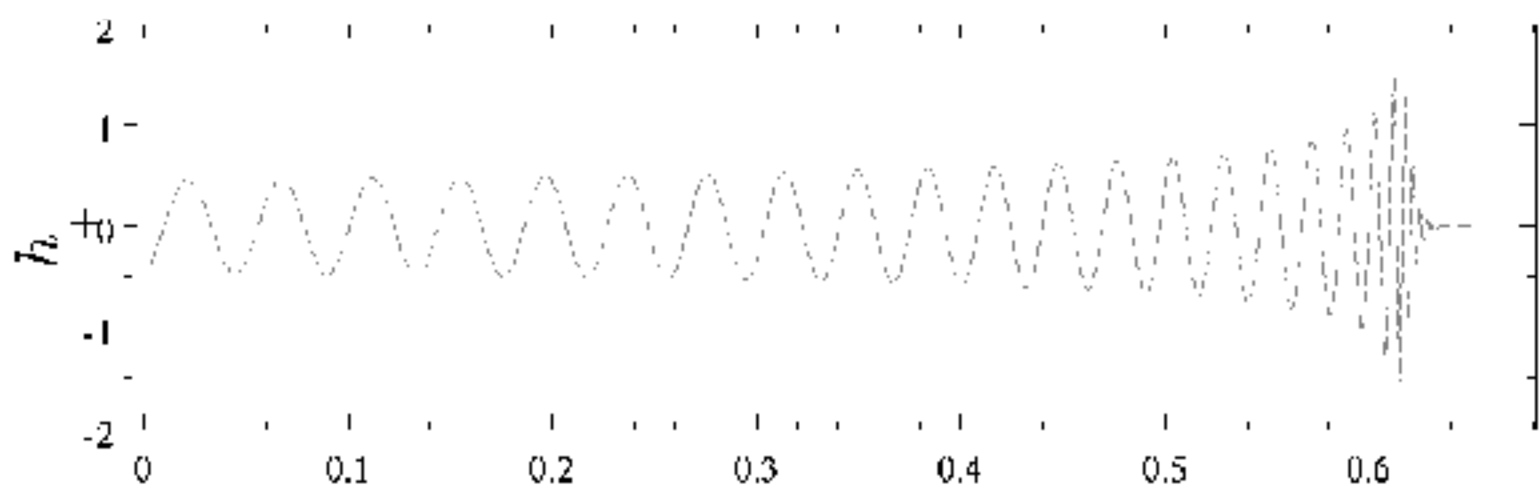
$$\frac{dp_{r_*}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{\rm EOB}}{\partial r},$$

$$\Omega \equiv \frac{d\varphi}{dt} = \frac{\partial \hat{H}_{\rm EOB}}{\partial p_\varphi}$$

$$\frac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi .$$

$$T_{\ell m} = \frac{\Gamma(\ell + 1 - 2i\hat{\hat{k}})}{\Gamma(\ell+1)} e^{\pi \hat{k}} e^{2i\hat{k}\log(2kr_0)},$$





$$h_{\ell m} \equiv h_{\ell m}^{(N,\epsilon)} \hat{h}_{\ell m}^{(\epsilon)} \hat{h}_{\ell m}^{\rm NQC}$$

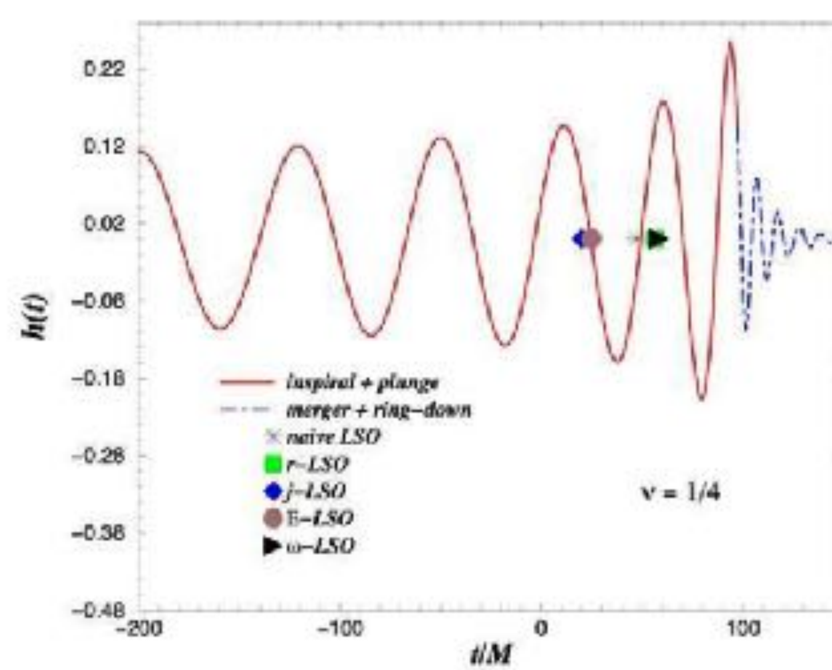
$$\hat{h}_{\ell m}^{(\epsilon)} = \hat{S}_{\rm eff}^{(\epsilon)} T_{\ell m} e^{i\delta_{\ell m}} \rho_{\ell m}^{\ell}$$

$$\mathcal{F}_\varphi \equiv -\frac{1}{8\pi\Omega} \sum_{\ell=2}^{\ell_{\max}} \sum_{m=1}^{\ell} (m\Omega)^2 |R h_{\ell m}^{(\epsilon)}|^2$$

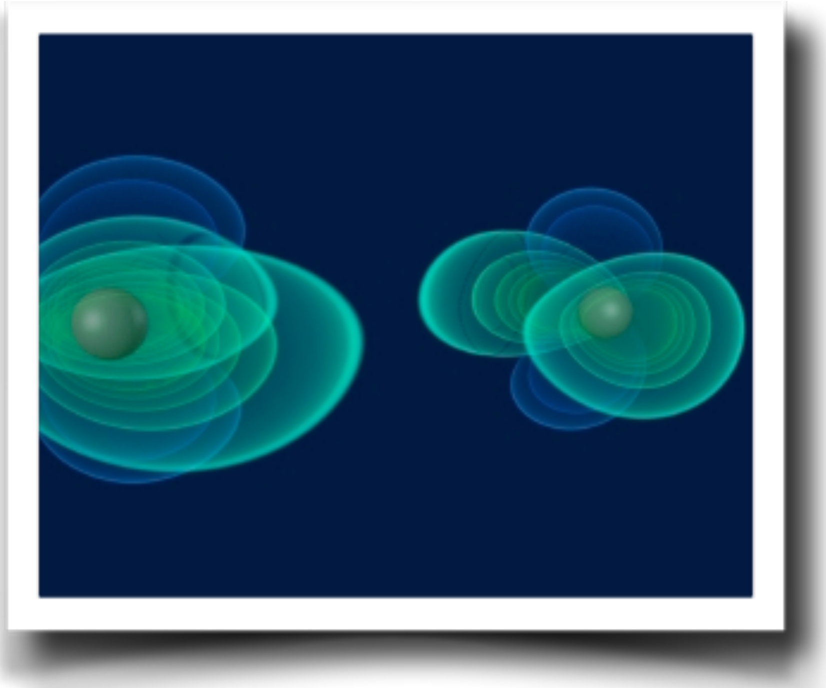
$$h_{\ell m}^{\rm EOB} = \theta(t_m - t) h_{\ell m}^{\rm insplunge}(t) + \theta(t - t_m) h_{\ell m}^{\rm ringdow}$$

$$h_{\ell m}^{\rm ringdown}(t) = \sum_N C_N^+ e^{-\sigma_N^+(t - t_m)}$$

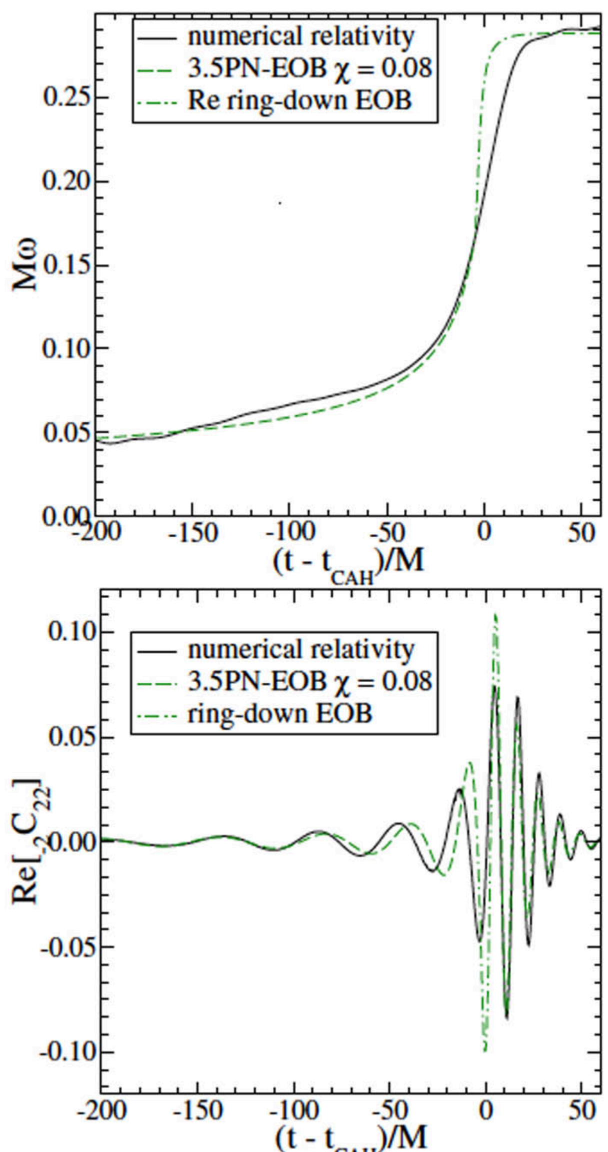
First complete waveforms for BBH coalescences: analytical EOB (Buonanno-Damour'00, Buonanno-Chen-Damour'05)



NR, EOB[NR] AND EOB MAIN RADIAL POTENTIAL A(R)



Buonanno-Cook-Pretorius 2007



Mathematical foundations : Darmois 27, Lichnerowicz 43, Choquet-Bruhat 52-

Breakthrough:

Pretorius 2005: generalized harmonic coordinates (Friedrich, Garfinkle); constraint damping (Brodbeck et al., Gundlach et al., Pretorius, Lindblom et al.); excision;

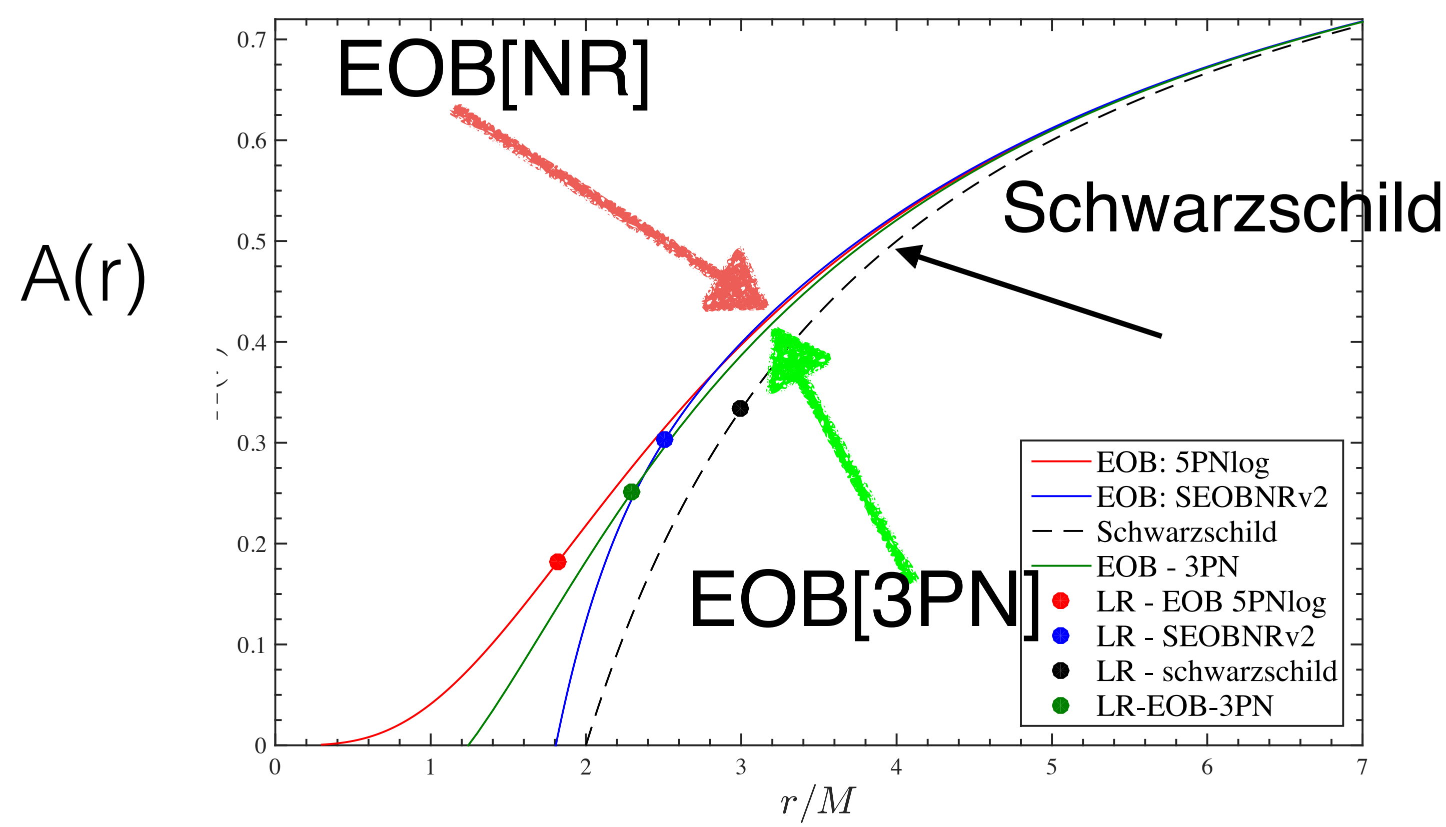
Moving punctures: Campanelli-Lousto-Maronetti-Zlochover 2006

Baker-Centrella-Choi-Koppitz-van Meter 2006

$$
\begin{aligned}
A(u;\nu,a_6^c) &= P_5^1\Bigg[1-2u+2\nu u^3+\nu\left(\frac{94}{3}-\frac{41}{32}\pi^2\right)u^4 \\
&+ \nu\left[-\frac{4237}{60}+\frac{2275}{512}\pi^2+\left(-\frac{221}{6}+\frac{41}{32}\pi^2\right)\nu+\frac{64}{5}\ln(16e^{2\gamma}u)\right]u^5 \\
&+ \nu\left[a_6^c(\nu)-\left(\frac{7004}{105}+\frac{144}{5}\nu\right)\ln u\right]u^6\Bigg]
\end{aligned}
$$

$$
a_6^{c\,\mathrm{NR\text{-}tuned}}(\nu) = 81.38 - 1330.6\,\nu + 3097.3\,\nu^2
$$

m1=m2 case

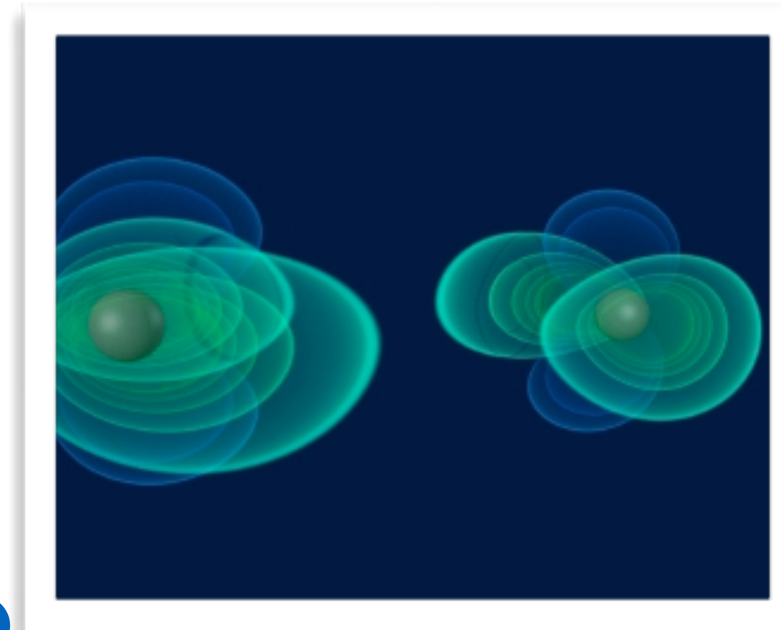


LIGO's bank of search templates
O1: 200 000 EOB + 50 000 PN
O2: 325 000 EOB + 75 000 PN

LISA's templates
via EOB[SF] ?

$v \sim c$

$R \sim GM/c^2$



PN

$v \ll c$

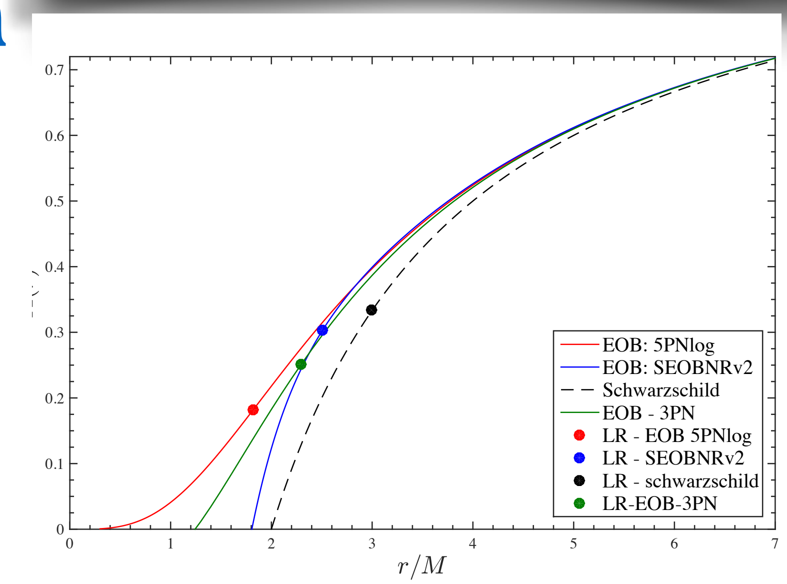
$R \gg GM/c^2$

EOB

PM

$R \gg GM/c^2$

NR



QFT
perturbation theory

STRING
perturbation theory

SF

$m_1 \ll m_2$

Quantum Scattering Amplitudes

Gravitational Scattering and the GR 2-body problem

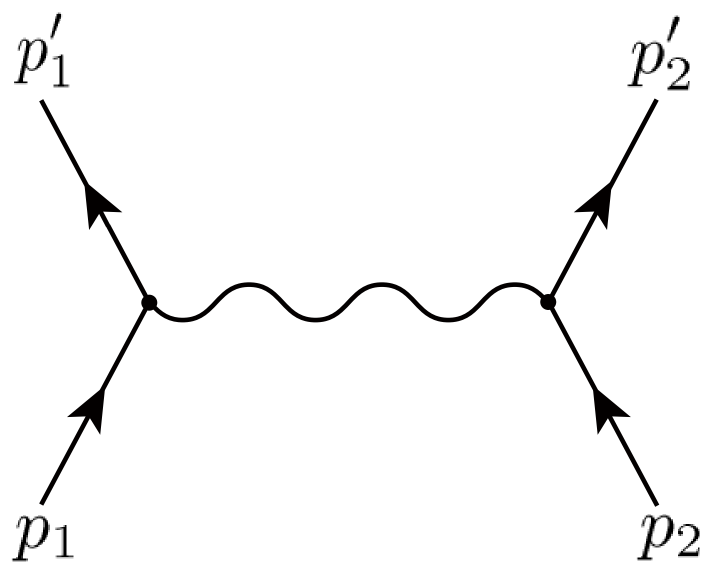
Beyond the PN approximation: (Possibly) High-energy Classical Scattering:
Post-Minkowskian (PM) approximation: expansion in G^n keeping all orders in v/c

Original EOB dictionary based on bound states.
New (equivalent) dictionary for scattering states:
applicable to the PM approximation (no restriction on v/c).
[Damour2016]

$$\boxed{\chi_{\text{eff}}(\mathcal{E}_{\text{eff}}, J) = \chi_{\text{real}}(\mathcal{E}_{\text{real}}, J);}$$

could recently exploit 'old' results by Bel-Martin '75-'81, Portilla '79,Westpfahl-Goller '79, Portilla '80, Bel-Damour-Deruelle-Ibanez-Martin'81,Westpfahl '85 to compute some pieces of the EOB dynamics to all orders in v/c.

G^1



$$\frac{1}{2}\chi_{\text{class}}(E, J) = \frac{1}{j}\chi_1(\hat{E}_{\text{eff}}, \nu) + \frac{1}{j^2}\chi_2(\hat{E}_{\text{eff}}, \nu) + O(G^3)$$

$$\frac{1}{2}\chi^{\text{real}}_{1PM} = 2\frac{G}{b\, p_{\text{c.m.}}}\frac{p_1^\alpha p_1^\beta P_{\alpha\beta;\alpha'\beta'} p_2^{\alpha'} p_2^{\beta'}}{\mathcal{D}}$$
$$= 2\frac{G}{J}\frac{p_1^\alpha p_1^\beta P_{\alpha\beta;\alpha'\beta'} p_2^{\alpha'} p_2^{\beta'}}{\mathcal{D}}.$$

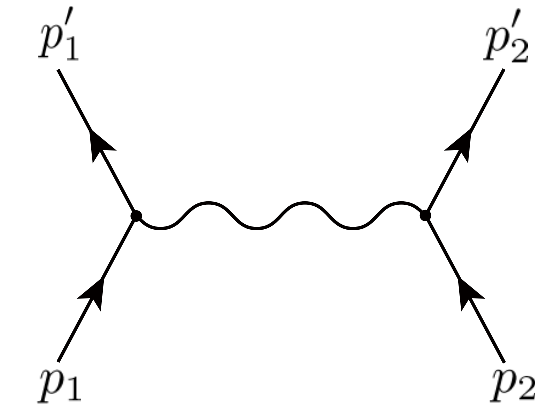
$$\mathcal{D}^2 = |p_1 \wedge p_2|^2 = -\frac{1}{2}(p_1^\mu p_2^\nu - p_1^\nu p_2^\mu)(p_{1\mu}p_{2\nu} - p_{1\nu}p_{2\mu})$$
$$= (p_1.p_2)^2 - p_1^2 p_2^2. \qquad (52)$$

$$\boxed{\frac{1}{2}\chi^{\text{real}}_{1PM} = \frac{G}{J}\frac{2(p_1.p_2)^2 - p_1^2 p_2^2}{\sqrt{(p_1.p_2)^2 - p_1^2 p_2^2}}.}$$

New results already at the 1PM order (linear in G)

Derivation of EOB energy map to all orders in v/c:

$$\boxed{\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)}.}$$



to order G^1, the relativistic dynamics of a two-body system (of masses m_1 , m_2) is equivalent to the relativistic dynamics of an effective test particle of mass $\mu = m_1 m_2/(m_1 + m_2)$ moving in a Schwarzschild metric of mass $M = m_1 + m_2$, i.e. the rather complicated 1PM Hamiltonian of Ledvinka-Schaefer-Bicak2010: with

$$\begin{aligned}
H_{\rm lin} = {} & \sum_a \overline{m}_a + \frac{1}{4} G \sum_{a,b\neq a} \frac{1}{r_{ab}} \left(7\, \mathbf{p}_a\cdot\mathbf{p}_b + (\mathbf{p}_a\cdot\mathbf{n}_{ab})(\mathbf{p}_b\cdot\mathbf{n}_{ab})\right) - \frac{1}{2} G \sum_{a,b\neq a} \frac{\overline{m}_a \overline{m}_b}{r_{ab}} \\
& \times \left(1 + \frac{p_a^2}{\overline{m}_a^2} + \frac{p_b^2}{\overline{m}_b^2}\right) - \frac{1}{4} G \sum_{a,b\neq a} \frac{1}{r_{ab}} \frac{(\overline{m}_a \overline{m}_b)^{-1}}{(y_{ba}+1)^2 y_{ba}} \Bigg[2\Big(2(\mathbf{p}_a\cdot\mathbf{p}_b)^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 \\
& - 2(\mathbf{p}_a\cdot\mathbf{n}_{ba})(\mathbf{p}_b\cdot\mathbf{n}_{ba})(\mathbf{p}_a\cdot\mathbf{p}_b)\mathbf{p}_b^2 + (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 \mathbf{p}_b^4 - (\mathbf{p}_a\cdot\mathbf{p}_b)^2 \mathbf{p}_b^2\Big) \frac{1}{\overline{m}_b^2} + 2\Big[-\mathbf{p}_a^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 \\
& + (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 + 2(\mathbf{p}_a\cdot\mathbf{n}_{ba})(\mathbf{p}_b\cdot\mathbf{n}_{ba})(\mathbf{p}_a\cdot\mathbf{p}_b) + (\mathbf{p}_a\cdot\mathbf{p}_b)^2 - (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 \mathbf{p}_b^2 \Big] \\
& + \Big[-3\mathbf{p}_a^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 + (\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 (\mathbf{p}_b\cdot\mathbf{n}_{ba})^2 + 8(\mathbf{p}_a\cdot\mathbf{n}_{ba})(\mathbf{p}_b\cdot\mathbf{n}_{ba})(\mathbf{p}_a\cdot\mathbf{p}_b) \\
& + \mathbf{p}_a^2 \mathbf{p}_b^2 - 3(\mathbf{p}_a\cdot\mathbf{n}_{ba})^2 \mathbf{p}_b^2 \Big] y_{ba} \Bigg], \qquad y_{ba} = \frac{1}{\overline{m}_b} \sqrt{m_b^2 + (\mathbf{n}_{ba}\cdot\mathbf{p}_b)^2}\,.
\end{aligned} \tag{6}$$

$$\overline{m}_a = (m_a^2 + \mathbf{p}_a^2)^{\frac{1}{2}}$$

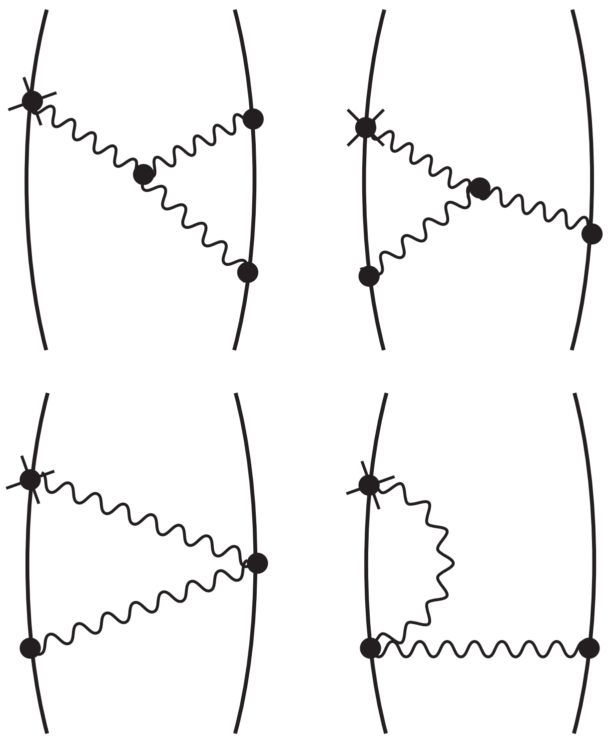
is fully described by the EOB energy map applied to

$$\boxed{ds^2_{\rm lin} = -(1 - 2\frac{GM}{r})dt^2 + (1 + 2\frac{GM}{r})dr^2 + r^2 d\Omega^2}$$

Classical Gravitational Scattering at the 2PM level (one-loop)

Damour'18, using Westpfahl-Goller '79, Bel-Damour-Deruelle-Ibanez-Martin'81, Westpfahl '85

G^2



$$\frac{1}{2}\chi_{\rm class}(E,J) = \frac{1}{j}\chi_1(\hat{E}_{\rm eff},\nu) + \frac{1}{j^2}\chi_2(\hat{E}_{\rm eff},\nu) + O(G^3)$$

$$\chi_1(\hat{\mathcal{E}}_{\rm eff},\nu) = \frac{2\hat{\mathcal{E}}^2_{\rm eff}-1}{\sqrt{\hat{\mathcal{E}}^2_{\rm eff}-1}},$$

$$\chi_2(\hat{\mathcal{E}}_{\rm eff},\nu) = \frac{3\pi}{8}\frac{5\hat{\mathcal{E}}^2_{\rm eff}-1}{\sqrt{1+2\nu(\hat{\mathcal{E}}_{\rm eff}-1)}}.$$

$$\hat{\mathcal{E}}_{\rm eff} \equiv \frac{\mathcal{E}_{\rm eff}}{\mu} \equiv \frac{(E_{\rm real})^2 - m_1^2 - m_2^2}{2m_1m_2} = \frac{s-m_1^2-m_2^2}{2m_1m_2}.$$

Effective EOB Hamiltonian transcription of chi2PM as a post-Schwarzschild Hamiltonian

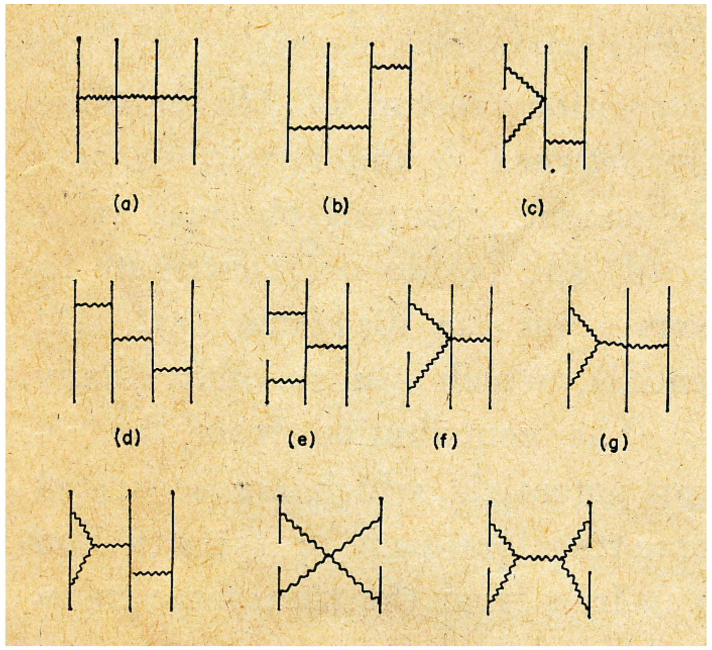
$$0 = g^{\mu\nu}_{\rm Schwarz}P_\mu P_\nu + Q(\mathbf{R},\mathbf{P}) \longrightarrow \frac{1}{2}(\chi(\mathcal{E}_{\rm eff},J) - \chi^{\rm Schw}(\mathcal{E}_{\rm eff},J)) = \frac{1}{4}\frac{\partial}{\partial J}\int d\sigma_{(0)}Q + O(G^4).$$

gauge-freedom $\quad Q'(\mathbf{R},\mathbf{P}) = Q(\mathbf{R},\mathbf{P}) + \frac{d}{d\sigma_{(0)}}G(\mathbf{R},\mathbf{P})$ $\quad$ use an « energy gauge »

$$\boxed{\widehat{H}^2_{\rm eff}(p_r,r,p_\varphi;\nu) = \widehat{H}^2_{\rm Schw} + \frac{3}{2}(1-2u)u^2\left(5\,\widehat{H}^2_{\rm Schw}-1\right)\left(1-\frac{1}{\sqrt{1+2\nu(\widehat{H}_{\rm Schw}-1)}}\right)}$$

$$\hat{H}^2_{\rm Schw}(p_r,r,p_\varphi) \equiv (1-2u)[1+(1-2u)p_r^2 + p_\varphi^2u^2],$$

Quantum Scattering Amplitudes and 2-body Dynamics



- Quantum Scattering Amplitudes —> Potential

one-graviton exchange :
Corinaldesi '56 '71,
Barker-Gupta-Haracz 66,
Barker-O'Connell 70, Hiida-Okamura72

Nonlinear: Iwasaki 71 [1PN],
Okamura-Ohta-Kimura-Hiida 73[2 PN]

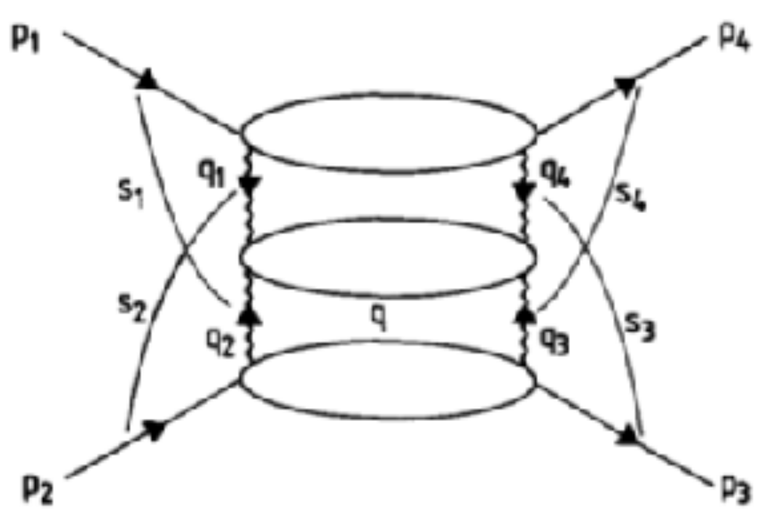


Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

Amati-Ciafaloni-Veneziano
Ultra-High-Energy Scattering at 2 loops

New technique: use EOB as a scattering-> Hamiltonian translation device

Progress in gravity amplitudes (Bern, Carrasco et al., Cachazo et al., Bjerrum-Bohr et al., Cachazo-Guevara,…) can be used (Damour '17) to improve the classical 2-body dynamics: need a quantum/classical dictionary.

High-energy limit of 2-body scattering and 2-body dynamics

Using the (eikonal) ultra-high-energy results of Amati-Ciafaloni-Veneziano: get HE information up to G^4

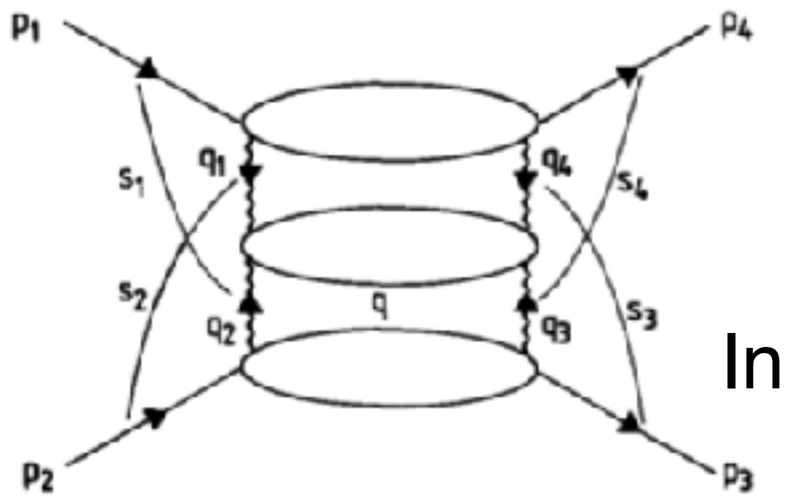


Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

$$\sin \frac{1}{2}\chi^{\mathrm{ACV}} \stackrel{\mathrm{HE}}{=} 2\alpha + (2\alpha)^3 + O(\alpha^5),$$

with $\alpha_{\mathrm{HE}} = \frac{G E_{\mathrm{real}}}{b}$

In HE limit the EOB energy map is such that

$$\alpha = \frac{G M E_{\mathrm{eff}}}{J} = \frac{G}{2} \frac{(E_{\mathrm{real}})^2 - m_1^2 - m_2^2}{J} \approx_{\mathrm{HE}} \frac{G E_{\mathrm{real}}}{b}$$

The masses disappear and the HE scattering is equivalent to a null geodesic in the « effective HE metric »

$$ds^2 = -A_{\mathrm{HE}}(u) dT^2 + \frac{dR^2}{1-2u} + R^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

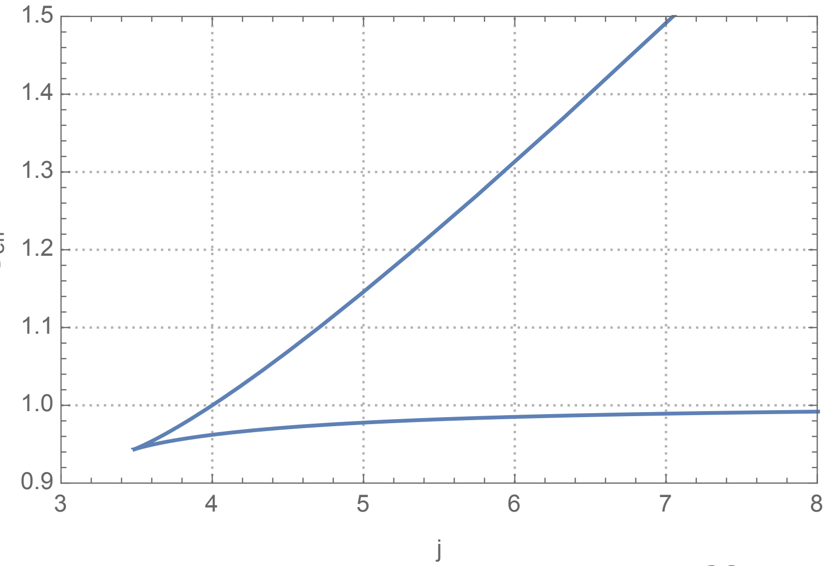
$$\boxed{A_{\mathrm{HE}}(u) = (1-2u)\left(1 + \frac{15}{2}u^2 - 3u^3 + \frac{1749}{16}u^4 + O(u^5)\right)}$$

2PM 3PM 4PM

Light-Ring Behavior: HE Regge Behavior

$$u \equiv \frac{GM}{R}$$

$$H^2_{\rm Schw} = \left(1 - \frac{2GM}{R}\right)\left[\mu^2 + (1 - \frac{2GM}{R})P_R^2 + \frac{J^2}{R^2}\right]$$



$$\boxed{\widehat{H}^2_{\rm eff}(p_r, r, p_\varphi; \nu) = \widehat{H}^2_{\rm Schw} + \frac{3}{2}(1-2u)u^2\left(5\,\widehat{H}^2_{\rm Schw} - 1\right)\left(1 - \frac{1}{\sqrt{1 + 2\nu(\widehat{H}_{\rm Schw} - 1)}}\right)}$$

High J —> H^2_eff ~ B(u) J^2

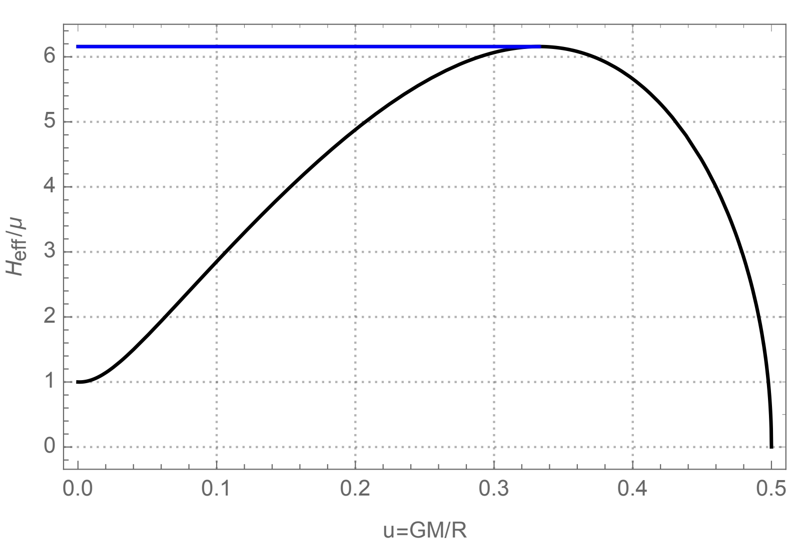
but

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1 + m_2)}\,.$$

HE unstable circular bound states: asymptotic constant Regge slope

$$s = E^2_{\rm real} \propto J$$

string-like, cf P. Goddard's talk



$$\frac{ds}{dJ} = \frac{2}{G}\frac{d\hat{\mathcal{E}}_{\rm eff}}{dj} \overset{\rm HE}{\approx} \frac{0.719\,964}{G}$$

$$E^2_{\rm real} \overset{\rm HE}{=} C\,\frac{J}{G}\,,$$

Translating quantum scattering amplitudes into classical dynamical information

How to translate a scattering amplitude into a classical Hamiltonian ?

$$\mathcal{M}(s,t) = \mathcal{M}^{(\frac{G}{\hbar})}(s,t) + \mathcal{M}^{(\frac{G^2}{\hbar^2})}(s,t) + \cdots \quad \mathcal{M}^{(\frac{G}{\hbar})}(s,t) = 16\pi \frac{G}{\hbar} \frac{2(p_1\cdot p_2)^2 - p_1^2 p_2^2}{-t}.$$

Problem: The domain of validity of the Born expansion is GE_1 E_2/(hbar v) << 1, while the domain of validity of the classical scattering is GE_1 E_2/(hbar v) >> 1! It is an accident that the Born approximation of a 1/r potential yields the exact cross section.

A way out is to quantize the classical EOB Hamiltonian dynamics. The 2PM-level mass-shell condition reads

$$\mathbf{p}^2 = p_\infty^2 + \bar{W}(\bar{u}) = p_\infty^2 + w_1 \bar{u} + w_2 \bar{u}^2 + O(\bar{u}^3),$$

$$p_\infty^2 = \hat{\mathcal{E}}_{\rm eff}^2 - 1,$$

$$w_1 = 2(2\hat{\mathcal{E}}_{\rm eff}^2 - 1),$$

$$w_2 = \frac{3}{2}\frac{5\hat{\mathcal{E}}_{\rm eff}^2 - 1}{h(\hat{\mathcal{E}}_{\rm eff})}.$$

$$-\hat{\hbar}^2 \Delta_{\mathbf{x}} \psi(\mathbf{x}) = \left[p_\infty^2 + \frac{w_1}{\bar{r}} + \frac{w_2}{\bar{r}^2} + O\left(\frac{1}{\bar{r}^3}\right)\right]\psi(\mathbf{x}).$$

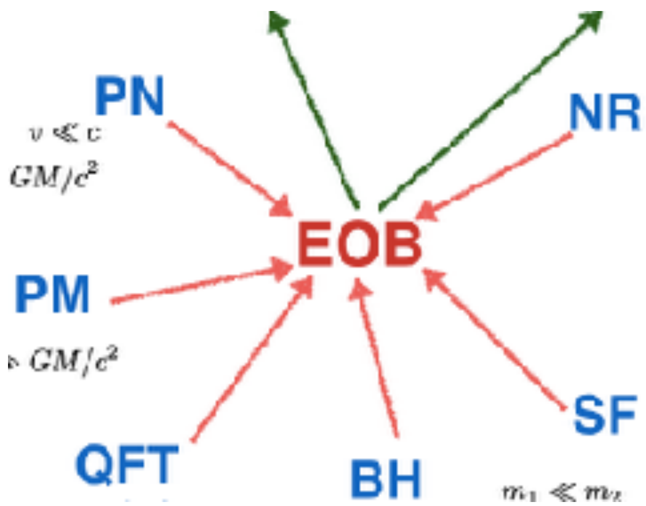
$$f^{+\mathrm{B}1}_{\mathbf{k}_a}(\mathbf{k}_b) = \frac{1}{\hat{\hbar}^2}\left[e^{\delta_C}\frac{w_1}{q^2} + \frac{\pi}{2}\frac{w_2}{q}\right],$$

$$\delta_C = i\frac{w_1}{2k\hat{\hbar}^2}\ln\left(\sin^2\frac{\theta}{2}\right) + 2i\arg\Gamma\left(1 - i\frac{w_1}{2k\hat{\hbar}^2}\right).$$

Quantum/classical dictionary:

OK with one-loop result of Guevara 1706.02314

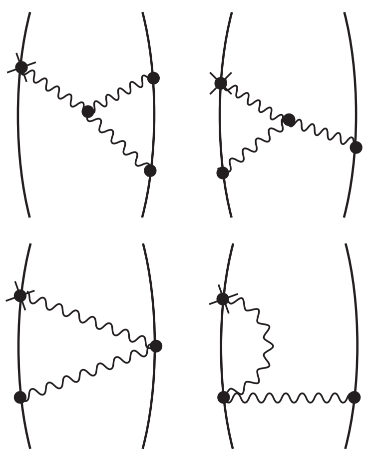
$$\boxed{\frac{f^+_{(1/q)}}{f^+_{(1/q^2)}} = \frac{3\pi}{8}\frac{5\hat{\mathcal{E}}_{\rm eff}^2 - 1}{2\hat{\mathcal{E}}_{\rm eff}^2 - 1}\frac{G(m_1+m_2)\sqrt{-t}}{\hbar} + O(G^2).}$$



Summary

The EOB formulation of 2-body dynamics is a useful tool for transcribing classical and quantum scattering information into bound-state information.

The classical one-loop (G^ 2) scattering has been transcribed in EOB theory thereby giving new vistas on high-energy gravitational interactions.



The Amati-Ciafaloni-Veneziano 2-loop HE result has been transcribed in EOB theory.

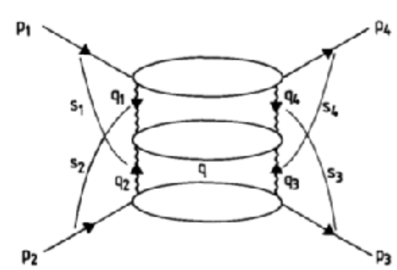
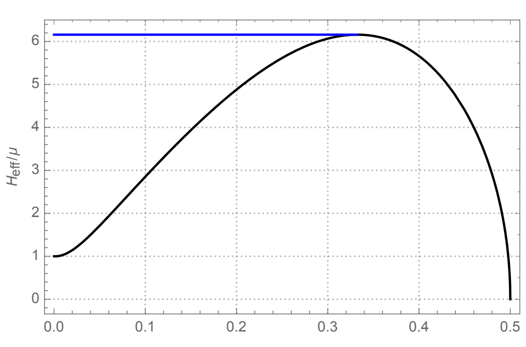


Fig. 3. The "H" diagram that provides the leading correction to the eikonal.

The HE gravitational EOB interaction predicts string-like (Regge-like) unstable « whirl » bound states: s ~ C J/G

A quantum/classical dictionary can be established.



It would be very valuable to have the two-loop quantum scattering amplitude of gravitationally interacting scalar particles so as to EOB-transcribe it.

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$$-\hat{\hbar}^2 \Delta_{\mathbf{x}} \psi(\mathbf{x}) = \left[p_\infty^2 + \frac{w_1}{\bar{r}} + \frac{w_2}{\bar{r}^2} + O\left(\frac{1}{\bar{r}^3}\right) \right] \psi(\mathbf{x}).$$