Entanglement and differential entropy for massive flavors

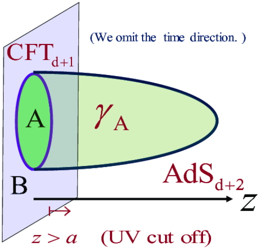
Marika Taylor

Mathematical Sciences and
STAG research centre, Southampton

April 16, 2015

Introduction

- There has been considerable interest recently in entanglement entropy.



(Takayanagi)

- $S_A = -\mathrm{Tr}(\rho_A \log \rho_A)$.
- Holographic Ryu-Takayanagi (RT) prescription: area of co-dimension two minimal surface homologous to A

$$S_A = \frac{\mathcal{A}}{4G_N}$$

STAG RESEARCH CENTRE

Key questions

1. Dependence of EE on shape and on field theory.
2. First law for EE.
3. Dependence of EE on the state in theory e.g. excited states.
4. Other bulk computables e.g. differential entropy and their roles in field theory.

(Balasubramanian et al, Hartman, Headrick, Hubeny, Liu, Mezei, Myers et al, Rangamani, Rosenhaus, Smolkin,Tayakanagi et al, $\cdots$)

Massive flavors

- Brane systems such as D3/D7 are a natural framework in which to explore these questions.
- Well-understood dual field theory and phenomenologically interesting (Aharony et al, Fayyazuddin et al '98, Karch and Katz '02, ...).

Based on:

- Peter Jones, Kostas Skenderis and Marika Taylor
  "Entanglement and differential entropy for massive flavors"
  arXiv:1505.xxxx

Outline

- **The D3/D7 system**
- Entanglement entropy
- Field theory interpretation
- Differential entropy

The D3/D7 system

- Consider $\mathcal{N}_c$ D3-branes and $\mathcal{N}_f \ll \mathcal{N}_c$ parallel coincident D7-branes.
- In the decoupling limit the D7-branes wrap an $AdS_5 \times S^3$ submanifold of $AdS_5 \times S^5$

$$ds^2 = \frac{1}{z^2}\left(dz^2 + dx \cdot dx\right) + d\theta^2 + \sin^2\theta d\Omega_3^2 + \cos^2\theta d\phi^2$$

  i.e. $\theta = \pi/2$.
- Dual to SYM coupled to $\mathcal{N} = 2$ massless hypermultiplets transforming in the bifundamental of $SU(\mathcal{N}_c) \times SU(\mathcal{N}_f)$.

STAG RESEARCH CENTRE

- Separating the D3 and D7 branes causes the hypers to become massive.
- From the brane probe perspective, the embedding is (Karch and Katz, '02)

$$\sin^2\theta = (1 - m^2 z^2),$$

i.e. the D7-branes extend to $z = 1/m$, with m the mass.

Deformation

- The corresponding deformation of the CFT is by a dimension three operator, the fermion mass,

$$I = I_{CFT} + m \int d^4x \sqrt{-g}\mathcal{O}_3$$

where the holographic normalization of the operator (brane holographic renormalization (Karch et al, '05)) is

$$\langle \mathcal{O}_3(x)\mathcal{O}_3(0)\rangle = 16 T_7 \mathcal{R}\left(\frac{1}{x^6}\right)$$

with T_7 the D7-brane tension.

STAG RESEARCH CENTRE

IR behaviour

- Integrating out the massive hypers leads to an effective IR theory

$$I = I_{SYM} + \frac{1}{m^2} \int d^4x \sqrt{-g}\mathcal{O}_6 + \cdots$$

  where $\mathcal{O}_6$ is a dimension six SYM operator, which breaks the R symmetry to $SO(4)$.
- The finite extent of the probe D7-brane tallies with the field theory behaviour at energy scales far smaller than m.

Outline

- The D3/D7 system
- **Entanglement entropy**
- Field theory interpretation
- Differential entropy

STAG RESEARCH CENTRE

Entanglement entropy

- To compute EE for D3/D7 we should find the full backreacted metric, asymptoting to $AdS_5 \times S^5$, extract the effective 5d Einstein metric and apply the RT formula.

This looks hard:

- Backreacted metric depends on $(z, \theta, \phi) \rightarrow$ cohomogeneity three problem.
- Smearing over the sphere simplifies problem (Bigazzi et al; Kontoudi and Policastro) but is obscure in field theory.

Probe approximation

As usual we work in the quenched approximation $\mathcal{N}_f \ll \mathcal{N}_c$.

- Effectively we need to solve

$$I = I_{\rm sugra} - \mathcal{N}_f T_7 \int d^8x \sqrt{-\gamma} + \cdots$$

  with γ_{ab} the induced brane metric perturbatively in $\mathcal{N}_f T_7$.
- EE is sensitive to the 5d Einstein metric so we cannot work just with D-brane action.

STAG RESEARCH CENTRE

Shortcuts?

Following (Jensen and O'Bannon; Karch et al):

1. Exploit CHM map for spherical entangling regions.
   - Intractable for finite mass, even at zero density.
2. Assume induced brane metric is a direct product of non-compact and compact parts.
   - Method is not applicable at finite density or for general brane embeddings.

STAG RESEARCH CENTRE

A systematic new method

Kaluza-Klein holography (Skenderis, M.T. '06):

- For any background which is a perturbation of $AdS_5 \times S^5$, i.e.
$$ds^2 = \frac{1}{z^2}\left(dz^2 + dx \cdot dx\right) + d\Omega_5^2$$
$$+\delta g_{mn}(z, x_\mu, \theta_i).$$
the pertubations can be decomposed in terms of spherical harmonics.
- Kaluza-Klein holography gives an algorithmic approach to extract the 5d Einstein metric from the perturbations.

STAG RESEARCH CENTRE

A systematic new method

- Given any brane embedding into *AdS* Schwarzschild $\times S^5$, i.e.

$$I_{DBI} = -T_p \int d^{p+1}x \sqrt{-(\gamma + \mathcal{B})} + T_p \int e^{\mathcal{B}} \wedge C$$

given the worldvolume metric γ and gauge field $\mathcal{B}$ one can compute the sources for ten dimensional sugra fields.

- Kaluza-Klein holography expresses the 5d Einstein metric in terms of specific spherical harmonics of these sources.
- No problems working at finite density!

Example: massive flavours

- The change in the 5d Einstein metric is particularly simple: for $z \leq 1/m$

$$\delta(ds^2) = \frac{1}{z^2}\left(f(z)dz^2 + h(z)dx \cdot dx\right)$$

with

$$\tilde{f}(z) = (f(z) + zh'(z)) = \frac{t_0}{12}(1 - m^2 z^2)^2$$

where $t_0 = \mathcal{V}_{S^3} T_7 = 2\pi^2 T_7$.

- The gauge invariant combination is $\tilde{f}(z)$.

STAG RESEARCH CENTRE

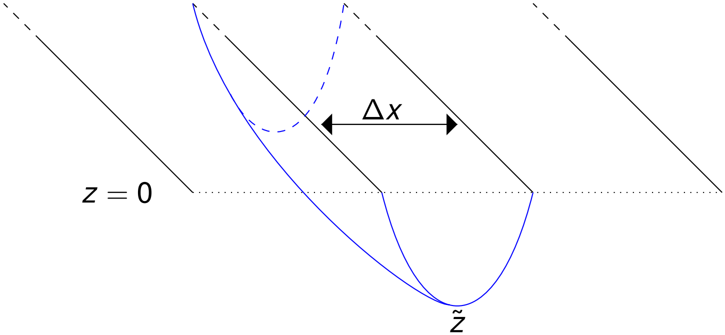
Change in EE

We explored three types of domains:

1. Slab in y, z plane, width $l = \Delta x$.
2. Half plane $x > 0$; $\Delta x \to \infty$ limit of slab.
3. Spherical region, of radius l; Casini-Huerta-Myers (CHM) case.

Focus on the first two in this talk.

Entangling surfaces



For the slab:

- AdS_5 result:

$$S = \frac{L^2}{2G_N}\left(\frac{1}{2\epsilon^2} + \frac{\sqrt{\pi}\Gamma(-\frac{1}{3})}{6\Gamma(\frac{1}{6}){z^*}^2}\right)$$

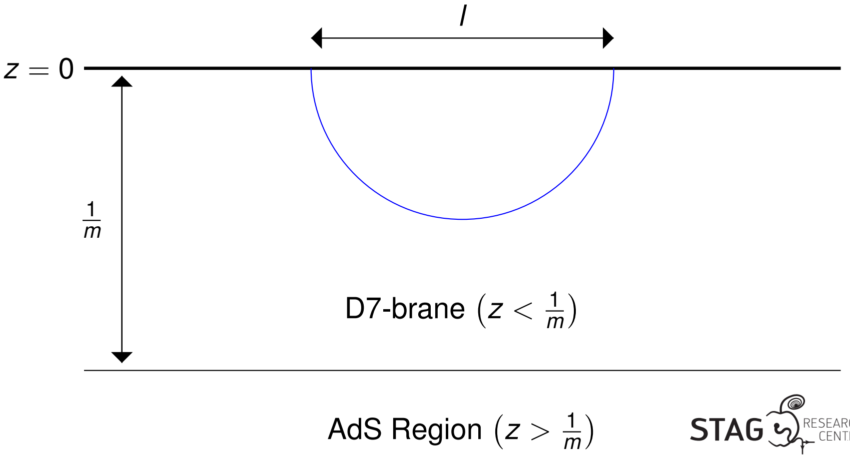
  with L^2 the regulated area of the y, z directions, ϵ the UV cutoff and z^* the turning point of the bulk entangling surface.

- The turning point is linearly related to the slab width

$$l = c_0 z^*.$$

Entangling surfaces

Since the D7-branes extend only to $z = 1/m$, the entanglement depends on whether the turning point of the entangling surface is as $z^* < 1/m$ or $z^* \geq 1/m$.



STAG RESEARCH CENTRE

Results for EE

- For $mz^* \leq 1$:

$$\delta S = \frac{t_0 L^2}{48 G_N} \left(\frac{1}{2\epsilon^2} + \frac{2}{3} m^2 + \frac{\sqrt{\pi}}{12 z^{*2}} \frac{\Gamma(-1/3)}{\Gamma(7/6)} + m^4 z^{*2} \frac{\sqrt{\pi}}{12} \frac{\Gamma(1/3)}{\Gamma(11/6)} + \frac{2}{3} m^2 \log(\epsilon^3 / 2 z^{*3}) \right) + \delta S_{\rm gauge}(m, \epsilon).$$

- For $mz^* \gg 1$:

$$\delta S = \frac{t_0 L^2}{48 G_N} \left(\frac{1}{2\epsilon^2} + 2 m^2 \log(m\epsilon) - \frac{1}{48 m^4 z^{*6}} + \cdots \right) + \delta S_{\rm gauge}(m, \epsilon)$$

STAG RESEARCH CENTRE

Half space limit

- The half space is obtained as the $l \to \infty$ limit at fixed m:

$$\delta S = \frac{t_0 L^2}{96 G_N} \left(\frac{1}{2\epsilon^2} + 2m^2 \log(m\epsilon) \right) + \delta S_{\rm gauge}(m, \epsilon)$$

Comments

- The entanglement entropy has a fourth order phase transition at $mz^* = 1$ (relic of probe approximation).
- The gauge dependent terms depend on our choice of $h(z)$, i.e. the gauge choice for the metric.
- The relation between the slab width l and the turning point z^* is corrected perturbatively:

$$l = (c_0 + t_0 c_1(z^*) + \cdots)z^*$$

STAG RESEARCH CENTRE

Scheme dependence and finite quantities

- EE is UV divergent.
- One is often interested in "universal" divergent terms....
- But one may also be interested in IR finite effects: e.g. finite mass in $d = 4$ (Hertzberg, Wilczek)

$$S_{HW} = m^4 \frac{\partial^2 S}{\partial (m^2)^2}$$

or finite slab width (Cardy et al)

$$S_l = l \frac{\partial S}{\partial l}$$

Renormalized entanglement entropy

- One can define a renormalized entanglement entropy via volume renormalisation (Witten, Graham), i.e.

$$S_{\text{ren}} = \frac{1}{4G_N}\int_{\gamma} d^{d-1}x\sqrt{g} - \frac{1}{4(d-2)G_N}\int_{\partial\gamma} d^{d-2}x\sqrt{H} + \cdots$$

  with H the induced boundary metric.

Many nice features:

- The metric gauge dependence cancels in the renormalized EE.
- Classification of when logarithmic terms arise (under relevant deformations).

Outline

- The D3/D7 system
- Entanglement entropy
- **Field theory interpretation**
- Differential entropy

Deformations of the CFT

- Recall that for the half space:

$$\delta S = \frac{t_0 L^2}{48 G_N}\left(\frac{1}{2\epsilon^2} + 2m^2 \log(m\epsilon)\right) + \delta S_{\rm gauge}(m,\epsilon)$$

- The $m \to 0$ limit follows from conformal invariance and agrees with the result for free massless hypermultiplets.

STAG RESEARCH CENTRE

- At finite mass the CFT is deformed as

$$I = I_{\rm CFT} + m \int d^4x \sqrt{-g} \mathcal{O}_3 .$$

- The change in the entanglement entropy under a relevant perturbation of dimension $\Delta = (d+2)/2$ has been argued to contain universal log divergences (Rosenhaus, Smolkin):

$$\delta S = \mathcal{N} m^2 \frac{(d-2)}{4(d-1)} \frac{\pi^{\frac{d+2}{2}}}{\Gamma(\frac{d+2}{2})} \mathcal{A} \log \left(\frac{\epsilon_{UV}}{\epsilon_{IR}} \right) ,$$

with $\mathcal{N}$ the operator normalisation and $\mathcal{A}$ the area of the slab.

STAG RESEARCH CENTRE

Deformations of the CFT

- Using the known operator normalisation we indeed obtain

$$\delta S = \frac{2\pi t_0}{3} m^2 \mathcal{A} \log \left(\frac{\epsilon_{UV}}{\epsilon_{IR}} \right)$$

  in agreement with our result, setting $\epsilon_{IR} = 1/m$.
- Moreover, the result agrees with the results for free massive hypers, i.e. there is a non-renormalisation theorem (which was not obvious given $\mathcal{N} = 2$ susy).

STAG RESEARCH CENTRE

Finite slab width

We can also understand the $ml \gg 1$ limit for the slab:

- The leading finite contribution is

$$\delta S = \frac{t_0 L^2}{48 G_N} \left(- \frac{1}{48 m^4 z^{*6}} \right) .$$

- Integrating out the massive flavors results in

$$I = I_{SYM} + \frac{1}{m^2} \int d^4 x \sqrt{-g} \mathcal{O}_6$$

with $\mathcal{O}_6$ an R-charged operator.

Finite slab width

- Symmetry implies that the leading contribution to the entanglement entropy is at order $1/m^4$.
- By translational invariance along the slab the EE scales as L^2.
- Hence
$$\delta S \sim \frac{L^2}{m^4 l^6}$$
on dimensional grounds, since there is no other scale in the theory.

We may also be able to match the coefficient (?)

Outline

- The D3/D7 system
- Entanglement entropy
- Field theory interpretation
- **Differential entropy**

Differential entropy

- The differential entropy is defined as

$$E = \sum_{k=1}^{\infty} [S(I_k) - S(I_k \cap I_{k+1})]$$

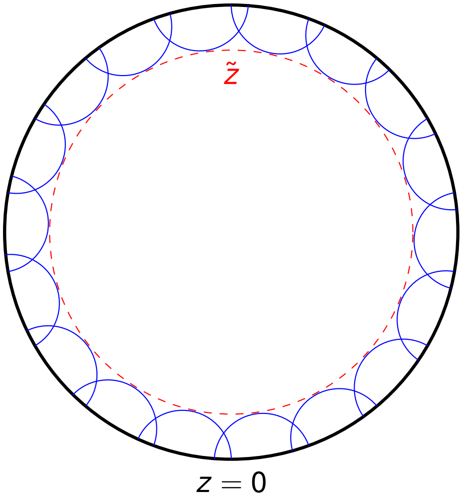
  where $\{I_k\}$ is a set of intervals partitioning the boundary.
- We will take $\{I_k\}$ to be slabs of width Δx, with intersection of width $(\Delta x - L_x/n)$, and take $n \to \infty$.

Holes and differential entropy

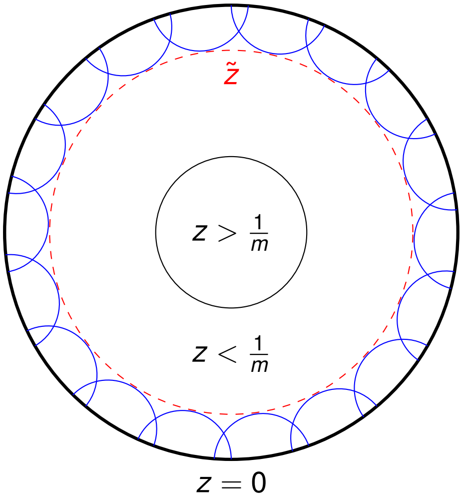
- In AdS_5 the differential entropy computes the area of a hole of radius z^*, the turning point of the entangling surface associated with each slab.
- This equivalence can be proved geometrically (Balasubramanian et al; Myers et al; Headrick et al).

Differential entropy

Witten diagram showing differential entropy: differential entropy computes area of red hole.



Differential entropy for massive flavor systems



Differential entropy for massive flavor systems

It still computes the area of a hole in the 5d Einstein metric.

- For $ml \gg 1$ the metric is just AdS_5, yet the differential entropy is changed:

$$E = \frac{V}{4G_N}\left(\frac{c_0^3}{(\Delta x)^3} + \frac{t_0 c_0^6}{384 m^4 (\Delta x)^7}\right)$$

  with c_0 the number such that $\Delta x = c_0 z^* + \cdots$.
- The metric is unchanged, but the relation between Δx and the turning points of the entangling surface z^* is changed.

Differential entropy for massive flavor system

- The change is consistent with the viewpoint of the IR theory as an irrelevant deformation of SYM.
- Differential entropy however tells us only about the 5d metric, not the 10d spacetime.
- The former does not generically have the same causal structure e.g. Coulomb branch geometries.

STAG RESEARCH CENTRE

Conclusions

- We have developed a systematic method for computing EE for probe brane systems.
- Finite terms in the EE may be obtained using volume renormalization for the minimal surfaces.
- Exact coefficients in the EE can be matched.
- Differential entropy computes the area in the 5d Einstein metric, not the 10d metric.

Additional results and outlook

- Phenomenology: finite density, phase transitions.
- General results for shape and field theory dependence (including irrelevant deformations).
- Interpretations of differential entropy in the field theory?