Diagrammatic Monte-Carlo for large-N non-Abelian lattice field theories based on the convergent weak-coupling expansion

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Diagrammatic Monte-Carlo for dense QCD and sign problem

So far lattice strong-coupling expansion:

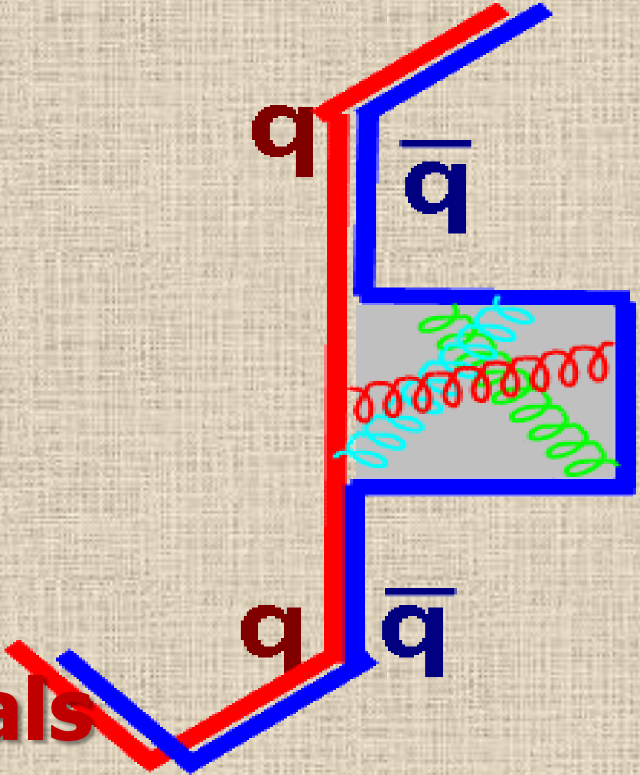
(leading order or few lowest orders)

[de Forcrand,Philipsen,Unger, Gattringer,...]

- **Worldlines of quarks/mesons/baryons**
- **Confining strings**

Very good approximation!

Physical degrees of freedom!



✓ **Phase diagram, tri-critical**

X **Hadron spectrum, potentials**

Lattice strong-coupling expansion

- **Confinement**
- **Dynamical mass gap generation**

ARE NATURAL, BUT…

Continuum physics is at weak-coupling!

DiagMC @ Weak-coupling?

Non-perturbative physics via Resurgence

DiagMC algorithms from Schwinger-Dyson

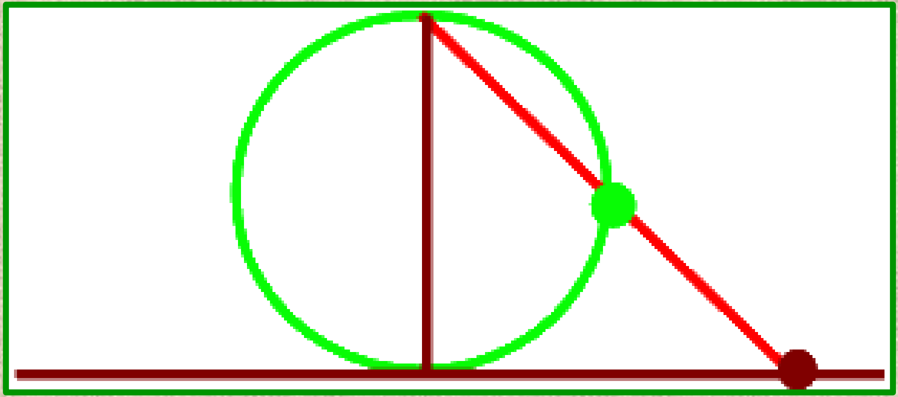
Perturbative DiagMC

it should be first-principle and automatic

- **Take N->∞ to reduce diagram space**
- **Small fluctuations of SU(N) fields**
- **Map SU(N) to Hermitian matrices**

Cayley map

$$g = \frac{1+i\alpha\phi}{1-i\alpha\phi}$$



$$\int\limits_{SU(N)} dg \Rightarrow \int\limits_{\mathbb{H}_{N\times N}} d\phi \, \det\left(1+\alpha^2\phi^2\right)^{-N} =$$

$$= \int\limits_{\mathbb{H}_{N\times N}} d\phi \, \exp\left(-N\alpha^2 \mathrm{Tr}\,\phi^2 + O\left(\alpha^4\phi^4\right)\right)$$

SU(N) principal chiral model

$$\mathcal{Z} = \int\limits_{U(N)} dg_x \exp\left(-\frac{N}{\lambda}\sum_{<x,y>} \mathrm{Tr}\,\left(g_x^\dagger g_y\right)\right)$$

$$\alpha^2 = \frac{\lambda}{8}$$

Expand action and Jacobian in *φ*

Infinitely many interaction vertices

$$S[\phi_x] = \frac{1}{2}\sum_{x,y}\left(D_{xy} + \frac{\lambda}{4}\delta_{xy}\right)\mathrm{Tr}\,(\phi_x\phi_y) +$$

$$+ \sum_{n=2}^{+\infty}\left(-\frac{\lambda}{8}\right)^{n-1}\left(\frac{\lambda}{8n}\sum_x \mathrm{Tr}\,\phi_x^{2n} + \right.$$

$$\left. \frac{1}{2}\sum_{l=1}^{2n-1}(-1)^{l-1}\sum_{x,y} D_{xy}\mathrm{Tr}\,\left(\phi_x^{2n-l}\phi_y^l\right)\right)$$

SU(N) principal chiral model

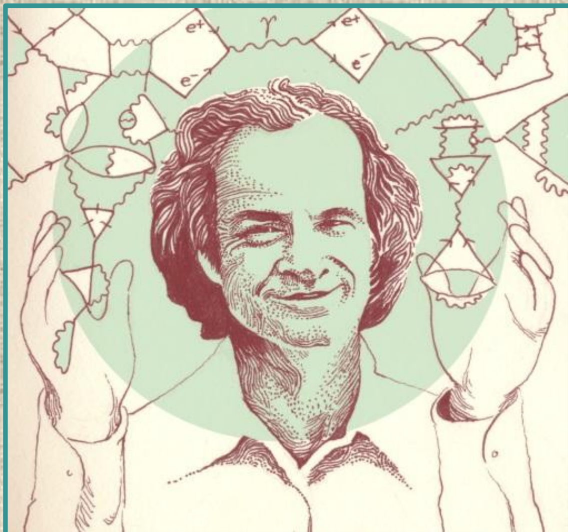
Power series in t'Hooft λ?

Factorial growth even at large N
due to IR renormalons … [Bali, Pineda]
Can be sampled, but resummation difficult

…Bare mass term ~λ from Jacobian???
[a-la Fujikawa for axial anomaly]

- ✓ **Massive planar fields**
- ✓ **Suitable for DiagMC**
- ? **How to expand in λ?**

Count vertices !??



Minimal working example:
2D O(N) sigma model @ large N

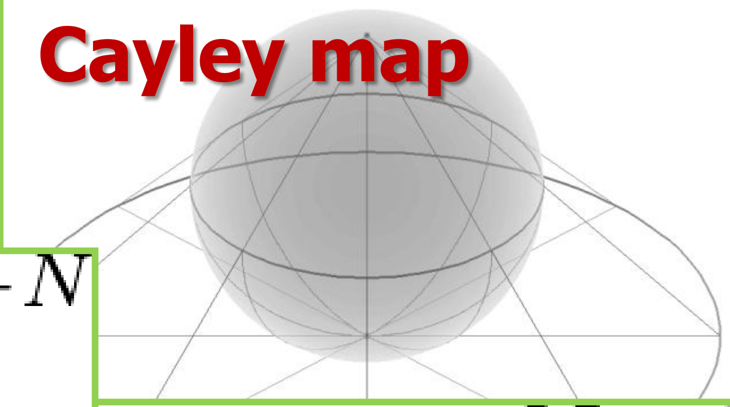
$$\int_{S_N} d\vec{n}_x \exp\left(-\frac{1}{\alpha^2}\sum_{<x,y>} \vec{n}_x\cdot\vec{n}_y\right)$$

$$\langle\, \vec{n}_x \cdot \vec{n}_y \,\rangle \sim$$
$$\sim \exp\left(-m^2|x-y|\right)$$

Non-perturbative mass gap

$$m^2 = 32\exp\left(-\frac{4\pi}{\alpha^2}\right)$$

Cayley map



Jacobian reads

$$\mathcal{D}n_x = \mathcal{D}\phi_x\left(1+\frac{\lambda}{4}\phi_x^2\right)^{-N}$$

$$S_N \to \mathbb{R}^{N-1}$$

Again, bare mass term from the Jacobian…

[PB, 1510.06568]

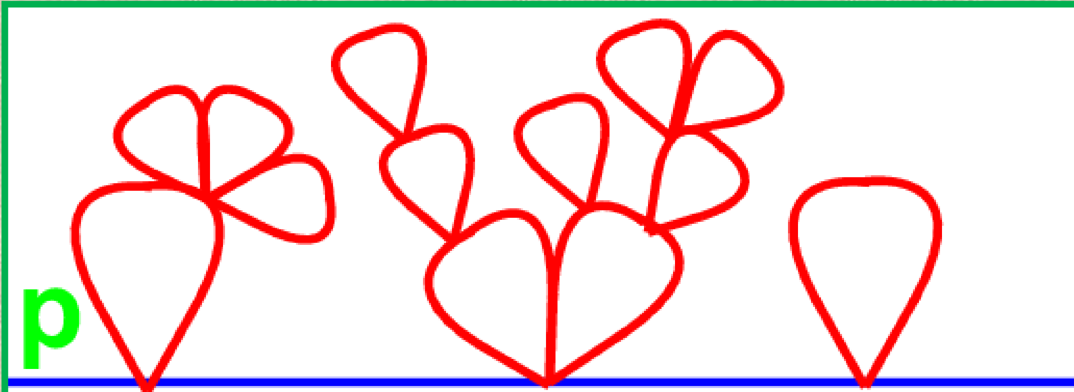
O(N) sigma model @ large N

Full action in new coordinates

$$S[\phi_x] = \frac{1}{2}\sum_{x,y}\left(D_{xy} + \frac{\lambda}{2}\delta_{xy}\right)\phi_x \cdot \phi_y +$$
$$+ \sum_{k=2}^{+\infty} \frac{(-1)^{k-1}\lambda^k}{4^k\, k} \sum_x \left(\phi_x^2\right)^k +$$
$$+ \sum_{\substack{k,l=0 \\ k+l\neq 0}}^{+\infty} \frac{(-1)^{k+l}\lambda^{k+l}}{2\cdot 4^{k+l}} \sum_{x,y} D_{xy}\left(\phi_x^2\right)^k \left(\phi_y^2\right)^l (\phi_x \cdot \phi_y)$$

We blindly do perturbation theory [with A.Davody]

Only cactus diagrams @ large N



Trans-series and Resurgence

From our perturbative expansion we get

$$m^2 = \sum_{p,q=0}^{+\infty} c_{p,q}\, \lambda^p \, (\log \lambda)^q$$

Same for PCM!!!

Resurgent trans-series [Écalle,81]

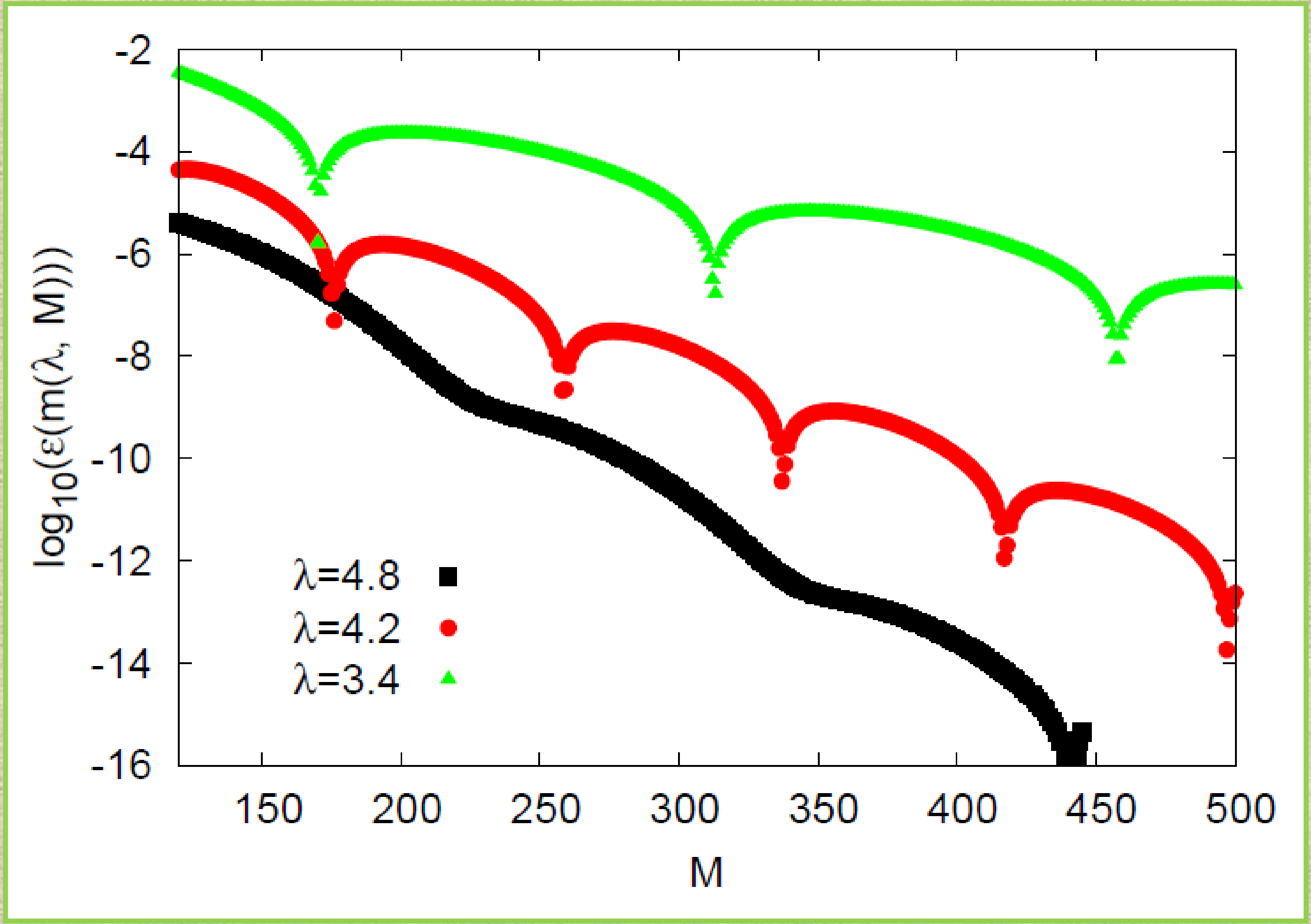
$$f(z) = \sum_{p,q,r} c_{p,q,r}\, z^p \, (\log z)^q \left(e^{-\frac{S}{z}}\right)^r$$

PT Zero modes Classical solutions

[Argyres,Dunne,Unsal,...,2011-present]

$$\exp\left(-\frac{1}{\lambda}\right) = \exp\left(-e^{-\log(\lambda)}\right) = \sum_k c_k \, (\log \lambda)^k$$

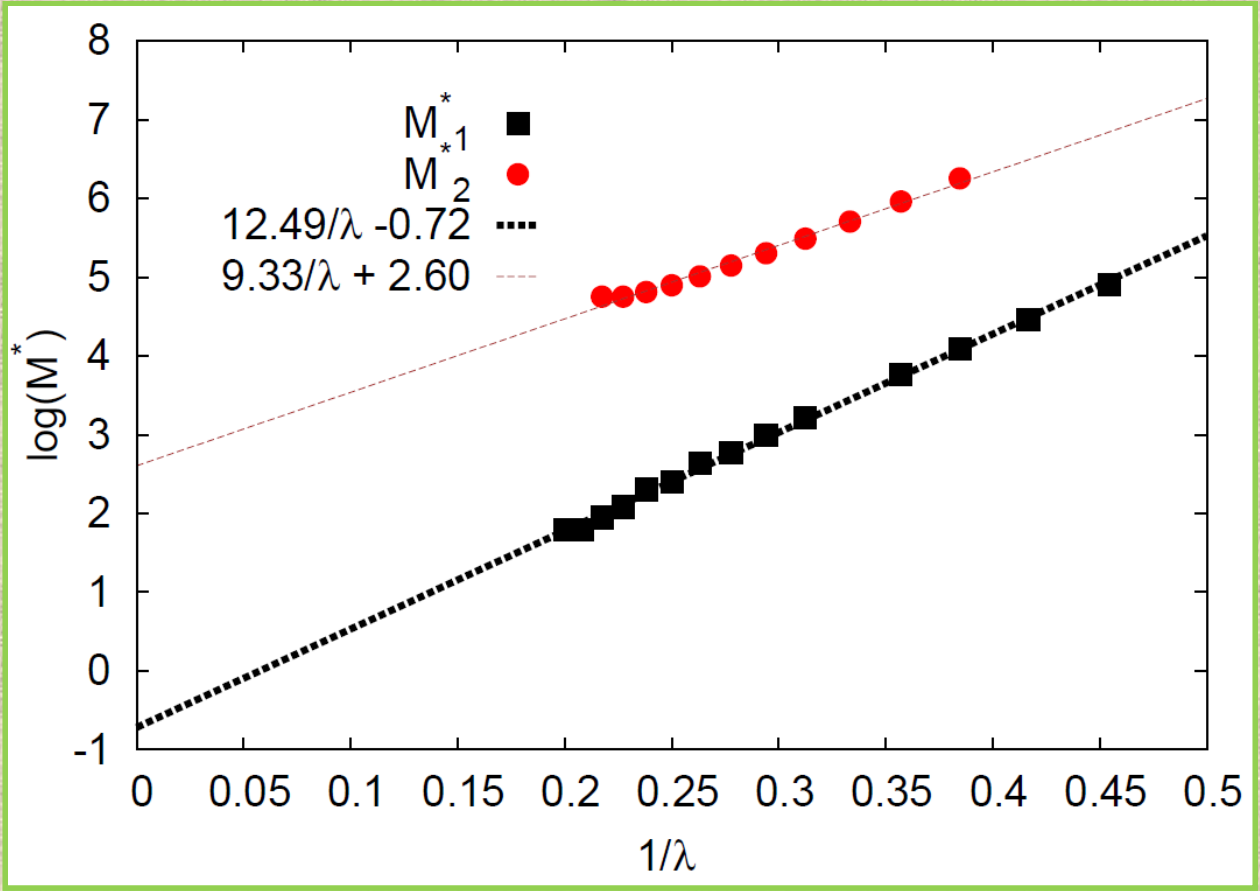
O(N) sigma model @ large N



Relative error of mass vs. order M

Numerical evidence of convergence!!!

O(N) sigma model @ large N



Convergence rate: first extremum in ε(M)

Has physical scaling!!!

Similar to critical slowing-down in Monte-Carlo

$$M_1^\star \sim l_c^2 \sim m^{-2} \sim \exp\left(\frac{4\pi}{\lambda}\right)$$

(Back to) Principal Chiral Model

Now we need DiagMC, all planar diagrams?

➡ **Basic idea: SD equations for disconnected correlators are linear, allow for stochastic solution ➡ DiagMC**

$$\phi(X) = b(X) + \sum_{Y} A(X|Y)\,\phi(Y),$$

φ(X): **all correlators of the form**

$$\langle\, \phi(x_1)\,\phi(x_2)\ldots\phi(x_n)\,\rangle$$

X: **all sequences** $\{x_1, \ldots, x_n\}$

b(X): **contact terms in SD equations**

DiagMC from SD equations

Solution is a formal series of the form

$$\phi(X) = \sum_{n=0}^{+\infty} \sum_{X_0} \ldots \sum_{X_n} \delta(X, X_n) A(X_n|X_{n-1}) \ldots A(X_1|X_0) b(X_0)$$

Sample sequences $\{X_n, \ldots, X_0\}$ with the weight

$$w \sim |A(X_n|X_{n-1})| \ldots |A(X_1|X_0)| |b(X_0)|$$

We use the Metropolis algorithm:

Two basic transitions:

$$\{X_n, \ldots, X_0\} \rightarrow \{X_{n+1}, X_n, \ldots, X_0\}$$

- **Add new index X_{n+1},**

$$\pi(X_{n+1}|X_n) = \frac{A(X_{n+1}|X_n)}{\mathcal{N}(X_n)}$$

- **Remove index**

$$\{X_n, X_{n-1}, \ldots, X_0\} \rightarrow \{X_{n-1}, \ldots, X_0\}$$

- **Restart** $\{X_0\} \rightarrow \{X_0'\}$ $\qquad \pi(X_0') = b(X_0') / \mathcal{N}_b$

$$\mathcal{N}(Y) = \sum_X A(X|Y), \quad \mathcal{N}_b = \sum_X b(X)$$

Stochastic diagram generation

Transformations $X_n \to X_{n+1}$:

Local and elementary updates of diagrams

* $A(X|Y)$ very sparse, no need to keep in RAM

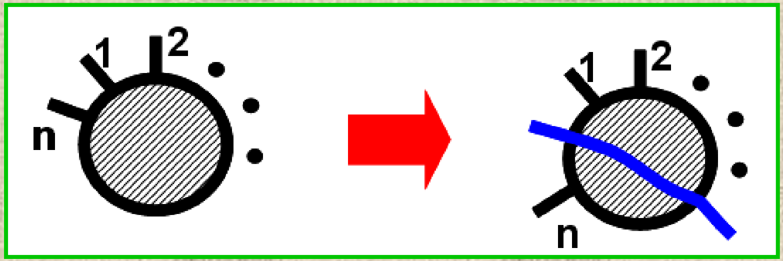
* Sequences $\{X_n, \ldots, X_0\}$ can be very compactly characterized by sequence of transitions $\{X_0 \to X_1, X_1 \to X_2, \ldots, X_{n-1} \to X_n\}$

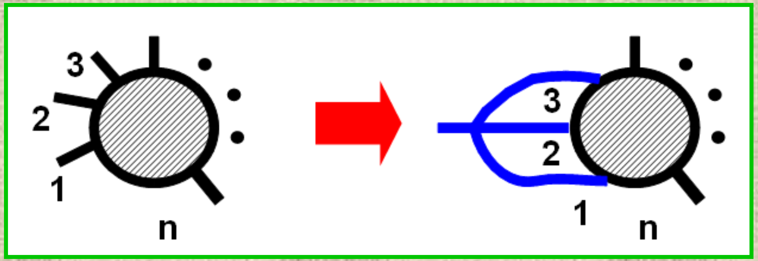
* Graphical editor for drawing diagrams,

* Terms in SD equations -> Drawing operations

* Drawing random diagram elements + Removing by "Undo" operations

* Planar structure automatically preserved





Schwinger-Dyson for PCM

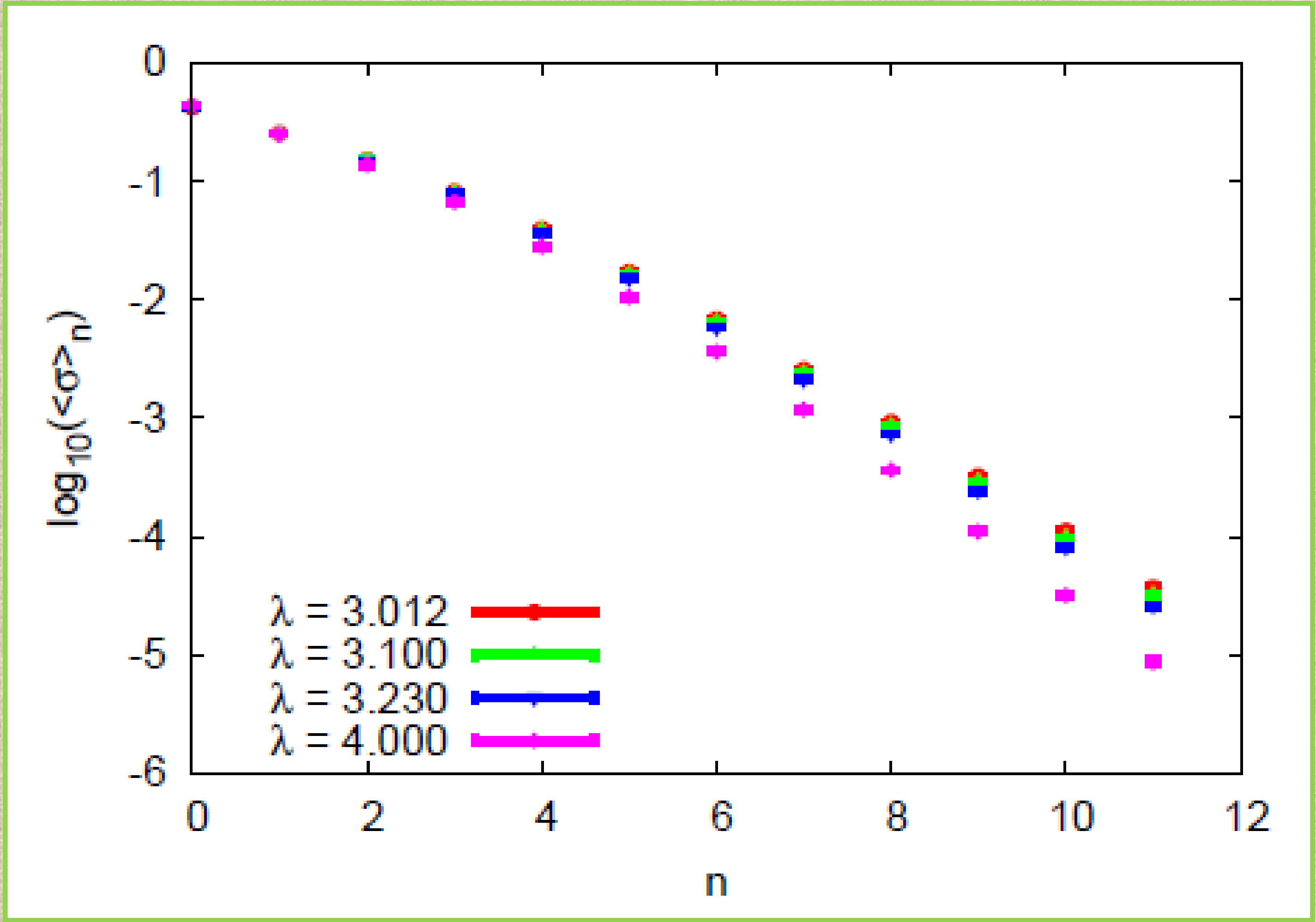
$$\langle\, p_1 p_2 \,\rangle_m = \delta_{m,0} \frac{\delta(p_1+p_2)}{V} G_0\,(p_1) -$$
$$- \left(1-\delta_{m,0}\right) G_0\,(p_1) \sum_{v=1}^{m} \sum_{q_1,\ldots,q_{2v+1}} \delta\,(p_1-\bar{q})\, V\,(q_1,\ldots,q_{2v+1})\,\langle\, q_1 \ldots q_{2v+1}\, p_2 \,\rangle_{m-v}$$

* Scalar field theory with infinitely many interaction vertices (momentum-dependent)
* Some interaction vertices have negative weight ➡ reweighting + sign problem?

$$\langle\, \tfrac{1}{N}\mathrm{Tr}\,\left(g_x^{\dagger}\, g_y\right) \rangle = 2\langle\, \tfrac{1}{N}\mathrm{Tr}\, g_x \,\rangle - 1 + 4 \sum_{k,l=1}^{+\infty} (-1)^{\frac{k-l}{2}} \left(\tfrac{\lambda}{8}\right)^{\frac{k+l}{2}} \langle\, \tfrac{1}{N}\mathrm{Tr}\,\left(\phi_x^k \phi_y^l\right) \rangle =$$
$$= \left(2\langle\, \tfrac{1}{N}\mathrm{Tr}\, g_x \,\rangle - 1\right) + 4 \sum_{k=1}^{+\infty} \left(-\tfrac{\lambda}{8}\right)^k \sum_{l=1}^{2k-1} (-1)^l \langle\, \tfrac{1}{N}\mathrm{Tr}\,\left(\phi_x^l \phi_y^{2k-l}\right) \rangle$$

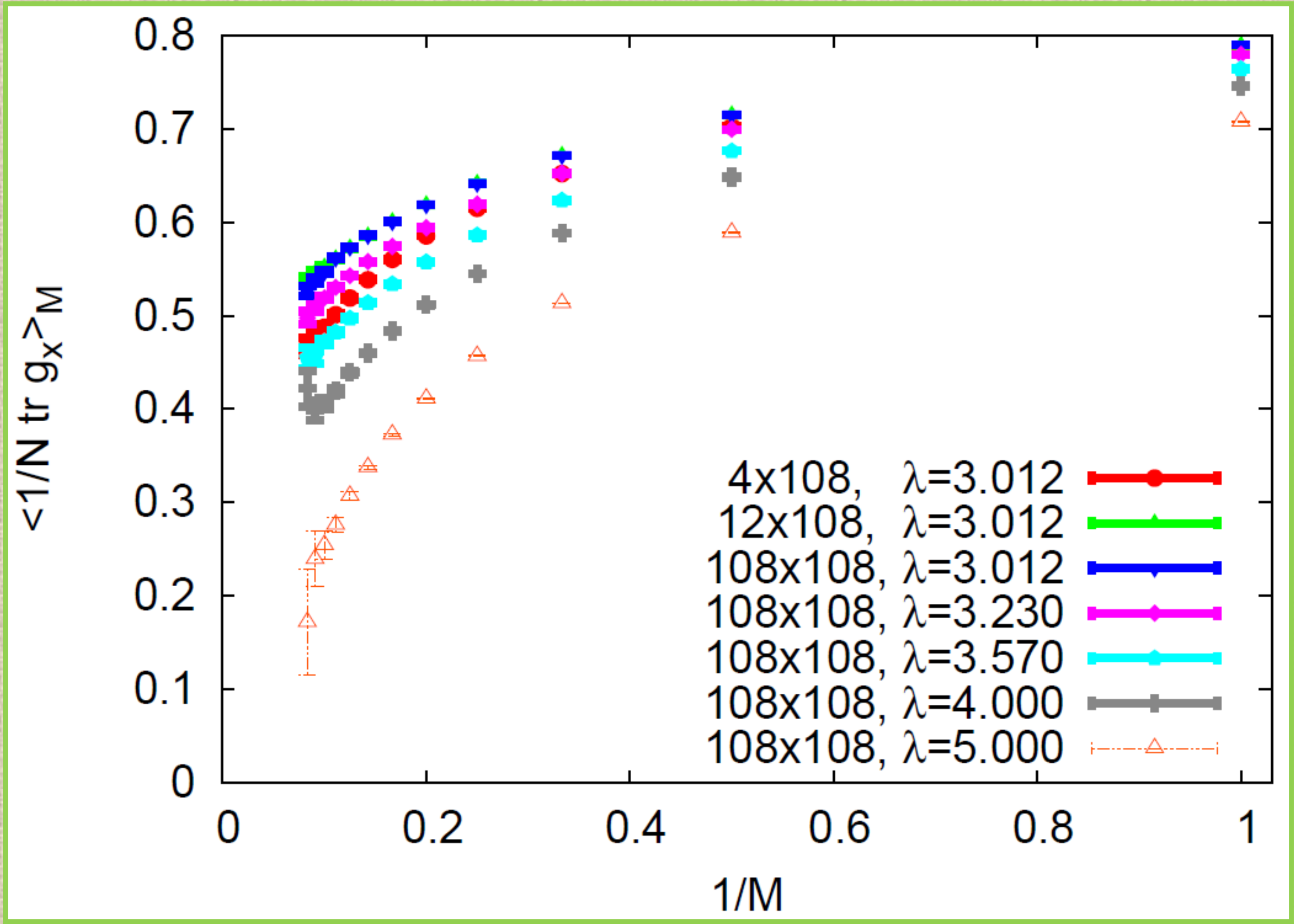
* Sign cancellations also in observables

Sign problem at high orders



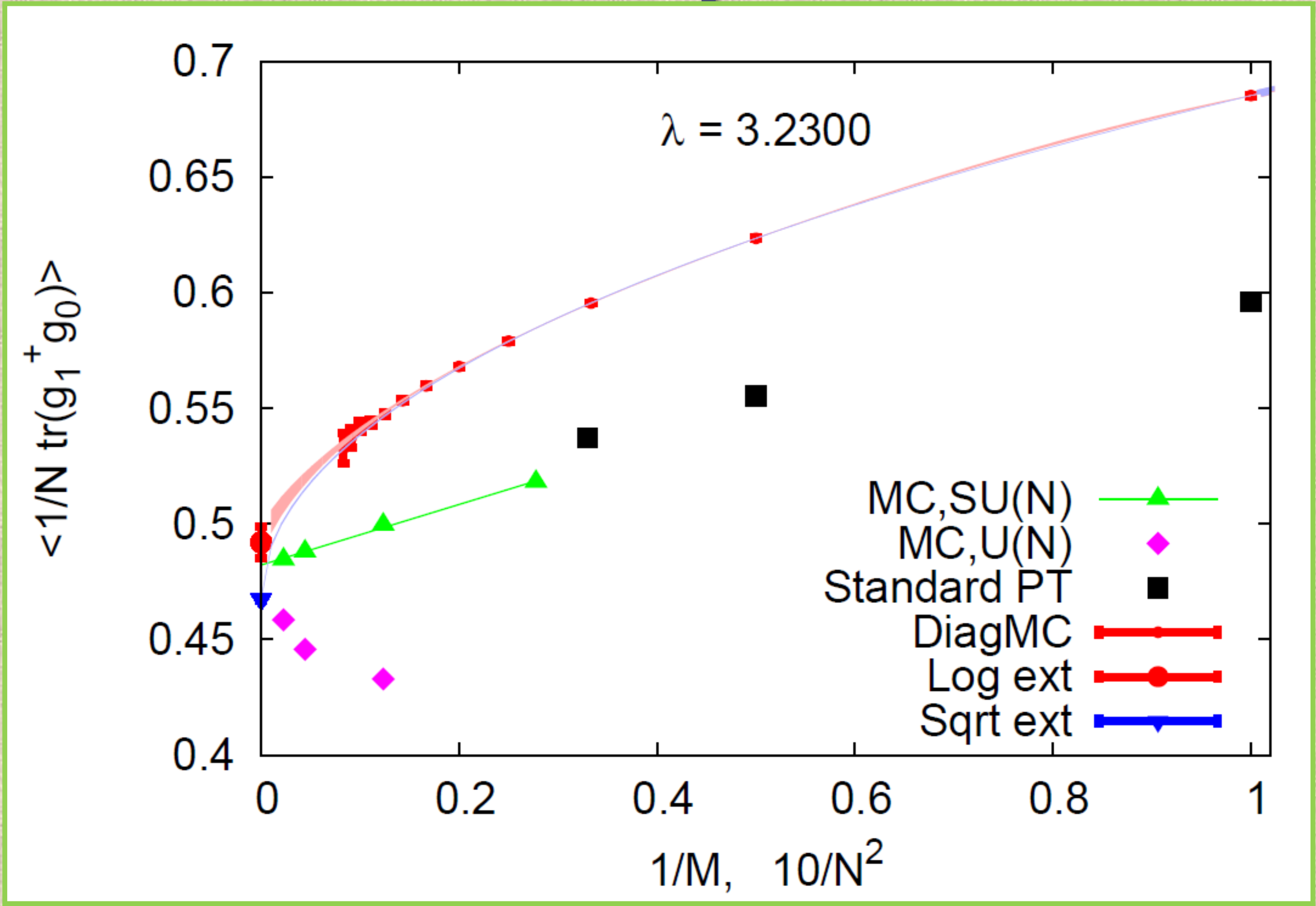
- Mean **sign decays exponentially** with order
- Limits practical simulations to **orders ~ 10**
- Sign problem depends on **spacing, not volume**

Restoration of SU(N)xSU(N) symmetry



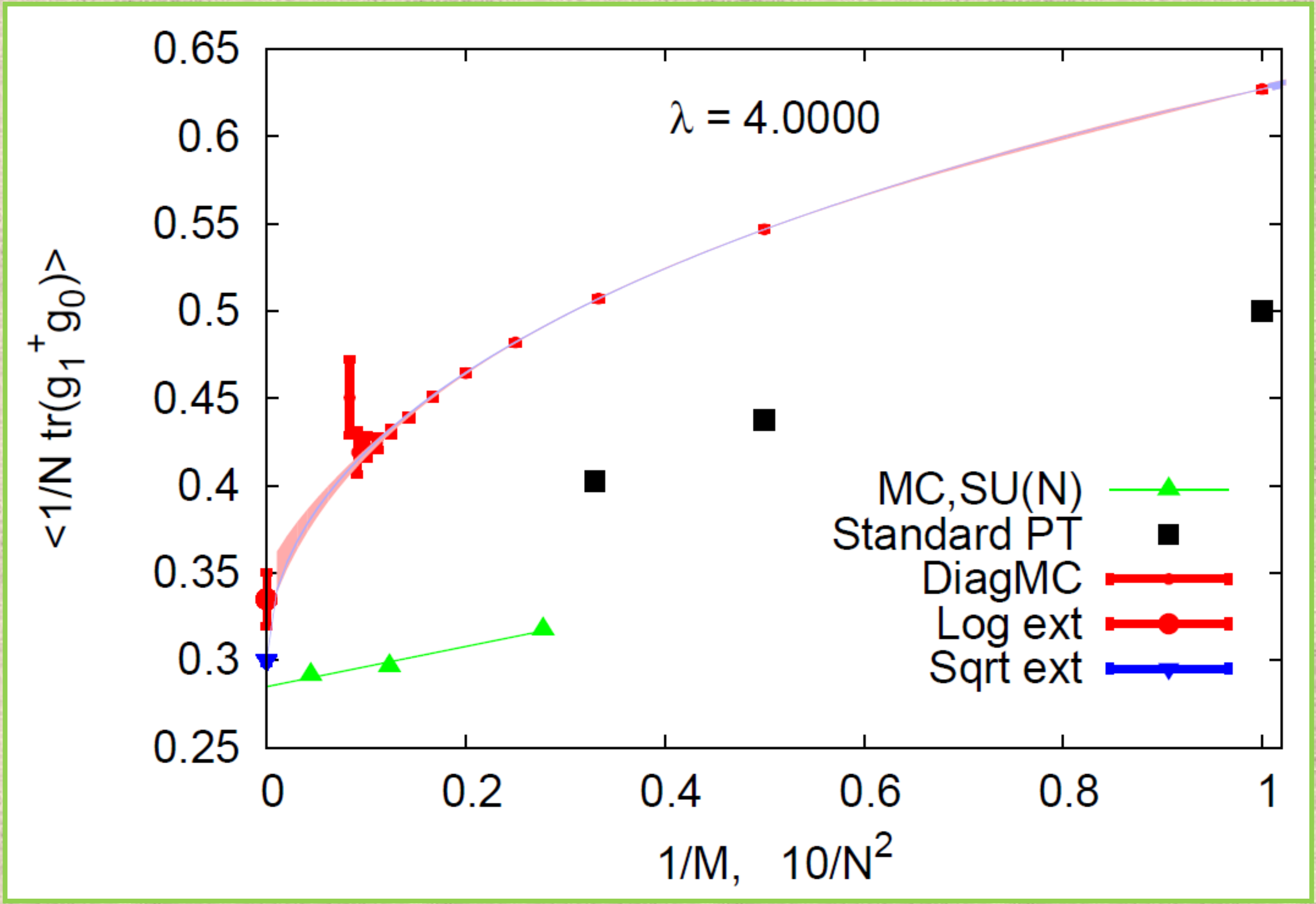
* Perturbative vacuum not SU(N)xSU(N) symm.
* Symmetry seems to be restored at high orders
* Restoration is rather slow

Mean link vs expansion order



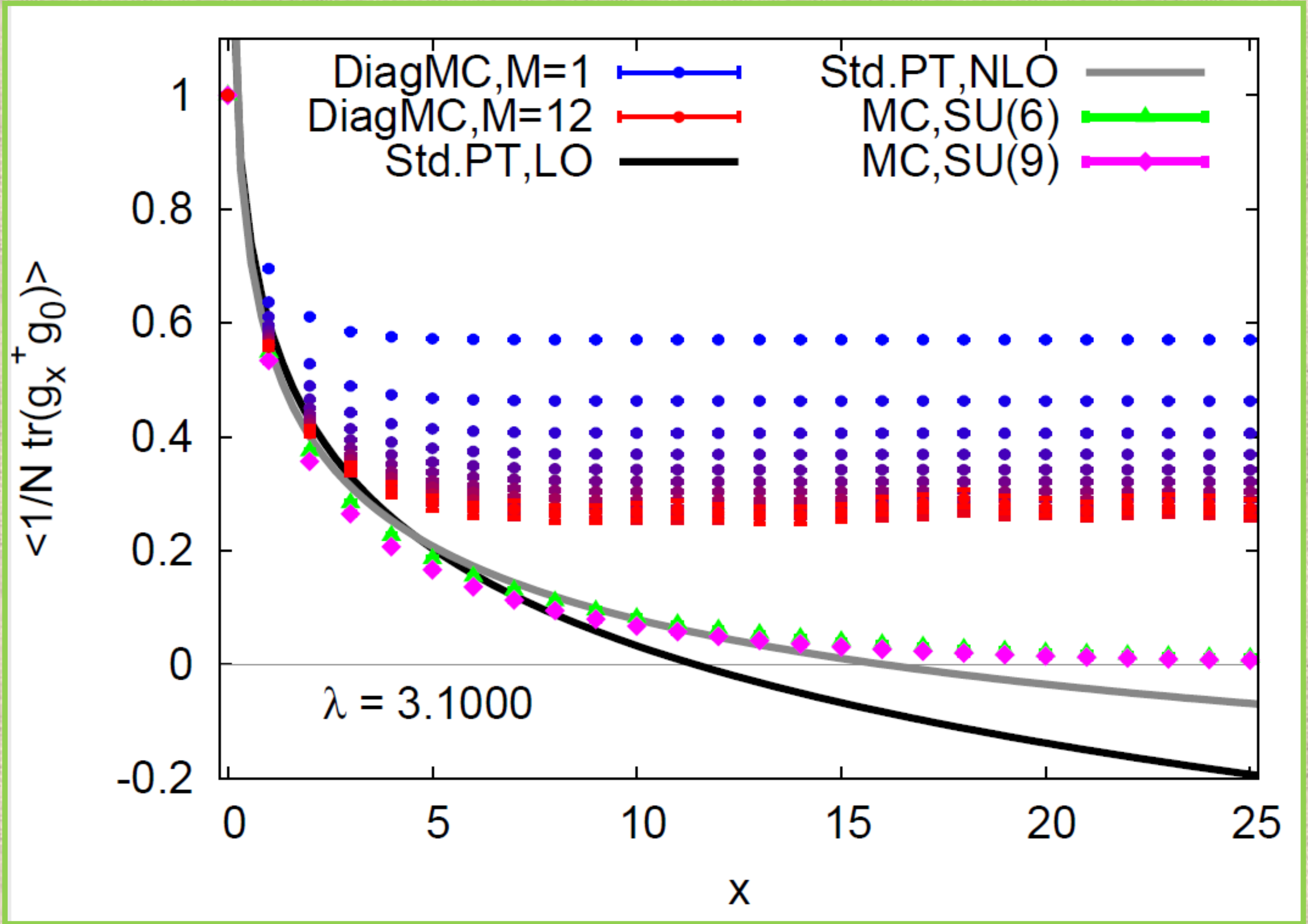
- Good agreement with ***N->∞ extrapolation***
- Convergence slower than for standard PT
- MC Data from [Vicari, Rossi, Campostrini'94-95]

Mean link vs expansion order



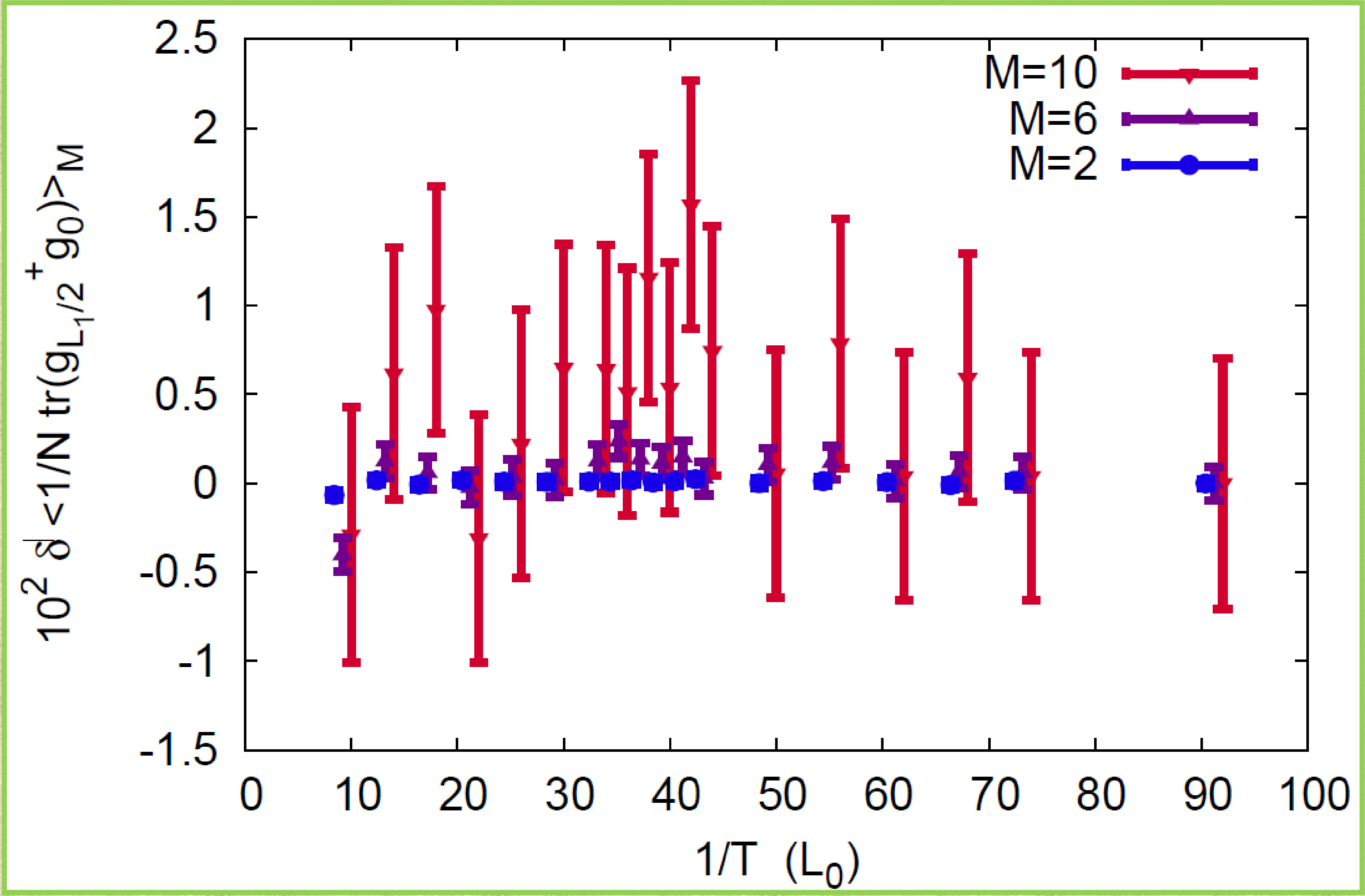
- **Wrong or no convergence after large-N phase transition (*$\lambda > \lambda_c = 3.27$*) [hep-lat/9412102]**

Long-distance correlators



- Constant values at large distances, consistent with *$<1/N\, tr\, g_x> > 0$*
- Converges slower than standart PT, but IR-finite

Finite temperature (phase) transition?



Weak enhancement of correlations at L0 ~ 35-40 [Poster of S. Valgushev]

Resume

Weak-coupling DiagMC in the large-N limit:

\+ IR-finite, convergent series

\+ Volume-independent algorithm

\- Sign problem vs. Standard MC

\- Slower convergence than standard PT

\- Starts with symmetry-breaking vacuum

\- Finite-density matter: complex propagators

$$G\left(p\right) \sim \frac{1}{(p_0 - i\mu)^2 + \vec{p}^2 + m^2}$$

Outlook

Resummation of logs:

$$\sum_{k,m} c_{km} \log(\lambda)^k \lambda^l \to \exp\left(-\beta_0/\lambda\right)$$

Easy in mean-field-approximation
(for O(N) sigma-model just one exponent)
DiagMC with mean-field???
... Not easy in non-Abelian case

$$\mathcal{Z} = \int dg_x \int d\xi_x \exp\left(-\frac{N}{\lambda}\sum_{x\neq y} D_{xy}\mathrm{tr}\,\left(g_x^\dagger g_y\right) - \frac{iN}{\lambda}\sum_x \mathrm{tr}\,\left(\xi_x g_x^\dagger g_x - \xi_x\right)\right) =$$
$$= \int d\xi_x \exp\left(N\mathrm{tr}\,\ln\left(D_{xy} + i\xi_x\delta_{xy}\right) + \frac{iN}{\lambda}\sum_x \mathrm{tr}\,\xi_x\right)$$

... Matrix-valued Lagrange multipliers

Outlook

DiagMC based on strong-coupling expansion?

+ Sign problem really reduced

+ Volume-independent

+ Correct vacuum from the very beginning

+ Hadrons/mass gap/confinement are natural

- No continuum extrapolation

? In practice, high-order SC expansion can work „reasonably" well even in the scaling region …

? DiagMC for SC expansion? Possible in loop space, but needs fixing the „Zigzag" symmetry