Hecke Relations in Rational Conformal Field Theory

(with Y. Wu, arXiv:1804.06860)

50 Years of the Veneziano Model

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SUMMARY I

Two-dimensional Conformal Field Theory started with studies of phase transitions, grew into a framework for classical solutions of string theory and now plays a central role in physics and mathematics: Topological insulators, AdS/CFT, Moonshine.

Rational CFT has a chiral algebra $\mathcal{A}$ large enough to reduce the Hilbert space to a finite number of representations of $\mathcal{A}$. RCFT has also been hugely influential.

There is a hidden symmetry relating characters of many different RCFTs based on the mathematical theory of Hecke operators.

Examples:

The Yang-Lee model is a non-unitary minimal model $\mathsf{M}(5,2)$ (RCFT) with two independent characters

$$\chi_0^{YL} = q^{-1/60} \sum_{n=0} c_0^{YL}(n) q^n$$

$$\chi_{1/5}^{YL} = q^{11/60} \sum_{n=0} c_{1/5}^{YL}(n) q^n$$

Affine G_2 is a unitary rational CFT with two independent characters

$$\chi_0^{G_2} = q^{-7/60} \sum_{n=0} c_0^{G_2}(n) q^n$$

$$\chi_{17/60}^{G_2} = q^{17/60} \sum_{n=0} c_{17/60}^{G_2}(n) q^n$$

Although apparently unrelated, there is a subtle relation between the character coefficients:

DATA

coefficients of q expansion of h=1/5 character of Yang-Lee model c=-22/5

$n\backslash k$	1	2	3	4	5	6	7
0	0	1	1	1	1	2	2
1	3	3	4	4	6	6	8
2	9	11	12	15	16	20	22
3	26	29	35	38	45	50	58
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24	4081118	4324669	4581514	4853375	5140036	5443362	5763114
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27	13343681	14095665	14888948	15723845	16604348	17530906	18507742

Table 4: Coefficients $c^{YL}_{1/5}(7n+k)$

coefficients of q expansion of vacuum character of affine G2 at level one (c=14/5) divided by 7

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102428	
166450	
266850	
422966	
662780	
7198178/7	1/7
1579853	
2406046	
3633082	
5443362	
8094202	
11952388	
122716344/7	2/7

RCFT

Hilbert space: $$\mathcal{H} = \bigoplus_{i,\bar{i}} \mathcal{N}_{i,\bar{i}} V_i \otimes \overline{V_{\bar{i}}}$$

Representation of chiral algebra, e.g. Virasoro for minimal models

Characters: $$\chi_i(\tau) = \mathrm{Tr}_{V_i} q^{L_0 - c/24}, \qquad q = e^{2\pi i \tau}$$

Partition function: $$Z(\tau) = \sum_{i \in \mathcal{I}, \bar{i} \in \bar{\mathcal{I}}} \mathcal{N}_{i,\bar{i}} \chi_i(\tau) \overline{\chi_{\bar{i}}(\tau)}$$

Examples: Ising model, Yang-Lee model, affine Lie algebras, Monster VOA, BM VOA

Modular Properties

Characters are weakly holomorphic weight zero vector-valued modular functions transforming according to

$$\rho : SL(2,\mathbb{Z}) \to GL(n,\mathbb{C})$$

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ (\tau \to -1/\tau) \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ (\tau \to \tau + 1)$$

$$(\rho(S))^2 = (\rho(S)\rho(T))^3 = C \qquad C^2 = 1$$

$$N = \mathrm{order}(\rho(T)) \qquad \Gamma(N) \subset \ker(\rho)$$ (Bantay)

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| \ a = d = 1 \bmod N,\ b = c = 0 \bmod N \right\}$$

Example

Ising Model

$$\chi_0^I = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) = q^{-1/48}(1 + q^2 + q^3 + 2q^4 + 2q^5 + \cdots),$$

$$\chi_{1/2}^I = \frac{1}{2}\left(\sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}\right) = q^{23/48}(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + \cdots),$$

$$\chi_{1/16}^I = \frac{1}{\sqrt{2}}\sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}} = q^{1/24}(1 + q + q^2 + 2q^3 + 2q^4 + 3q^5 + \cdots)\,.$$

$$\rho(S) = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \qquad \chi_i(-1/\tau) = \sum_j \rho(S)_{ij}\chi_j(\tau)$$

$$\rho(T) = e^{-2\pi i/48}\mathrm{diag}(1, e^{2\pi i/2}, e^{2\pi i/16}) \qquad \chi_i(\tau+1) = \sum_j \rho(T)_{ij}\chi_j(\tau)$$

$$N = 48$$

Galois Symmetry

K=Field extension of $\mathbb{Q}$ by matrix elements of $\rho(\gamma)$

De Boer & Goeree plus Kronecker-Weber:

$\longrightarrow$ $K \subseteq \mathbb{Q}[\zeta_N]$ N=conductor (also of RCFT).

$$\mathrm{Gal}(\mathbb{Q}[\zeta_N]) \cong (\mathbb{Z}/N\mathbb{Z})^\times$$

Primitive Nth root of unity

Group of units in Ring $\mathbb{Z}/N\mathbb{Z}$

Coste-Gannon: For each $\ell \in (\mathbb{Z}/N\mathbb{Z})^\times$

$$f_{N,\ell} : \rho(T) \to \rho(T)^\ell$$

$$\rho(S) \to G_\ell \rho(S)$$

$$f_{N,\ell} : \rho(S)_{i,j} \to \varepsilon_\ell(i)\rho(S)_{\pi_\ell(i),j} = \varepsilon_\ell(j)\rho(S)_{i,\pi_\ell(j)}$$

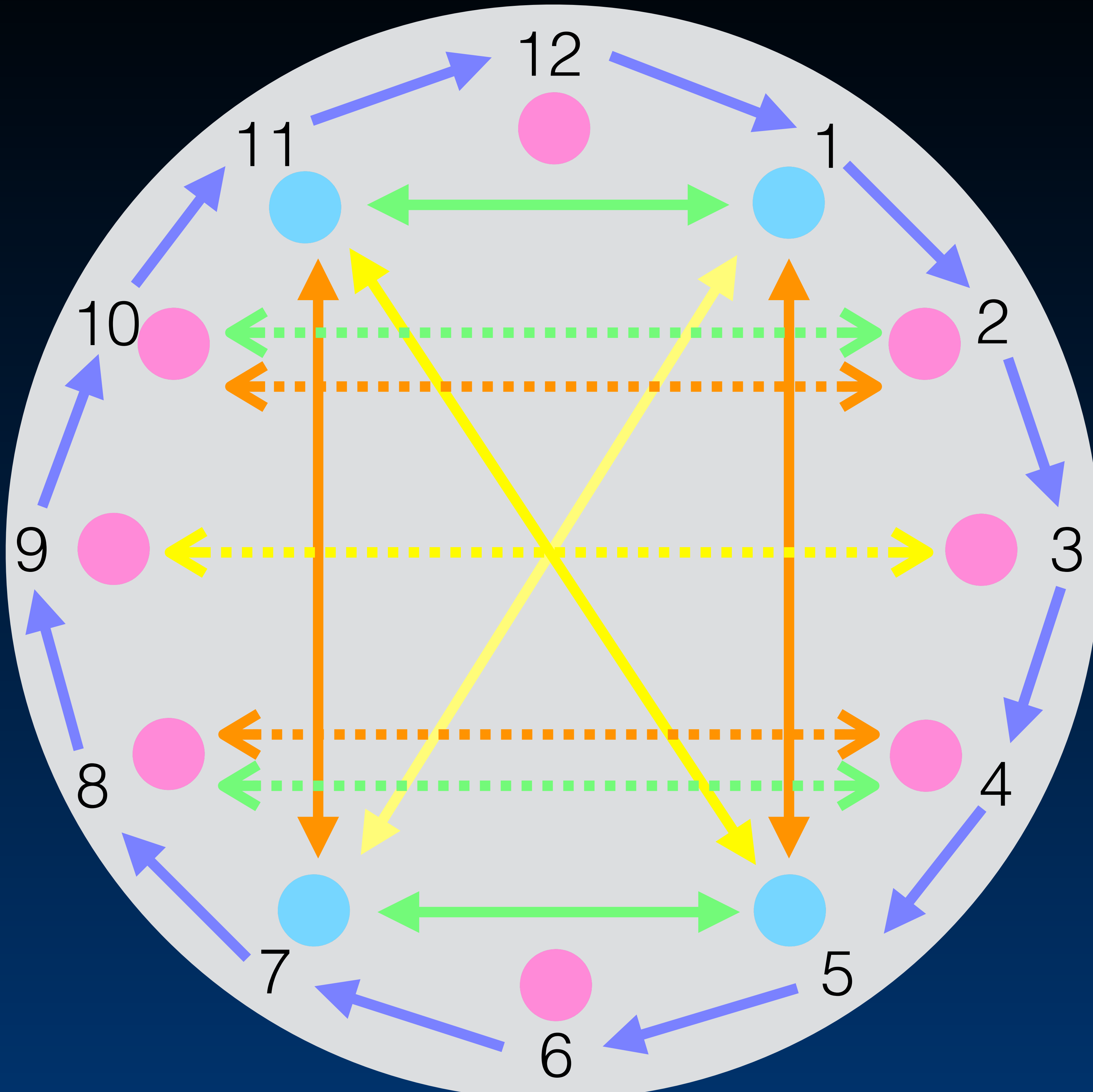
permutation of indices $\varepsilon = \pm 1$

$\mathbb{Z}/12\mathbb{Z}$

$(\mathbb{Z}/12\mathbb{Z})^\times$

$5 \quad 7 \quad 11$

$f_5 \quad f_7 \quad f_{11}$



Examples of Galois RCFT Relations

$$\rho(S)^{YL} = \begin{pmatrix} -\frac{1}{2\sin(\pi/5)} & \frac{1}{2\sin(2\pi/5)} \\ \frac{1}{2\sin(2\pi/5)} & \frac{1}{2\sin(\pi/5)} \end{pmatrix} \qquad \rho(S)^{G_2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rho(S)^{YL}$$

$$\xrightarrow{f_{60,7}}$$

$$\rho(T)^{YL} = \mathrm{diag}(e^{2\pi 11/60}, e^{-2\pi i/60}) \qquad \rho(T)^{G_2} = (\rho(T)^{YL})^7$$

Similarly Yang-Lee $\xrightarrow{f_{60,13}}$ F_4

Three character RCFT Ising $\xrightarrow{f_{48,47}}$ Baby Monster

These are relations between modular representations.
We will extend them to relations between **characters**.

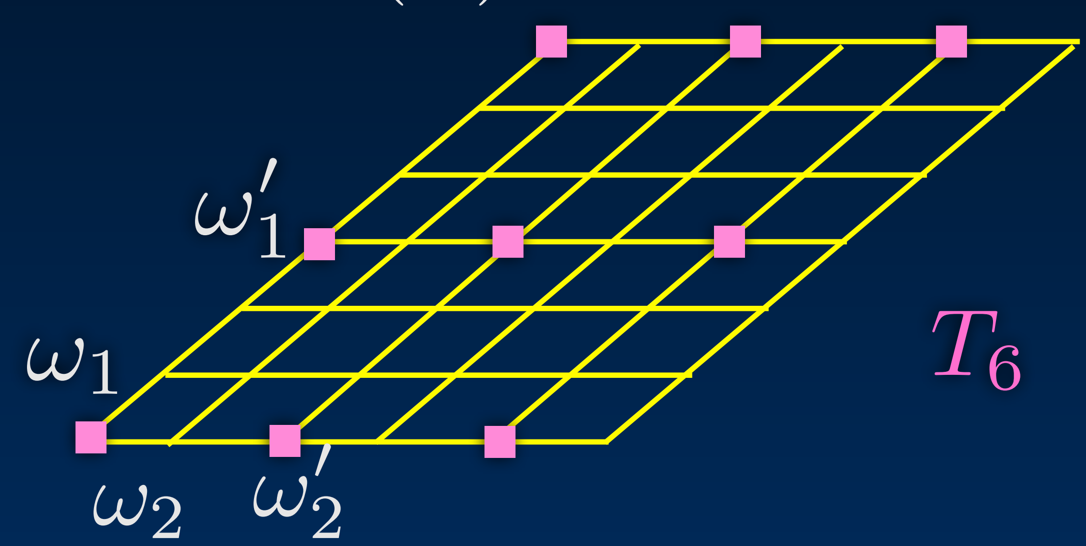
Scalar Hecke

Modular forms can be thought of in two ways:

Functions of $\tau \in \mathbb{H}$ $\quad f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$

Functions of rank 2 lattices $\quad F(\lambda L) = \lambda^{-k} F(L)$

$$F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \omega_2^{-k} f(\omega_1/\omega_2)$$



Hecke operator: $\quad (T_n F)(L) = \sum_{\substack{L' \subset L \\ |L/L'| = n}} F(L')$

$$\omega_1' = a\omega_1 + b\omega_2$$
$$\omega_2' = c\omega_1 + d\omega_2 \qquad ad - bc = n$$

Hecke Operators

When translated into an action on modular forms as a function of τ or $q = e^{2\pi i \tau}$ this leads to

$$f = \sum_n a(n) q^n \qquad (T_p f)(\tau) = \sum_n a^{(p)}(n) q^n$$

Action on Fourier coefficients:

$$a^{(p)}(n) = \begin{cases} p^k a(pn) & \text{if } p \nmid n, \\ p^{k-1}\big(p\, a(pn) + a(n/p)\big) & \text{if } p|n. \end{cases}$$

To generalize Hecke operators to RCFT characters:

Use Hecke operators for $\Gamma(N)$ (Rankin, Chap. 9)

Use modular representation properties of RCFT characters.

This leads to the following formula for Hecke images of RCFT characters for (p,N)=1:

On Fourier coefficients:
$$\chi_i(\tau) = \sum_n b_i(n) q^{n/N}$$
$$(\mathsf{T}_p\chi)_i(\tau) = \sum_n b_i^{(p)}(n) q^{n/N}$$

$$b_i^{(p)}(n) = \begin{cases} p b_i(np) & p \nmid n \\ p b_i(np) + \sum_b \rho_{ij}(\sigma_p) b_j(n/p) & p | n \end{cases}$$

The $(\mathsf{T}_p\chi)_i$ are again modular forms for $\Gamma(N)$ but transform under a different representation of $SL(2,\mathbb{Z})$

where σ_p is the pre-image of $\begin{pmatrix} \bar{p} & 0 \\ 0 & p \end{pmatrix}$ under the mod N map $SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}/N\mathbb{Z})$ and $\bar{p}p = 1 \bmod N$

Example: $N = 60, p = 7, \bar{p} = 43$ $\qquad \sigma_7 = \begin{pmatrix} 27343 & -33780 \\ 480 & -593 \end{pmatrix}$

Representation of Hecke image $\qquad \rho^{(p)}(S) = \rho(\sigma_p S), \qquad \rho^{(p)}(T) = \rho(T^{\bar{p}})$

The change of representation under Hecke is the same as that under Galois for $\ell = p,\ (p, N) = 1$.

The equivalence relies on the identities

$$f_{N,p}(\rho(S)) = \rho(\sigma_{\bar{p}} S)$$
$$f_{N,p}(\rho(T)) = \rho(T^p)$$

Applications and Examples

The example from the first DATA slide:

$$\chi_0^{YL} = q^{-1/60} G(q) = q^{-1/60} \sum_{n=0}^{\infty} c_0^{YL}(n) q^n$$

$$\chi_{1/5}^{YL} = q^{11/60} H(q) = q^{11/60} \sum_{n=0}^{\infty} c_{1/5}^{YL}(n) q^n$$

Then we have the Hecke relation $\chi^{G_2} = \mathsf{T}_7 \chi^{YL}$

$$c_0^{G_2}(n) = \begin{cases} 7c_{1/5}^{YL}(7n-1) & \text{if } 7 \nmid n, \\ 7c_{1/5}^{YL}(7n-1) + c_0^{YL}(\frac{n}{7}) & \text{if } 7|n; \end{cases} \qquad \rho^{YL}(\sigma_7) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$c_{2/5}^{G_2}(n) = \begin{cases} 7c_0^{YL}(7n+2) & \text{if } 7 \nmid (n-1), \\ 7c_0^{YL}(7n+2) - c_{1/5}^{YL}(\frac{n-1}{7}) & \text{if } 7|(n-1). \end{cases}$$

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Modular Linear Differential Equation

MLDEs are a useful tool for studying and classifying characters of RCFT (Anderson-Moore, Eguchi-Ooguri, Mathur, Mukhi, Sen; Kaneko-Zagier,…)

quasi modular weight 2 Eisenstein series

Ramanujan-Serre: $\mathcal{D}_k = d/d\tau - \frac{1}{6} i\pi k E_2 : \quad M_k(\Gamma) \to M_{k+2}(\Gamma)$

Second order: $\mathrm{D}^2 f - \frac{E_2}{6}\mathrm{D} f - \frac{\mu E_4}{4} f = 0 \qquad \mathrm{D} = \frac{1}{2\pi i}\frac{d}{d\tau}$

Solutions ($\ell = 0$) completely classified by Mathur, Mukhi, Sen

$$X = \{YL, A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_{7\frac{1}{2}}\}$$

Yang-Lee model

Affine level 1 characters Deligne exceptional series

Characters of Intermediate Vertex Subalgebra (Kawasetsu)

More Examples

Two character theories:

$$\chi^{F_4} = \mathsf{T}_{13}\chi^{YL} \qquad \chi^{E_7} = \mathsf{T}_7\chi^{A_1} \qquad \chi^{E_{7\frac{1}{2}}} = \mathsf{T}_{19}\chi^{YL}$$

Three character theories: No classification I am aware of. Examples include

Minimal models: $\mathcal{M}_{4,3}$ (Ising), $\mathcal{M}_{5,2}^{\otimes 2}$ $(\mathrm{YL}^{\otimes 2})$, $\mathcal{M}_{7,2}$

Models w/o Kac-Moody symmetry (Hampapura-Mukhi):

$$\mathrm{BabyMonster} \quad c = 47/2 = 47 \times 1/2$$
$$c = 164/5 = 41 \times 4/5$$
$$c = 236/7 = 59 \times 4/7$$

"Duality" of H-M and relations implied by Hecke relations

One can also use Hecke images to construct families of possible RCFT characters. As an example, if

$$p = 7, 13, 47, 53 \bmod 60$$

then $T_p \chi^{YL}$ obey

1. Have non-negative integer coefficients in q expansion.

2. The vacuum appears with degeneracy one.

3. The fusion coefficients from Verlinde are non-negative.

Consistency with Virasoro requires

$$\chi_{N,p} = \mathsf{T}_{60*N+p} \chi^{YL} + \sum_{k=0}^{N-2} d(k) \mathsf{T}_{p+60k} \chi^{YL}$$

SUMMARY II

Many RCFT with the same n=dimension of modular rep, N=order of $\rho(T)$ and with effective central charges $c^{(1)}_{eff} = pc^{(2)}_{eff}$ have modular representations and characters related by Hecke operators $\mathsf{T}_p, \ (p, N) = 1$ This generalizes and extends Galois symmetry.

We have found and studied such relations for n=2,3 involving minimal models, affine Lie algebras, Gaussian models and their orbifolds and certain moonshine CFTs. These relations appear to be very common.

Questions

Do these Hecke operators have a natural physical origin?

Can one understand more systematically which RCFT with the same n,N have characters related by Hecke operators?

Does this indicate there are new symmetries acting on the space of RCFTs?

RCFTs provide building block for construction of SCFTs describing special points in CY moduli space. Are there new geometrical relations following from Hecke relations?

RCFTs and their fusion algebras, modular tensor categories, characters etc. appear in many places in condensed matter physics:

Boundary modes of QHE systems and topological insulators.

Tool for computing and studying Entanglement Entropy.

Quantum computation.

It will be interesting to see if these new Hecke relations have implications in the real world.

THANK YOU
and
Happy 50th to Gabriele Veneziano and his wonderful amplitude

$$A(s,t,u) = \frac{\bar{\beta}}{\pi}\left[B(1-\alpha(t), 1-\alpha(s)) + B(1-\alpha(t), 1-\alpha(u)) + B(1-\alpha(s), 1-\alpha(u))\right] \qquad (3)$$

where we have introduced the Euler B-function $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

Hecke and MLDE

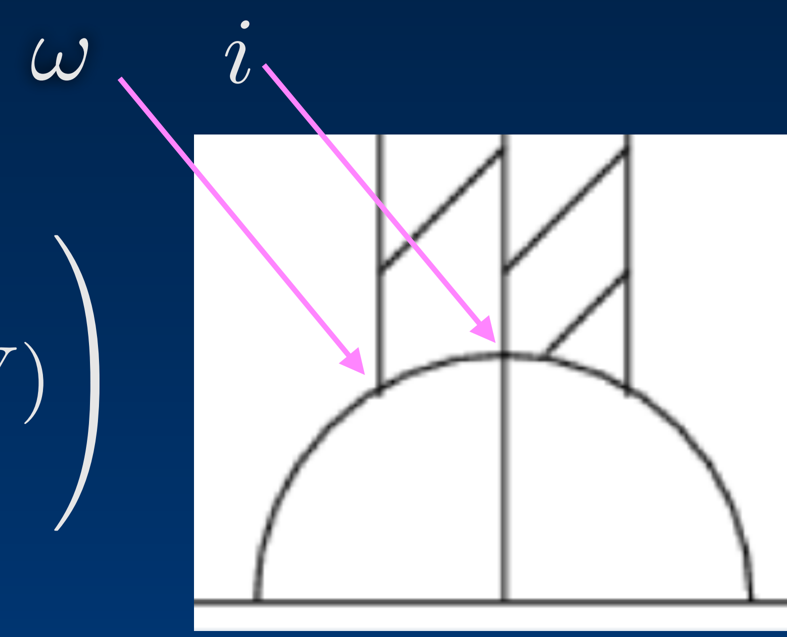
$$\mathcal{D}^n f + \sum_{k=0}^{n-1} \phi_k(\tau)\mathcal{D}^k f = 0 \qquad \phi_k = (-1)^k W_k/W$$

$$W_k = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ \mathcal{D}f_1 & \mathcal{D}f_2 & \cdots & \mathcal{D}f_n \\ \vdots & \vdots & & \vdots \\ \mathcal{D}^{k-1}f_1 & \mathcal{D}^{k-1}f_2 & \cdots & \mathcal{D}^{k-1}f_n \\ \mathcal{D}^{k+1}f_1 & \mathcal{D}^{k+1}f_2 & \cdots & \mathcal{D}^{k+1}f_n \\ \vdots & \vdots & & \vdots \\ \mathcal{D}^{n}f_1 & \mathcal{D}^{n}f_2 & \cdots & \mathcal{D}^{n}f_n \end{vmatrix},$$

$$W = W_n$$

$$\mathrm{ord}_\infty(W) + \frac{\ell(W)}{6} = \frac{k}{12}$$

$$\ell(W) = 6\left(\frac{1}{2}\mathrm{ord}_i(W) + \frac{1}{3}\mathrm{ord}_\omega(W) + {\sum_{p\in\mathcal{F}}}' \mathrm{ord}_p(W)\right)$$



Mathur, Mukhi & Sen classified n=2, $\ell(W) = 0$ solutions, but solutions exist for $\ell(W) > 0$ and are Hecke images of $\ell(W) = 0$ solutions since T_p changes $\mathrm{ord}_\infty(W)$

p	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71
$\ell^{YL}(p)$	0	2	0	2	0	2	2	0	0	2	0	2	2	2	6	6	8

Table 1: Number of zeros in the modular Wronskian for Hecke images under T_p of Yang-Lee characters for small values of p.

- ○ These Hecke images have negative coefficients
- ○ These Hecke images have positive coefficients and as RCFT characters appear in work of Naculich and Hampapurma & Mukhi

p	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49	53	57
$\ell^{I}(p)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	6	6

Table 1: Number of zeros in the modular Wronskian for Hecke images under T_p of Ising characters for small values of p.

In math these are often applied to weight k>0 modular forms and used to construct cusp forms which are eigenfunctions of the Hecke operators. E.g. the unique weight 12 cusp form (i.e. vanishing as $\tau \to i\infty$

$$\Delta = \eta^{24} = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} c(n) q^n$$

$$T_n \Delta = c(n) \Delta$$

The modularity theorem used in the proof of Fermat's last theorem associates a weight 2 Hecke eigenform to each elliptic curve over the rationals.

In RCFT we interested in the action on weight 0, weakly holomorphic modular functions which are never Hecke eigenfunctions.