Dirac's class two difficulties
and
Recent progress in external field QED

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Dirac '75: Most physicists are very satisfied with the situation. They say: 'Quantum electrodynamics is a good theory and we do not have to worry about it any more.'

I must say that I am very dissatisfied with the situation, because this so-called 'good theory' does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way.

This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small – not neglecting it just because it is infinitely great and you do not want it!

Overview

I Dirac's class two difficulties

(1) Classical field theory

(2) Quantum field theory

(a) Ultraviolet and infrared divergences

(b) Dirac sea divergences

II Recent progress in external field theory

I. The class two difficulties of field theory

(I) Classical field theory:

Classical electrodynamics comes in two parts:

- **Lorentz force**: Motion of charges in prescribed external fields

$$m \ddot{z}^{\mu}_{\ell} = e\, F^{\mu\nu}(z_\ell)\, \dot{z}_{\ell\nu} \quad , \quad \ell = 1, 2, \dots$$

- **Maxwell equations:** Dynamics of fields for given charge trajectories

$$\partial_\nu F^{\mu\nu} = \mu_0 \sum_{\ell=1}^{N} j_\ell \qquad F_{[\alpha\beta,\gamma]} = 0$$

> The fully coupled system fails to be well-defined for point charges:
>
> - The fields of a charge diverge $\sim \frac{1}{\text{distance}^2}$ on the charge trajectory.
> - The Lorentz force would need to evaluate the fields exactly there.

Every solution of Maxwell's equations is of the form:

$$E_t(x) = E^{\text{free}}_t(x) + \sum_{\ell=1}^{N} \Bigg[\overbrace{\frac{(n_\ell - v_\ell)(1 - v_\ell^2)}{\|x - z_\ell\|^2 (1 - n_\ell \cdot v_\ell)^3} + \frac{n_\ell \wedge [(n_\ell - v_\ell) \wedge a_\ell]}{\|x - z_\ell\| (1 - n_\ell \cdot v_\ell)}}^{\text{ret. field of charge } k} \Bigg]$$

ret. acceleration

some free Maxwell solution

ret. position

ret. normal $\frac{x - z_\ell}{\|x - z_\ell\|}$

ret. velocity

$$B_t(x) = n \wedge E_t(x)$$

- Mathematical remedy: Replace point charges by extended charges.

$$\frac{d}{dt}\begin{pmatrix} q_{\ell,t} \\ p_{\ell,t} \\ E_t \\ B_t \end{pmatrix} = \begin{pmatrix} v(p_{k,t}) \\ \sum_{\ell=1}^{N} e \int dx\, \varrho(x-q_{\ell,t}) \left[E_t(x) + v(p_{\ell,t}) \wedge B_t(x) \right] \\ \frac{1}{\varepsilon_0 \mu_0} \nabla \wedge B_t(x) - \sum_{\ell=1}^{N} \frac{e}{\varepsilon_0} v(p_{k,t})\, \varrho(x-q_{\ell,t}) \\ -\nabla \wedge E_t \end{pmatrix}$$

$$\text{and} \quad \nabla \cdot E_t(x) = \sum_{\ell=1}^{N} \frac{e}{\varepsilon_0} \varrho(x-q_{\ell,t}) \,, \quad \nabla \cdot B_t = 0 .$$

  - Mathematically consistent (Komech, Spohn, Bauer, Dürr, Deckert), however:
    - Unwanted arbitrariness of the geometrical extension $\varrho(x-q_{\ell,t})$
    - Acausal overlaps for accelerations $\sim \frac{1}{\text{diameter}}$
    - Generation of electrodynamic mass

- Hence, point limit $\varepsilon := \operatorname{diam} \varrho \longrightarrow 0$ desirable:

  - Abraham-Lorentz-Dirac equation

ret. Liénard-Wiechert field of z

adv. LW field of z

$$m \ddot{z}^{\mu} = e\left[F_{\text{free}} + F_{-}^{\varepsilon}\right]^{\mu\nu}(z)\, \dot{z}_{\nu}$$

$$= e\Big[F_{\text{free}} + \underbrace{\tfrac{1}{2}\left(F_{-}^{\varepsilon} - F_{+}^{\varepsilon}\right)}_{\text{radiation field}} + \tfrac{1}{2}\left(F_{-}^{\varepsilon} + F_{+}^{\varepsilon}\right)\Big]^{\mu\nu}(z)\, \dot{z}_{\nu}$$

$$\approx e F_{\text{free}}^{\mu\nu}(z)\, \dot{z}_{\nu} + \frac{2}{3}\frac{e^2}{c^5}\left(\dddot{z}^{\mu} \dot{z}^{\nu} - \dddot{z}^{\nu} \dot{z}^{\mu}\right) \dot{z}_{\nu} - \underbrace{\frac{1}{2}\frac{e^2}{c^2 \varepsilon}}_{\text{electrodynamic mass}} \ddot{z}^{\mu} + \mathcal{O}(\varepsilon)$$

setting bare mass $m = m_{exp} - \frac{1}{2}\frac{e^2}{c^2 \varepsilon}$ suggests for the limit $\varepsilon \to 0$

$$m_{exp} \ddot{z}^{\mu} = F_{\text{free}}^{\mu\nu}(z)\, \dot{z}_{\nu} + \frac{2}{3}\frac{e^2}{c^5}\left(\dddot{z}^{\mu} \dot{z}^{\nu} - \dddot{z}^{\nu} \dot{z}^{\mu}\right) \dot{z}_{\nu}$$

However:

- run-aways, pre-acceleration
  - asymptotic conditions (Dirac, Rohrlich, ...)
    - no runaways but complicated functional differential equation
  - Landau-Lifshitz equation, critical manifold (Spohn, Kunze-Spohn)
    - good approximation, no pre-acceleration, no runaways
- all these approaches are informal Taylor expansions of ill-defined equations of motion of ML

Bottom line:

There is no consistent relativistic theory of classical electrodynamics.

Open program:

A. A fresh start with well-defined equations of motion.

B. Derivation of an effective equation for radiation damping from first principles.

C. Test the predictions experimentally beyond Lamor's formula.

A promising fresh start: Wheeler-Feynman electrodynamics

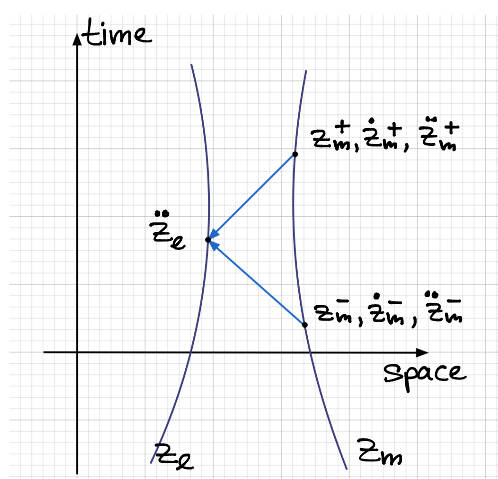
Instead of starting with ill-defined equations and tempering with them,
Wheeler and Feynman suggested to start anew:

$$m\ddot{z}_\ell^\mu = e\sum_{m\neq\ell}\frac{1}{2}\left(F_-[z_m]+F_+[z_m]\right)^{\mu\nu}(z_\ell)\,\dot{z}_{\ell\nu}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and

$$A_\pm[z]^\mu(x) = e\,\frac{\dot{z}^\mu_\pm}{(x-z_\pm)_\alpha\,\dot{z}^\alpha_\pm}$$

No singularities in the equations of motion.



However:

- No self-interaction, so how can it describe radiation damping?

Suppose no radiation leaks to infinity

$$\sum_{m=1}^{N} \left(F_-[z_m] - F_+[z_m] \right) \approx 0$$

one "morally" finds

$$m \ddot{z}_\ell^\mu = e \sum_{m \neq \ell} \frac{1}{2} \left(F_-[z_m] + F_+[z_m] \right) + 0$$

$$\approx e \left[\frac{1}{2} \left(F_-[z_\ell] - F_+[z_\ell] \right) + \sum_{m \neq \ell} F_-[z_m] \right]^{\mu\nu} (z_\ell) \dot{z}_{\ell\nu}$$

$$\approx \underbrace{\frac{2}{3} \frac{e^2}{c^5} \left(\dddot{z}^\mu \dot{z}^\nu - \dddot{z}^\nu \dot{z}^\mu \right) \dot{z}_\nu}_{\text{Abraham \& Dirac radiation friction term}} + \underbrace{e F_{ext}^{\mu\nu} (z_\ell) \dot{z}_{\ell\nu}}_{\text{effective external field solving Maxwell's equations}}$$

as an effective equation without running into singularities.

However:

- Advanced and retarded arguments in equations of motion make it hard to analyze.

- Solution theory only in some situations under control
  - see Schild, Driver, Bauer, Dürr, Deckert, Hinrichs

- Derivation of radiation friction completely open.

Possibly very hard, nevertheless, only mathematical problems.

(2) Quantum field theory:

(a) Infrared and ultraviolet divergences

Toy model

$$\overbrace{(\Box + \mu^2)}^{\text{Klein-Gordon dispersion}} \varphi(x) = -g \sum_{\ell=1}^{N} \overbrace{\delta(x - x_\ell)}^{\text{fixed charges at } x_\ell}$$

of a Klein-Gordon field with sources x_ℓ has Hamiltonian

$$H = \int dx \left([\Pi_t(x)]^2 + [\nabla \varphi_t(x)]^2 + \mu^2 \varphi_t(x)^2 \right) + \underbrace{g}_{\text{coupling constant}} \sum_{\ell=1}^{N} \varphi_t(x_\ell)$$

By requirement $[\varphi(x), \varphi(y)] = i\,\Delta(x-y)$ one finds

$$H = \int dk\, \underbrace{\omega_k}_{\omega_k = \sqrt{k^2+\mu^2}}\, a_k^\dagger a_k + g \sum_{\ell=1}^{N} \underbrace{\varphi(x_\ell)}_{\varphi(x) = \varphi(0,x)}$$

for the field

$$\varphi(t,x) = \int dk\, \underbrace{\gamma_k}_{\gamma_k = (2\pi)^{-3/2} \frac{1}{\sqrt{2\omega_k}}} \left(a_k\, e^{-i\omega_k t + ikx} + a_k^\dagger\, e^{i\omega_k t - ikx} \right)$$

$a_k\,|0\rangle$, $[a_k, a_m^\dagger] = \delta(k-m)$

span Fock space $\mathcal{F}$

The ground state can be computed explicitly

$$|\Omega\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(-g \sum_{\ell=1}^{N} \int dk\, \frac{\gamma_k}{\omega_k}\, e^{ikx_\ell}\, a_k^\dagger \right)^n \underbrace{|0\rangle}_{\text{true vacuum}}$$

and the eigenvalue is given by

$$E = \sum_{\ell=1}^{N} \sum_{m=1}^{N} \frac{1}{(2\pi)^3} \int dk\, \frac{\gamma_k^2}{\omega_k}\, e^{ik(x_\ell - x_k)}$$

One observes the difficulties:

(i) One consequence is that the informally computed ground state energy diverges

$$E = \sum_{\ell=1}^{N} \sum_{m=1}^{N} \frac{e^{-\mu |x_\ell - x_m|}}{|x_\ell - x_m|} = \underbrace{N \left. \frac{e^{-\mu|x|}}{|x|} \right|_{x=0}}_{\text{self-interaction}} + \underbrace{\sum_{\ell \neq m} \frac{e^{-\mu|x_\ell - x_m|}}{|x_\ell - x_m|}}_{\text{interaction between charges}}$$

not surprisingly, as the classical Hamiltonian diverges exactly in the same way.

QFT inherits the classical ultraviolet divergence.

(ii) $\int dk\, f(k)\, a_k^+ |\Psi\rangle \in \mathcal{F} \iff f \in L^2$

Recall $H = \int dk\, \omega_k a_k^+ a_k + g \sum_{\ell=1}^{N} \int dk\, \gamma_k \left(a_k e^{ikx_\ell} + a_k^+ e^{-ikx_\ell} \right)$

but $\gamma_k = \mathcal{O}_{|k|\to\infty}(|k|^{-1/2}) \Rightarrow \gamma_k \notin L_2$

Hence, the Hamiltonian H and the equation of motion

$$i\partial_t |\Psi_t\rangle = H |\Psi_t\rangle$$

are ill-defined, which is referred to as ultraviolet divergence.

(iii) In fact, the situation is slightly worse than in classical field theory:

$$|\Omega\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(-g \sum_{\ell=1}^{N} \int dk\, \frac{\gamma_k}{\omega_k} e^{ikx_\ell} a_k^+ \right)^n |0\rangle$$

but $\frac{\gamma_k}{\omega_k} = \mathcal{O}_{|k|\to\infty}(|k|^{-3/2}) \Rightarrow \frac{\gamma_k}{\omega_k} \notin L^2 \Rightarrow |\Omega\rangle \notin \mathcal{F}$

Hence, as another manifestation of the ultraviolet divergence, the ground state is not a Fock vector.

(iv) And even if we impose an ultraviolet cut-off $0 \leq |k| \leq \Lambda < \infty$

$$\omega_k = \sqrt{k^2 + \mu^2} \quad \text{for } \mu = 0 \text{ gives}$$

$$\frac{\gamma_k}{\omega_k} = \underset{|k| \to 0}{O}\left(k^{-3/2}\right) \Rightarrow \frac{\gamma_k}{\omega_k} \mathbb{1}_{|k| \leq \Lambda} \notin L^2 \Rightarrow |\Omega_\Lambda\rangle \notin \mathcal{F}$$

which is referred to as the infrared divergence.

The representation problem $|\Omega\rangle \notin \mathcal{F}$ seems rather a mathematical issue.

- In this toy model it can be solved by switching to a Fock space with vacuum $|\Omega\rangle$
- Many advances concerning the infrared problem

  Bloch, Nordsieck, Pauli, Fierz, Fröhlich, Pizzo, Chen, Arai, Spohn, Hiroshima, Hirokawa, Lörenzini, Minlos, Bach, Ballesteros

Infinities in the equations of motion and the physical relevant quantities are more worrisome.

- One needs to implement a renormalization program as in classical field theory.
- In the toy model we may simply drop the infinities in the energy but what if charges move?

- Mass renormalization as in classical electrodynamics?
  - The Pauli-Fierz model:

$$H = \frac{\left[-i\nabla - eA(x)\right]^2}{2m} + \sum_{\lambda=1,2} \int_{|k| \leq \Lambda} dk\, \omega_k\, a^+_{k,\lambda} a_{k,\lambda}$$

    - Conjecture about the effective velocity

$$v_{eff}(P) = \left|\frac{\partial E_P}{\partial p}\right| \sim \frac{|P|}{m} \Lambda^{-c(e)} = \underset{\Lambda \to \infty}{O}(1)$$

for mass renormalization $m \sim m_{exp} \Lambda^{-c(e)} \xrightarrow[\Lambda \to \infty]{} 0$

  - The Yukawa model:

$$H = \sqrt{(-i\nabla)^2 + m^2} + \int dk\, \omega_k a_k^+ a_k + g \int dk\, \gamma_k \left(a_k e^{ikx} + a_k^+ e^{-ikx}\right)$$

- Here it was proven that (Deckert, Pizzo)

$$v_{\text{eff}}(P) = \left|\frac{\partial E_P}{\partial P}\right| \sim \frac{P}{\sqrt{P^2+m^2}}\, \Lambda^{-g^2 C} + \mathcal{O}(g^{1/2})$$

Hence, effective velocity vanishes independently of any mass renormalization!

Open program???

A. Learn from the possible "fresh start" in the classical counterpart.
B. Understand better the nature of relativistic interaction (Petrat's talk)
C. Find well-defined equation of motion, then proceed (Barut)

(c) Dirac sea divergences

- So far we mainly regarded one electron with its field alone.
- Let us now neglect the interaction with the field and describe the motion of electrons only.

Dirac's electrons obey the Schrödinger equation for the Dirac Hamiltonian

$$H = -i\alpha\cdot\nabla + \beta m \quad \text{on} \quad \mathcal{D}(H) \subset L^2(\mathbb{R}^3, \mathbb{C}^4) =: \mathcal{H}$$

which admits the spectrum $\sigma(H) = (-\infty, -m] \cup [m, \infty)$

and we shall use the notation $P^- := P_{(-\infty,-m]}$, $P^+ := 1 - P^-$, $\mathcal{H}^\pm := P^\pm \mathcal{H}$

Electrons in the negative spectrum move differently than usually observed:
- Schrödinger's zitterbewegung
- Klein's paradox

Furthermore, there is the phenomenon of pair creations, which led Dirac to propose:

The vacuum, when left alone, should be represented by a quantum state of the form

$$|\Omega\rangle = \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \cdots \quad \text{for orthonormal basis } (\varphi_n)_{n\in\mathbb{N}} \text{ of } \mathcal{H}^-$$

antisymmetric $\otimes$

a infinite sea of electrons occupying all negative energy states.

Subject to an external potential an electron can be described with the Hamiltonian

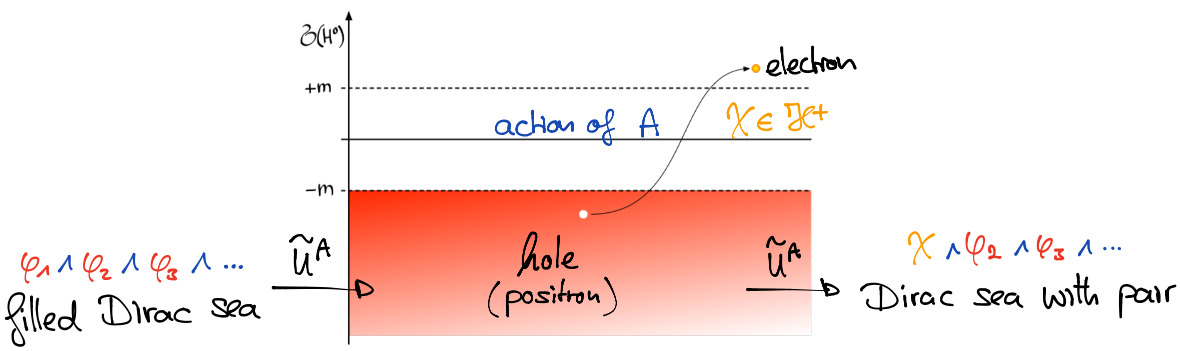
$$H^A = \alpha\cdot(-i\alpha\cdot\nabla - \underline{A}) + \beta m + A^0$$

which generates the one-particle time evolution operator U^A.

To lift U^A to a time evolution operator $\tilde{U}^A$ acting on Dirac seas it is natural to require

$$\tilde{U}^A|\Omega\rangle = U^A\varphi_1 \wedge U^A\varphi_2 \wedge U^A\varphi_3 \wedge \dots$$

The heuristic picture of pair creation then is



This idea can be wrapped up in a neat notation with creation/annihilation operators

$$\psi^+(f)\,\psi_1\wedge\psi_2\wedge\psi_3\wedge\dots = f\wedge\psi_1\wedge\psi_2\wedge\psi_3\wedge\dots$$

$$\psi(f)\,\psi_1\wedge\psi_2\wedge\psi_3\wedge\dots = \langle f|\,\psi_1\wedge\psi_2\wedge\psi_3\wedge\dots$$

which fulfill $\{\psi(f),\psi^+(g)\} = i\langle f,g\rangle$ and span the free Fock space $\mathcal{F}(\mathcal{H}^-;\mathcal{H})$ w.r.t.

$$\langle \psi_1\wedge\psi_2\wedge\dots,\ \phi_1\wedge\phi_2\wedge\dots\rangle = \det\langle\psi_n,\phi_m\rangle_{n,m}$$

The first adjustment to be made

$$\langle\Omega|j^\mu(x)|\Omega\rangle = \langle\Omega|\overline{\psi}(x)\gamma^\mu\psi(x)|\Omega\rangle = \sum_{n=1}^{\infty} e\,\overline{\varphi}_n(x)\gamma^\mu\varphi_n(x) = -\infty$$

This is expected since there are infinitely many electrons, all inducing a current.
Nevertheless, following Dirac's intuition:

Dirac 1934: Admettons que dans l'Univers tel que nous le connaissons, les états d'energie négative soient presque tous occupés par des électrons, et que la distribution ainsi obtenue ne soit pas accessible à notre observation à cause de son uniformité dans toute l'etendue de l'espace.

so we set

$$:j^\mu(x): \; := \; j^\mu(x) - \sum_{n=1}^{\infty} \overline{\varphi_n}(x)\gamma^\mu \varphi_n(x) \quad \Longrightarrow \quad \langle \Omega | :j^\mu(x): | \Omega \rangle = 0$$

Next problem would be to compute the current for a given external potential A^μ that is switched on and left constant.

$U^{A*}\Psi(x)U^A$

$$\underbrace{\langle \Omega | :\hat{U}^{A*} j^\mu(x) \hat{U}^A: | \Omega \rangle}_{= j^A} = \langle \Omega | \overline{\Psi}^A(x)\gamma^\mu \Psi^A(x) | \Omega \rangle - \langle \Omega | \overline{\Psi}(x)\gamma^\mu \Psi(x) | \Omega \rangle$$

$$= \infty + \infty - \alpha\, j^{ext} * (\infty - \Delta) + O(\alpha^3)$$

not Lorentz invariant

not gauge invariant

constant

Serge Bouillon

dropping the terms violating Lorentz and gauge invariance one is left with

$$\hat{j}^A(q) = -\alpha\, \hat{j}^{ext}(q)\left(R_\Lambda - \hat{\Delta}(q)\right) + O(\alpha^3)$$

$O(\log \Lambda)$, Λ being an ultraviolet cut-off

j^A is the answer to external disturbance j^{ext} but experimentally they are inseparable.

$$\alpha\underbrace{\left[\hat{j}^A(q) + \hat{j}^{ext}(q)\right]}_{=:\ \hat{j}^{tot}(q)} = \alpha\,\hat{j}^{ext}(q) - \alpha^2\,\hat{j}^{tot}(q)\left(R_\Delta - \hat{\Delta}(q)\right) + \hat{F}_4\left[\alpha\left(j^A + j^{ext}\right)\right](q)$$

$$\Rightarrow\quad (1+\alpha R_\Delta)\,\alpha\,\hat{j}^{tot}(q) = \alpha\,\hat{j}^{ext}(q) + \alpha^2\,\hat{j}^{tot}(q)\,\hat{\Delta}(q) + \hat{F}_4\left[\alpha j^{tot}\right](q)$$

$$\Leftrightarrow\quad \alpha\,\hat{j}^{tot}(q) = \frac{\alpha}{1+\alpha R_\Delta}\,\hat{j}^{ext}(q) + \frac{\alpha}{1+\alpha R_\Delta}\,\alpha\,\hat{j}^{tot}(q)\,\hat{\Delta}(q) + \hat{F}_4\left[\alpha\hat{j}^{tot}\right](q)$$

$\hat{j}^{tot,0}(0)$ = total charge $\qquad$ $\hat{j}^{ext,0}(0)$ = external charge

for $q=0$: $\quad \alpha\,Q^{tot} = \dfrac{\alpha}{1+\alpha R_\Delta}\,Q^{ext}$

because $\hat{\Delta}(0) = 0 = \hat{F}_4\left[\alpha j^{tot}\right](0)$

fixing the scale $\alpha_{exp}\,\hat{j}^{tot}_{exp} = \alpha\,\hat{j}^{tot}$ we get

$$\alpha_{exp} = \frac{\alpha}{1+\alpha R_\Delta}$$

$$\alpha_{exp}\,\hat{j}^{tot}_{exp}(q) = \alpha_{exp}\,\hat{j}^{ext}(q) + \alpha_{exp}^2\,\hat{j}^{tot}_{exp}(q)\,\hat{\Delta}(q) + \hat{F}_4\left[\alpha_{exp}\,\hat{j}^{tot}_{exp}\right](q)$$

a self-consistent equation for of the vacuum polarization without infinities.

However:

$$\alpha_{exp} = \frac{\alpha}{1+\alpha R_\Lambda} \iff \alpha = \frac{\alpha_{exp}}{1-\alpha_{exp} R_\Lambda} \qquad \text{recall} \quad R_\Lambda \sim \log \Lambda$$

Our bare parameter α became a running parameter.

But $\alpha_{exp} = \frac{1}{137}$, which means that $\alpha \to \infty$ already for a finite value of Λ, long before it can be sent to infinity.

This behavior is referred to as the Landau pole.

See Dirac, Heisenberg, Pauli, Rose, Uehling, Serber, Schwinger, Feynman, Landau, Lifshitz...
Modern literature on Hartree-Fock model: Chaix, Iracane, Hainzl, Lewin, Séré, Solovay

What is the source of these infinities?

- There are simply infinitely many electrons.
- After the thermodynamic limit only admissible differences/relations can survive.

Open program:

A. Take Dirac's idea seriously.
B. Don't hide the infinitely many electrons in the vacuum state.
C. Deal with them, as one would do in statistical mechanics.
D. Find a well-defined equation of motion, then start over.

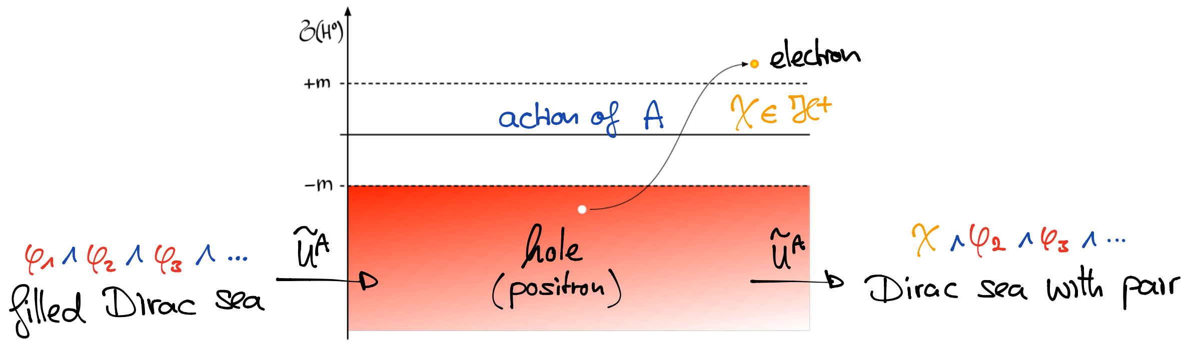
II Recent progress in external field theory

According to

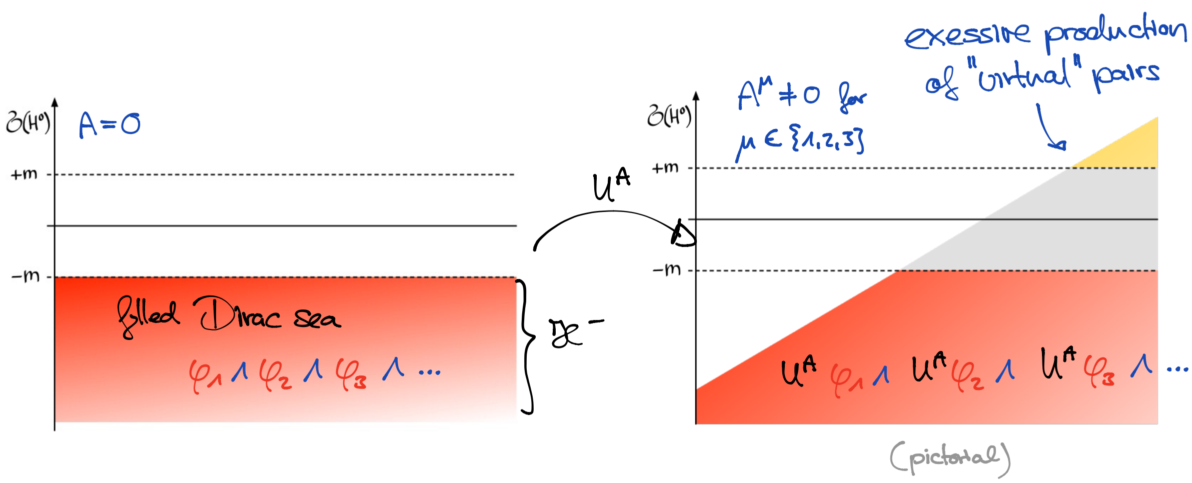
Theorem (Shale-Stinespring '65, Ruijsenaars '77):

Lift $\hat{U}^A$ exists on free Fock space $\iff$ $\underline{A} = 0$

the heuristic picture is only asymptotically correct



More to the point at intermediate times:



Reason:

$$H = \alpha \cdot (-i\nabla - A) + \beta m + A^0$$

"rotates" the spinors

Consequence:

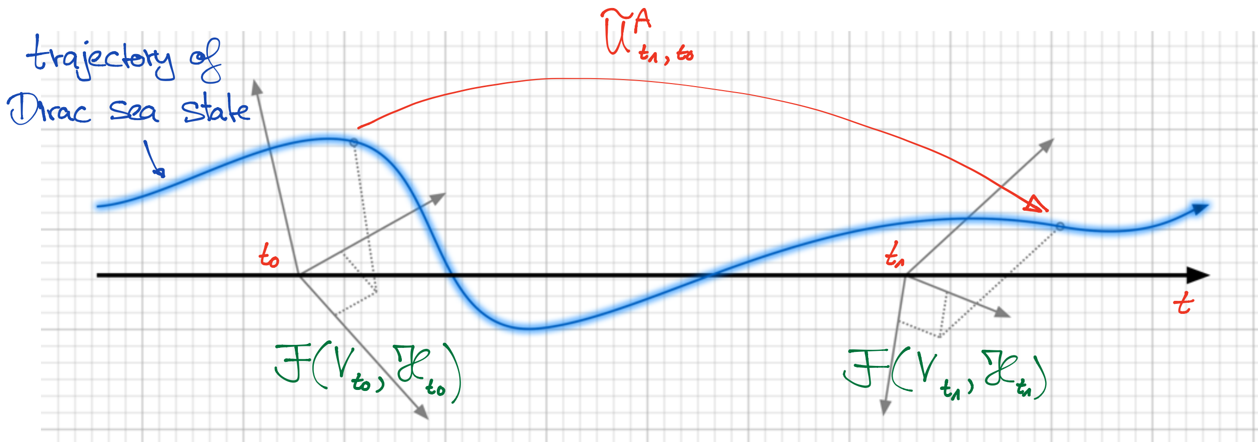
- As the polarization changes the Dirac sea cannot be represented in free Fock space

$$\mathcal{F}(\mathcal{H}^-; \mathcal{H}) \qquad \text{recall } \mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4), \quad \mathcal{H}^- := P^- \mathcal{H}$$

- The polarization $\mathcal{H}^-$ has to be adapted to an admissible choice V_t according to A.

$$\mathcal{F}(V_t, \mathcal{H}_t) \qquad V_t \subset \mathcal{H}_t = L^2(\Sigma_t, \mathbb{C}^4), \quad \Sigma_t \text{ space-like Cauchy surface}$$

Heuristically, the situation can be depicted as:



Status (D.-Dürr-Merkl-Schottenloher & D.-Merkl):

$\mathcal{H}_t = L^2(\Sigma_t, \mathbb{C}^4)$,

Σ_t space-like Cauchy surface

- Constructed the evolution operator

$$\tilde{U}^A_{t_1,t_0} : \mathcal{F}(V_{t_0}, \mathcal{H}_{t_0}) \longrightarrow \mathcal{F}(V_{t_1}, \mathcal{H}_{t_1})$$

  for admissible polarizations V_{t_0} and V_{t_1} between general Cauchy surfaces.

- The classes of admissible polarizations are given and shown to depend on $A|_{T\Sigma_t}$ only.

- The construction is unique up to a phase. Therefore:
  - Transition probabilities are shown to be free of infinities and unique.
  - But the current is not uniquely determined.

- However, for each admissible choice of a phase, the expression for the corresponding is well-defined.

- The hope is that the freedom of choice in the phase is restricted by gauge and Lorentz invariance to of freedom of choosing a real number, the charge of the electron.

Open program

A. Identify the physical current.

B. Allow back reaction of the polarization current.

C. Derive assumptions:

  a. Product $|\Omega\rangle$ close to true ground state.

  b. Validity of neglecting the pair interaction in $|\Omega\rangle$
  (Mitrouskas' talk)

D. Couple to a quantized photon field?

Thank you!

A possible fresh start: Wheeler-Feynman like interaction

Recall, the basic problem in classical electrodynamics was the ill-defined self-interaction.

$$m\ddot{z}^\mu = e\,F^{\mu\nu}[z](z)\,\dot{z}_\nu$$

$$\text{where}\quad F^{\mu\nu} = \Big(F_{\text{free}} + \overbrace{\tfrac{1}{2}(F_- - F_+)}^{\text{inducing radiation friction}} + \overbrace{\tfrac{1}{2}(F_- + F_+)}^{\text{carries the divergence}}\Big)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,, \qquad A^\mu_\pm(x) = \int dy\, \underbrace{G^\pm(x-y)}_{\text{adv./ret. Green's function}}\, j^\mu(y)\,, \qquad j^\mu(x) = \overbrace{e\int_{\tau\mapsto z(\tau)} dz\,\delta(x-z)}^{\text{charges' current}}$$

Copying that idea

$$(i\partial\!\!\!/ - m)\Psi(x) = e\,A\!\!\!/(x)\,\Psi(x) \qquad \text{where}\quad A^\mu = A^\mu_{\text{free}} + \tfrac{1}{2}(A_- - A_+)^\mu$$

$$\text{for}\quad A^\mu_\pm(x) = \int dy\, \underbrace{G^\pm(x-y)}_{\text{adv./ret. Green's function}}\, j^\mu(y)\,, \qquad j^\mu(x) = \bar{\Psi}(x)\gamma^\mu\Psi(x)$$

No singularities in the equations of motion.

- Barut, Dowling, Kraus, Ünal, Salamin studied this equation in over 15 publications
- Main claim: Lamb shift, g-factor can be computed without renormalization.

- This radiation damping equation can be derived starting from a Hartree-Fock model of the Dirac sea with a similar argument as in WF electrodynamics (Deckert)

- However:
  - The model is non-linear and should be seen as mean-field approximation.

Many open questions:

- Mathematical theory of solutions?
- Can Barut's computations be made rigorous?
- Is a linear version possible?
- Can this idea be implemented for a Dirac sea of charges?

Again very hard, nevertheless, only mathematical problems.