Canonical Approach for Exploring Finite Density QCD

V. G. Bornyakov, D. L. Boyda, V. A. Goy, H. Iida,
A. V. Molochkov, A. Nakamura,
A. A. Nikolaev, M. Wakayama
and V. I. Zakharov
FEFU (Far Eastern Federal Univ.), Vladivostok
XQCD2017

You know
Sign Problem.
Then skip to Page 7.

QCD at finite density

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int \mathcal{D}U \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-\beta S_G - \bar{\psi}\Delta\psi}$$

$$= \int \mathcal{D}U \prod_f \det \Delta(m_f)\, e^{-\beta S_G}$$

$$\Delta(\mu) = D_\nu \gamma_\nu + m + \mu\gamma_0$$

$$\boxed{\Delta(\mu)^\dagger = -D_\nu\gamma_\nu + m + \mu^*\gamma_0 = \gamma_5 \Delta(-\mu^*)\gamma_5}$$

$$\Rightarrow (\det \Delta(\mu))^* = \det \Delta(\mu)^\dagger = \det \Delta(-\mu^*)$$

$(\det \Delta(\mu))^* = \det \Delta(\mu)^\dagger = \det \Delta(-\mu^*)$

For $\mu = 0$

$$(\det \Delta(0))^* = \det \Delta(0)$$

$\det \Delta$ ⇒ *Real*

For $\mu \neq 0$ (in general)

$\det \Delta$ ⇒ *Complex*

$$Z = \int \mathcal{D}U \prod_f \det \Delta(m_f, \mu_f) e^{-\beta S_G}$$

Complex ⇒ Sign Problem

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}U \, O \, \det\Delta \, e^{-\beta S_G}$$

In Monte Carlo simulation, configurations are generated according to the Probability:

$$\det\Delta \, e^{-\beta S_G} / Z$$

$\det\Delta$: *Complex*

Monte Carlo Simulations very difficult !

$$\langle O \rangle = \frac{\int DU O \det \Delta e^{-S_G}}{\int DU \det \Delta e^{-S_G}}$$

$$\det \Delta = |\det \Delta| e^{i\theta}$$

$$\langle O \rangle = \frac{\int DU O |\det \Delta| e^{i\theta} e^{-S_G}}{\int DU |\det \Delta| e^{-S_G}} \times \frac{\int DU |\det \Delta| e^{-S_G}}{\int DU |\det \Delta| e^{i\theta} e^{-S_G}}$$

$$= \frac{\langle O e^{i\theta} \rangle_{|\det|}}{\langle e^{i\theta} \rangle_{|\det|}}$$

Canonical Approach

Not so well-known

From Experiments

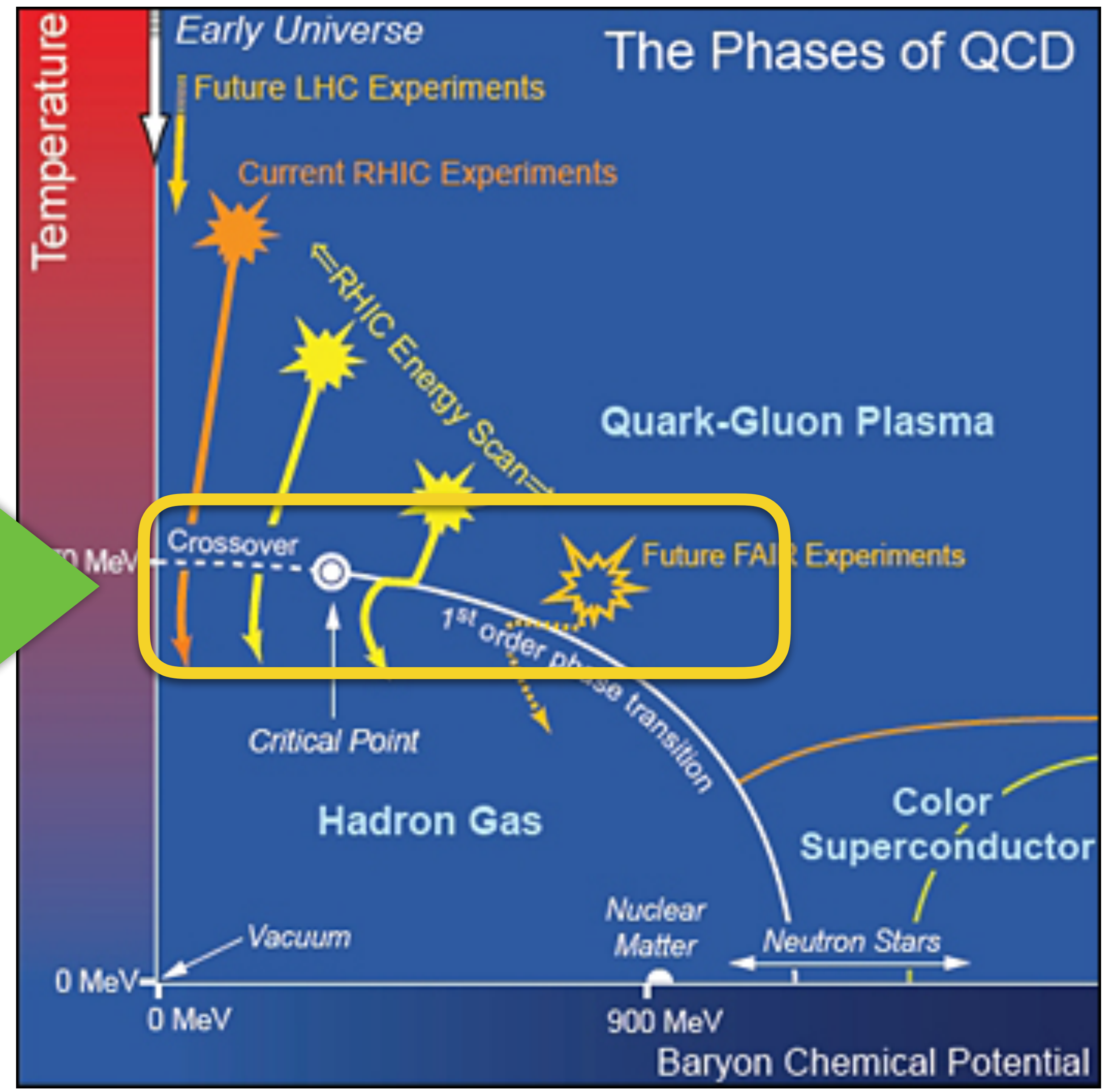
Z_N

From Lattice

Canonical Partition Functions

Objective of Vladivostok Group

Study here with Zn



Neutron Star: No
Color SC: No

http://www.bnl.gov/rhic/news2/news.asp?a=1870&t=today

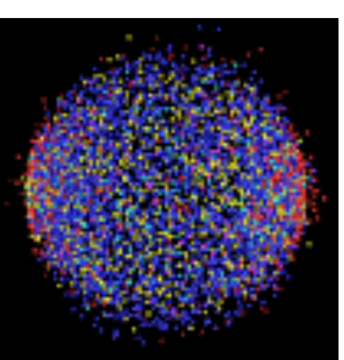
Statistical Description is good at least as a first approximation

with Two Parameters Chemical Potential, μ
and Temperature, T

$Z_{GC}(\mu, T)$ Grand Canonical Partition Function

Alternative: Number, n and Temperature, T

$Z_C(n, T)$ Canonical Partition Function



or Z_N

They are equivalent and related as

$$Z(\xi, T) = \sum_n Z_n(T)\, \xi^n$$

$\xi \equiv e^{\mu/T}$ Fugacity

Quick Proof of
Fugacity Expansion

$$\boxed{Z(\mu,T)=\sum_n Z_n(T)(e^{\mu/T})^n}$$

(Left Hand Side)= $\mathrm{Tr}\, e^{-(H-\mu\hat{N})/T}$

$$=\sum_n \langle n|e^{-(H-\mu\hat{N})/T}|n\rangle$$

If $[H,\hat{N}]=0$

$$=\sum_n \boxed{\langle n|e^{-H/T}|n\rangle}\, e^{\mu n/T}$$

$Z_n(T)$

This is a very useful relation.

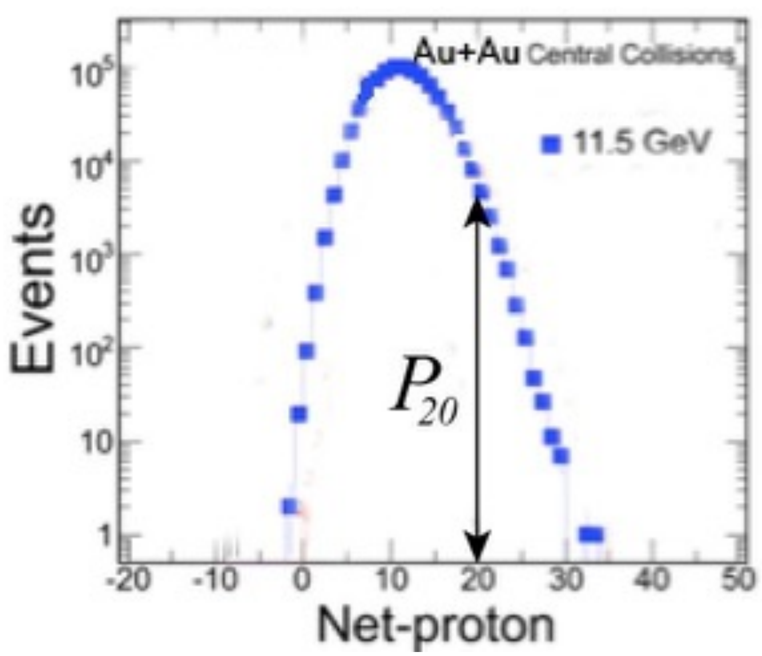
The partition function stands for the Probability

$$Z_{GC}(\mu, T) = \sum_n \boxed{Z_n(T)\xi^n}$$

System with μ and T

Probability to find (net-)baryon number=n

We extract Z_n from experimental multiplicity at RHIC



$$P_n = Z_n \xi^n \qquad \left(\xi \equiv e^{\mu/T}\right)$$

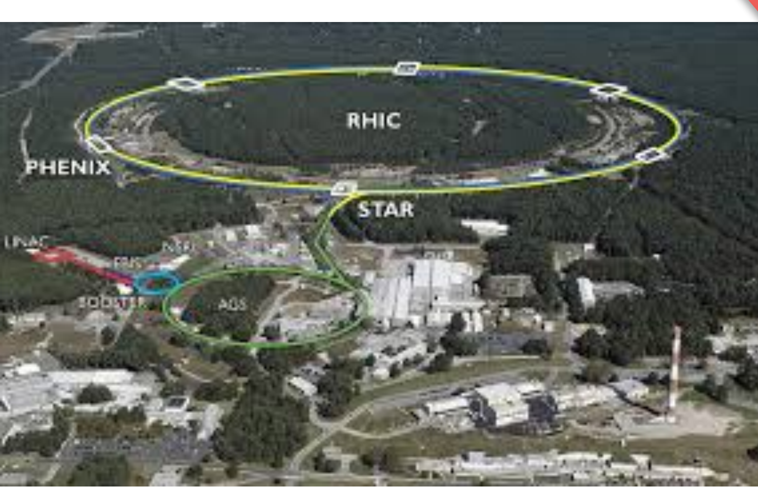
$\boxed{\xi}$ unknown

$$Z_n = P_n / \xi^n$$

Z_n satisfies

$$\boxed{Z_{+n} = Z_{-n}}$$

RHIC tells us Z_n

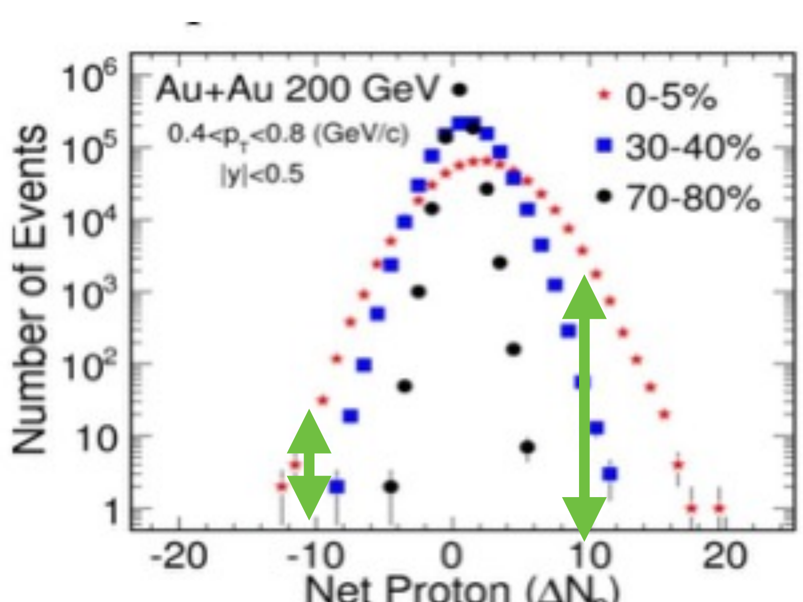


(Particle-AntiParticle Symmetry)

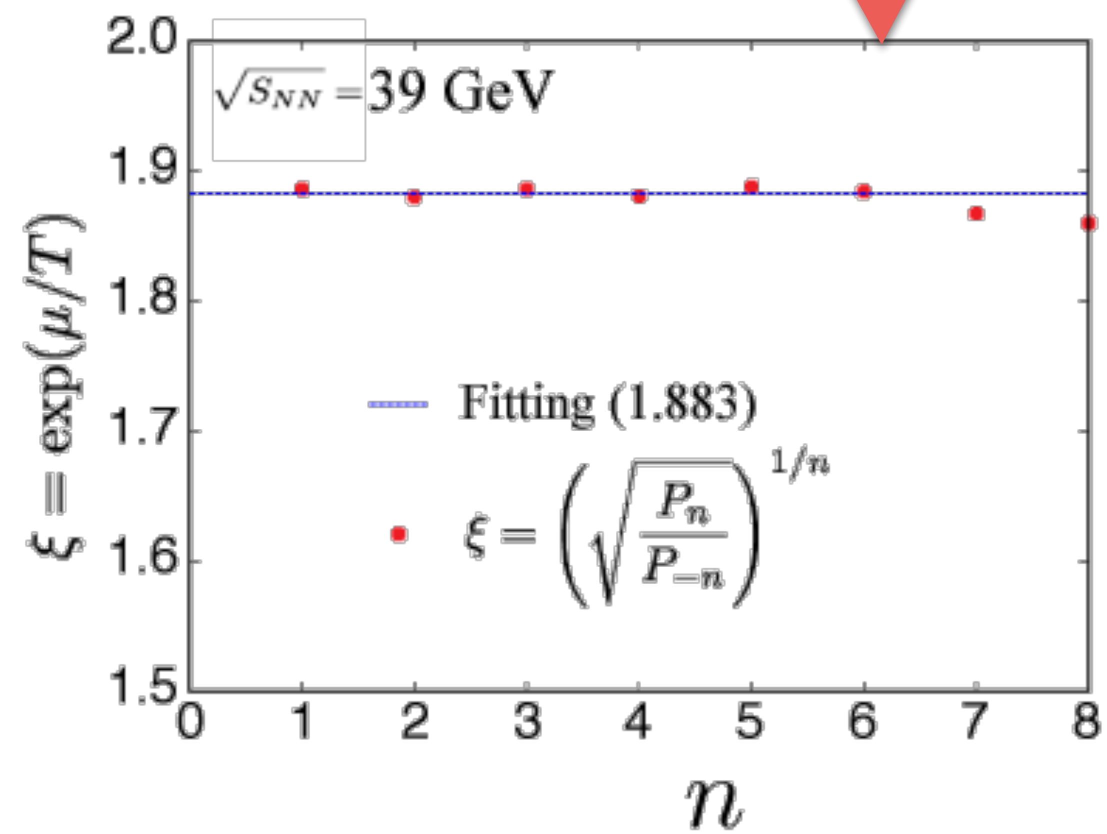
$$\boxed{Z_{+n} = Z_{-n}}$$

$$\boxed{\begin{aligned} P_n &= cZ_n\xi^n \\ P_{-n} &= cZ_{-n}\xi^{-n} \end{aligned}} \Rightarrow P_nP_{-n} = c^2Z_nZ_{-n} = c^2Z_n^2$$

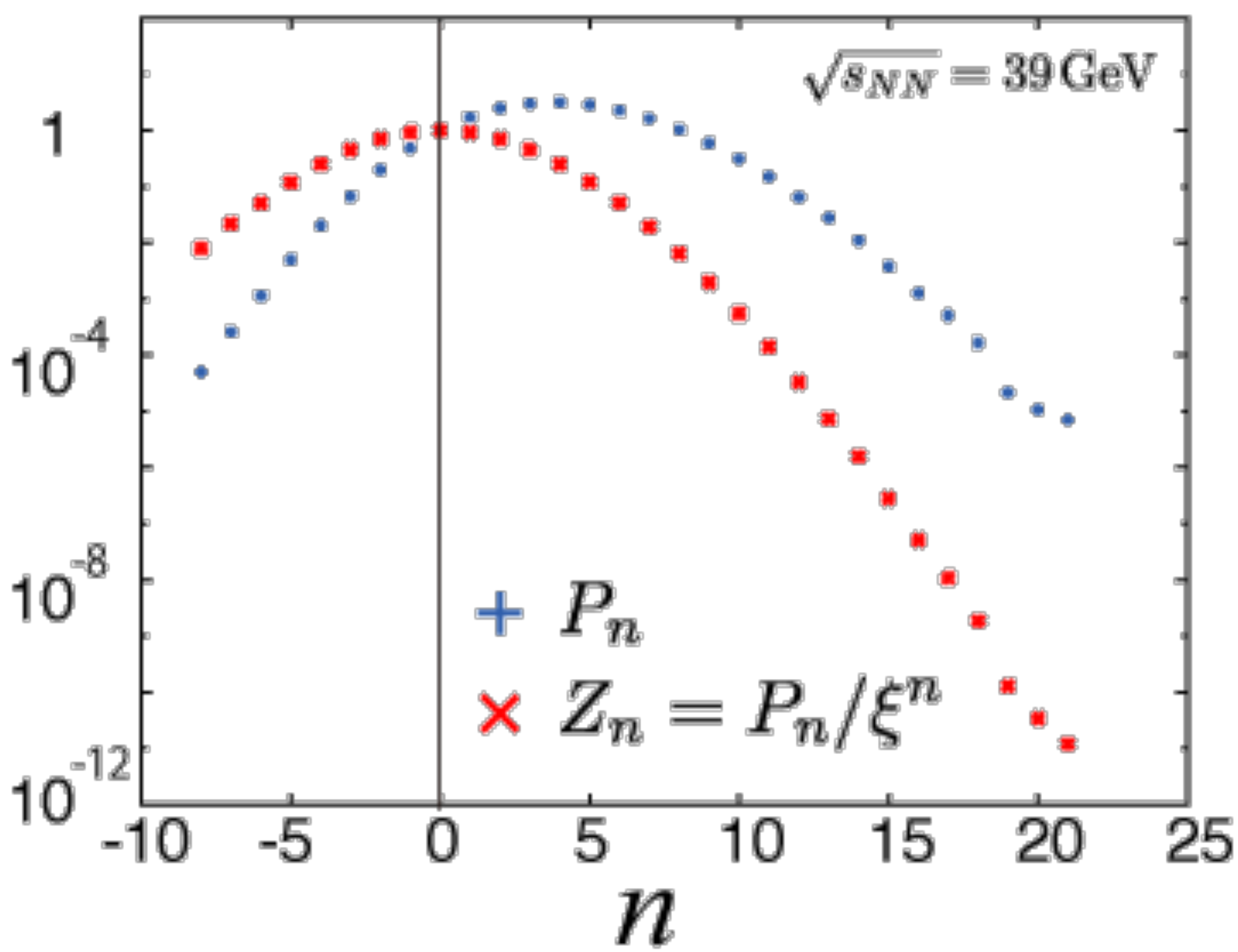
$$\Rightarrow \frac{P_n}{\sqrt{P_nP_{-n}}} = \xi^n \Rightarrow \xi = \left(\sqrt{\frac{P_n}{P_{-n}}}\right)^{1/n}$$

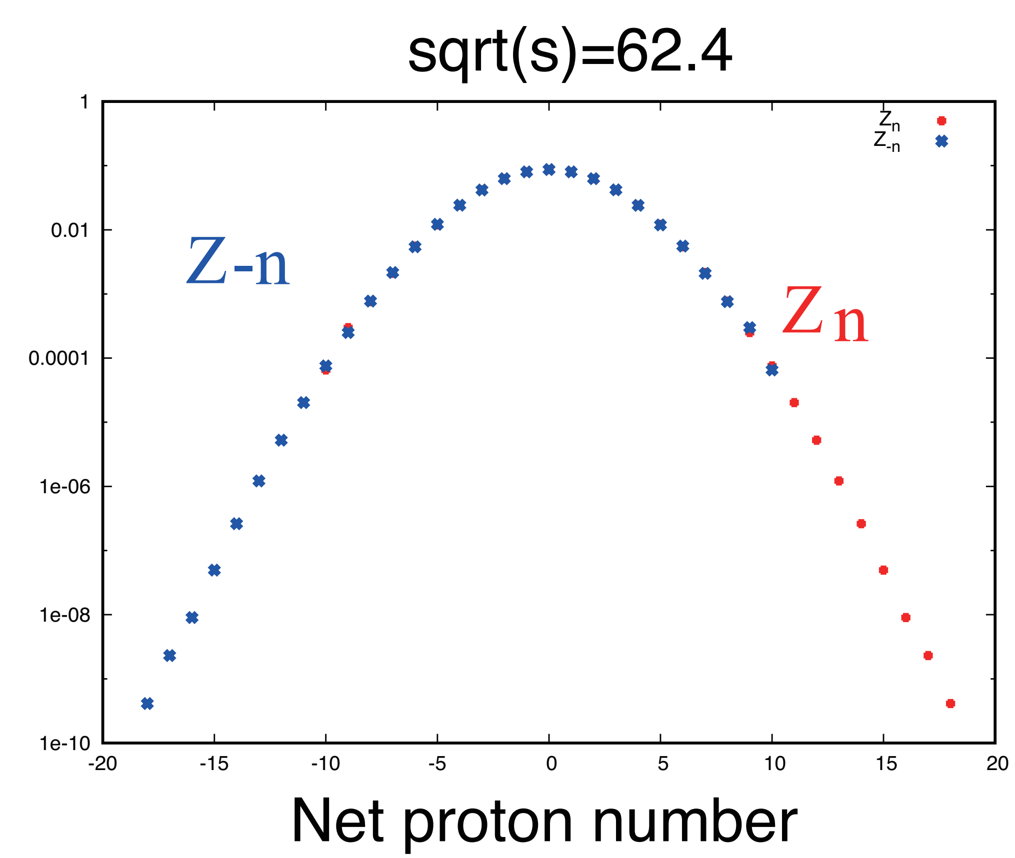


P_{-10} P_{+10}



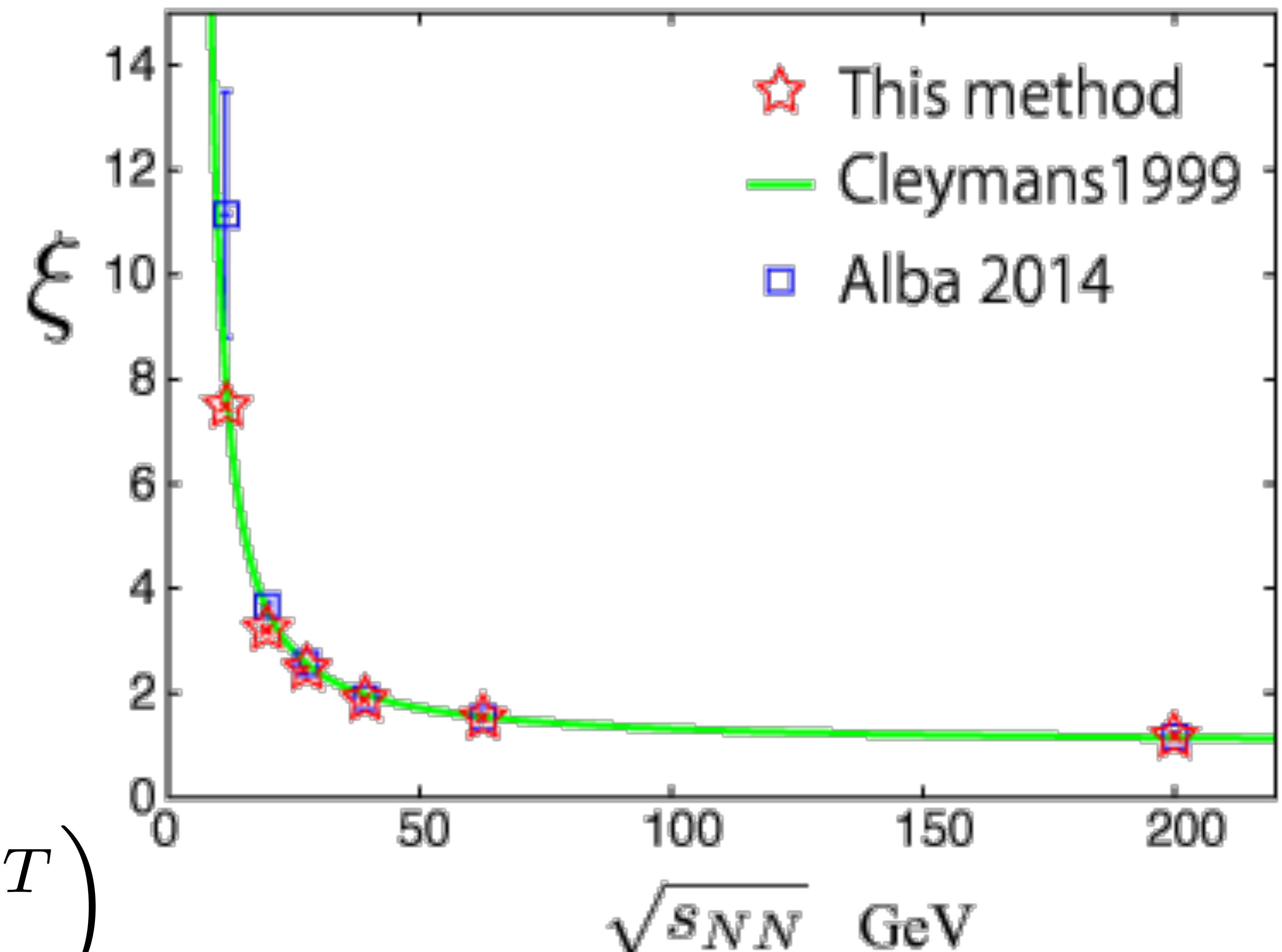
Here we demand $\boxed{Z_{+n} = Z_{-n}}$





$\xi = 1.88336$

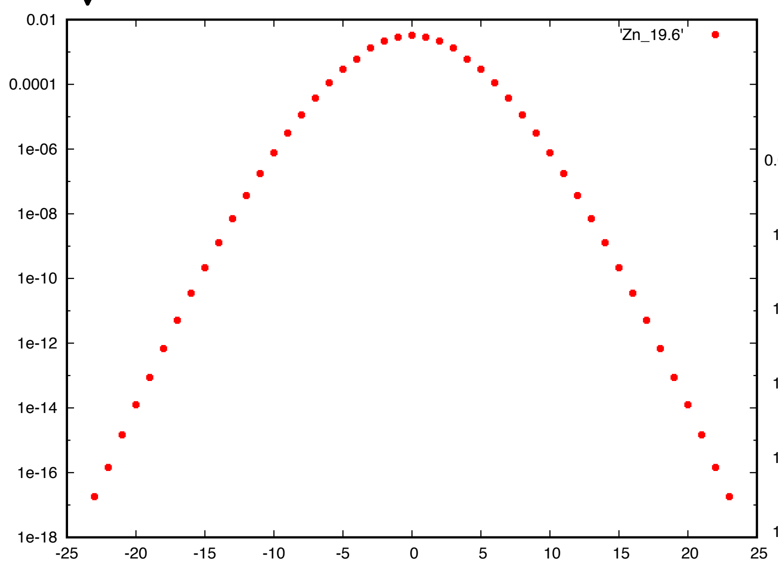
Fitted ξ are very consistent with those by Freeze-out Analysis.



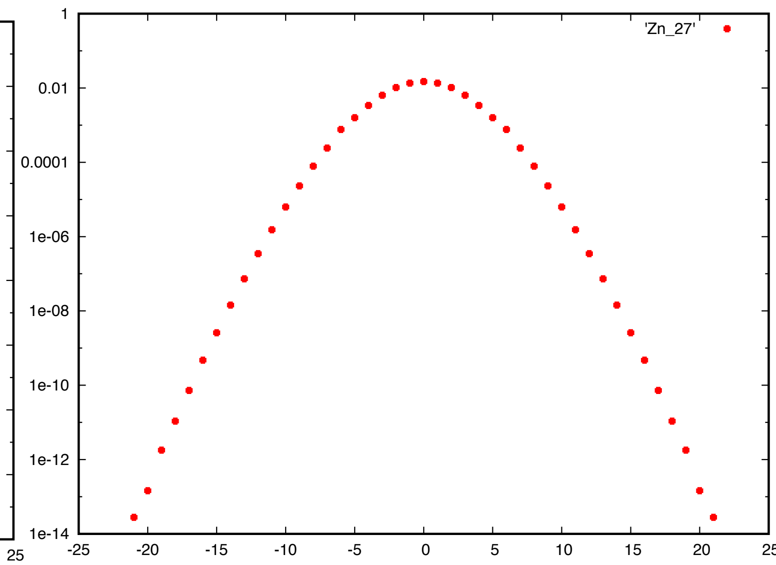
$$\left(\xi \equiv e^{\mu/T}\right)$$

Z_n from RHIC data

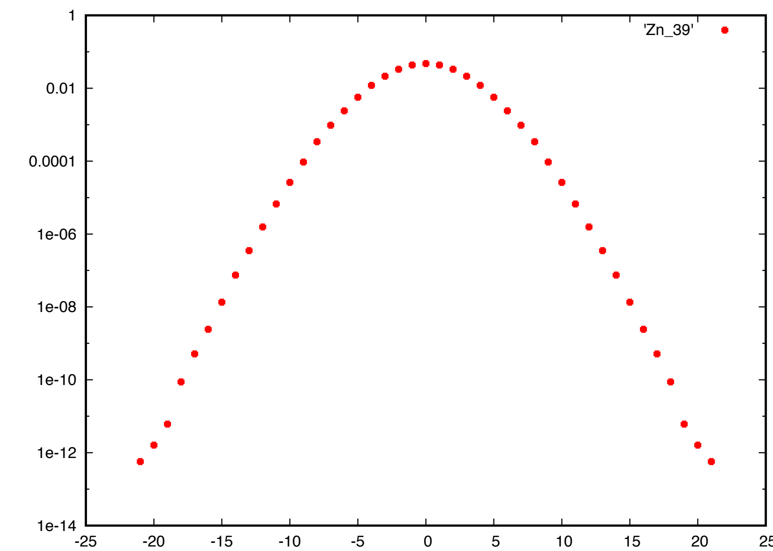
$\sqrt{s} = 19.6\mathrm{GeV}$



$\sqrt{s} = 27\mathrm{GeV}$

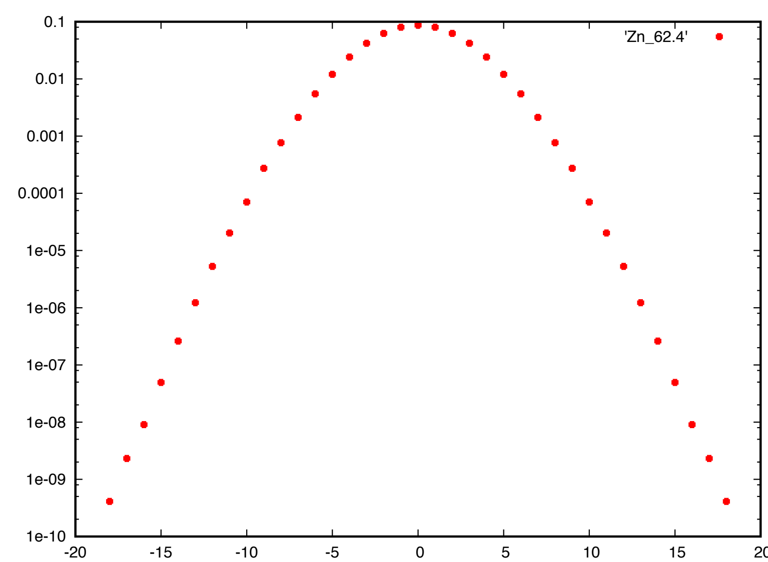


$\sqrt{s} = 39\mathrm{GeV}$

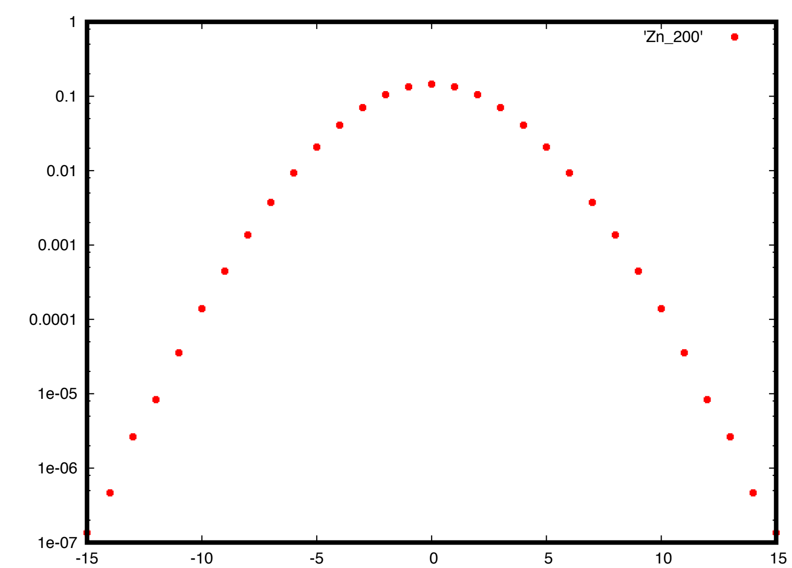


Can I see Difference?

$\sqrt{s} = 62.4\mathrm{GeV}$



$\sqrt{s} = 200\mathrm{GeV}$



Yes, You Can !
We will see it.

Yes, very useful, because

$$Z(\xi, T) = \sum_n Z_n(T)\, \xi^n$$

$(\xi \equiv e^{\mu/T}$: Fugacity$)$

$Z_n(T)$

$Z(\xi, T)$ at some ξ and T

$Z(\xi, T)$ at ANY ξ

for both Experiments and Lattice

2) N_{max} is not very large.

$$Z(\xi, T) = \sum_{n=-N_{max}}^{+N_{max}} Z_n(T)\, \xi^n$$

Lower estimation of larger density contribution.

We can calculate Z_n also by Lattice QCD

But Sign Problem on Lattice ?

$$Z_{GC}(\mu, T) = \int \mathcal{D}(\text{Gluon Fields}) \times \boxed{\det D(\mu)}\ e^{-(\text{Gluon Action})}$$

Complex if μ is real.

Our Lattice

- Clover improved Wilson action
- Iwasaki gauge action
- Lattice 4×16^3 (L ≈ 3.2fm, a ≈ 0.2fm)
- $m_\pi/m_\rho = 0.8$ ($m_\pi = 0.7$GeV)
  T/T_c = 0.84, 0.93,0.99, 1.08, 1.20, 1.35
- 20 - 40 points Imμ,
  1800 - 3800 configurations at each point
- Parameters were taken from
  S. Ejiri et. al., PRD 82, 014508 (2010)
- Our cluster: Vostok1 (20 GPU K40)

For Pure Imaginary μ ➡ $\det D$ **real**

A.Hasenfratz and Toussant, 1992

$$Z_C(n,T) = \int \frac{d\theta}{2\pi} e^{i\theta n} Z_{GC}(\theta \equiv \frac{\mathrm{Im}\mu}{T}, T)$$

Great Idea ! But practically it did not work.

We must develop several Engineering Methds.

1) Integration method
2) Multi-Precision Calculations

Integration Method

$$n_B = \frac{1}{3V} T \frac{\partial}{\partial \mu} \log Z_G$$

$$= \frac{N_f}{3N_s^3 N_t} \int \mathcal{D}U e^{-S_G} \mathrm{Tr}\Delta^{-1} \frac{\partial \Delta}{\partial \mu} \det \Delta$$

(For pure imaginary μ, n_B is also imaginary)

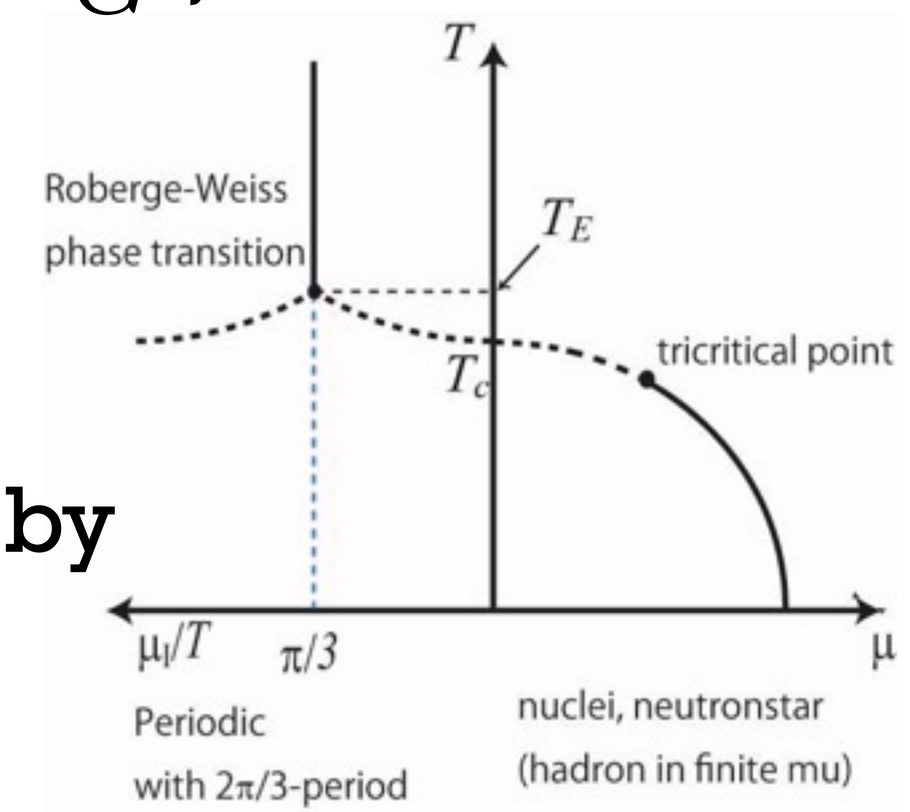
Then, for fixed T

$$Z(\theta \equiv \frac{\mu}{T}) = \exp(V \int_0^\theta n_B d\theta')$$

$$Z_k = \frac{3}{2\pi} \int_{-\pi/3}^{+\pi/3} d\theta \exp\left(i\, k\theta + \int_0^\theta n_B d\theta' \right)$$

We map Information in Pure Imaginary Chemical Potential to Real ones.

- We measure the number density at many pure imaginary chemical potential $n_B(\mu_I)$.
- We construct Grand Partition Function Z_G, by integrating $n_B(\mu_I)$
- By Fourier transformation, we get Z_n
- Then we can calculate Real μ regions by



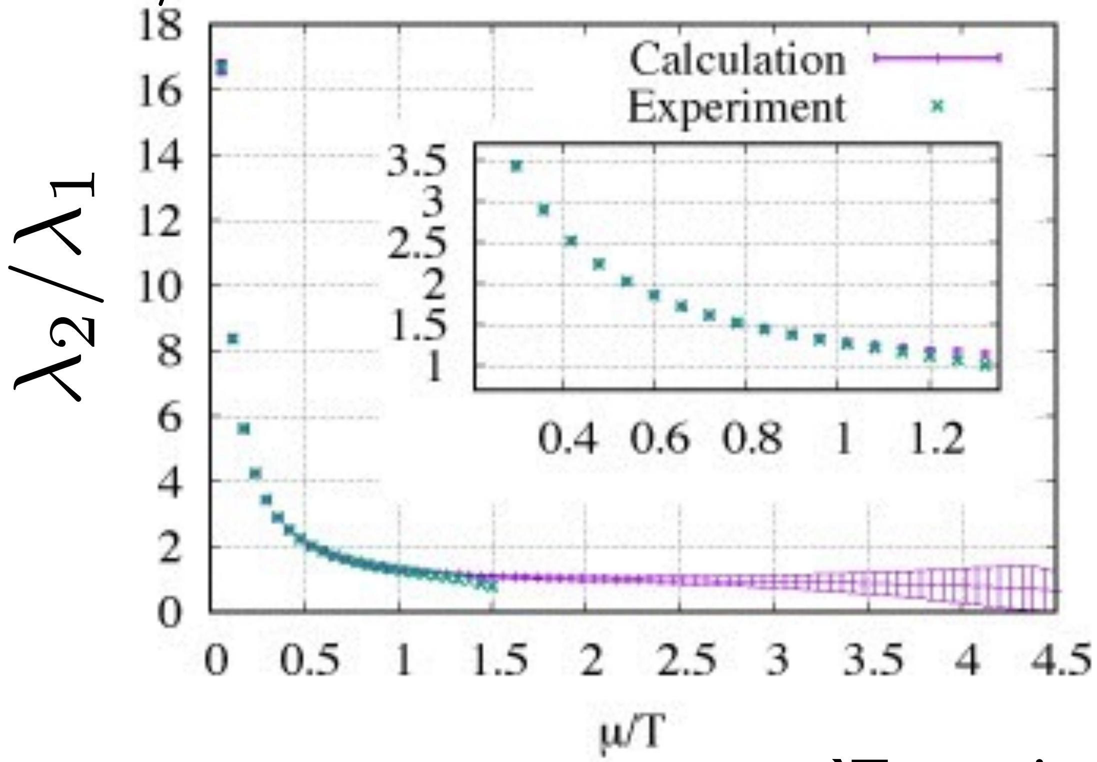
$$Z(\xi, T) = \sum_n Z_n(T)\, \xi^n$$

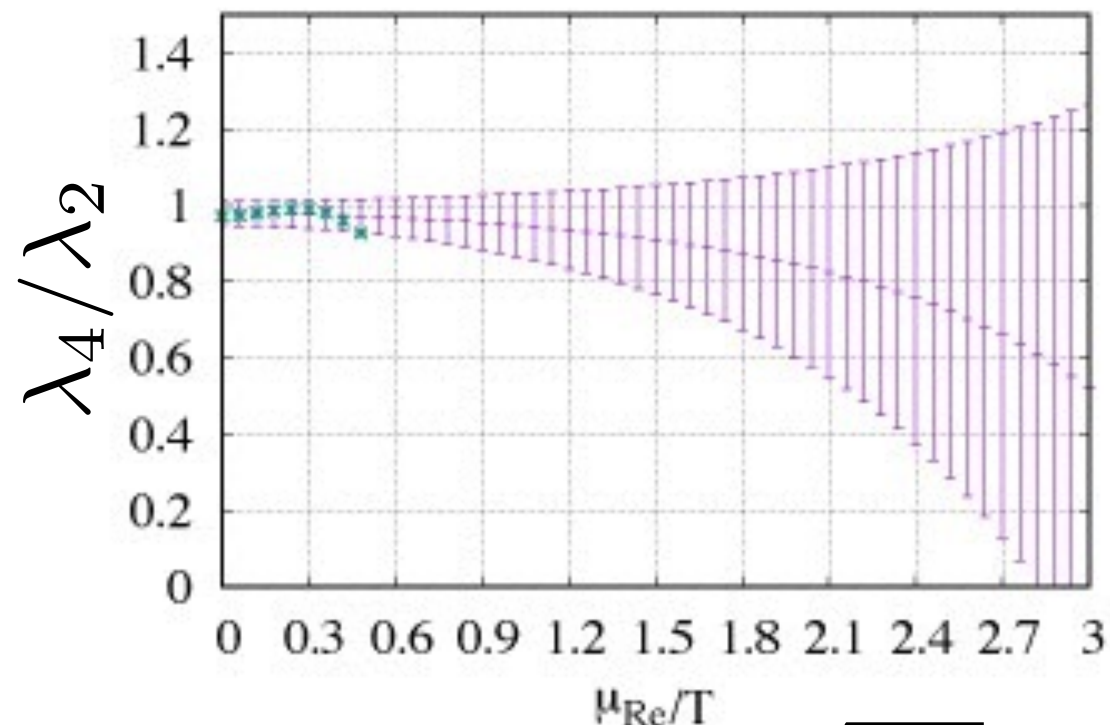
$\xi \equiv e^{\mu/T}$ **Fugacity**

Moments $\lambda_k = (T\frac{\partial}{\partial\mu})^k \log Z$

D.Boyda

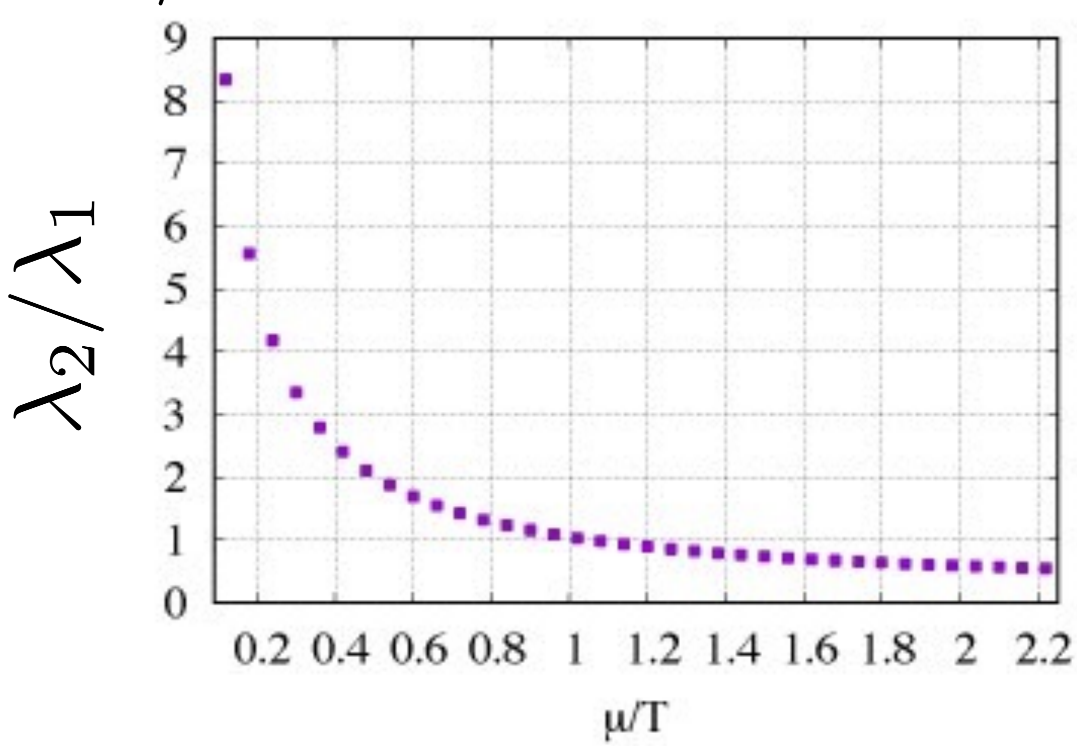
$T/T_c = 0.93$

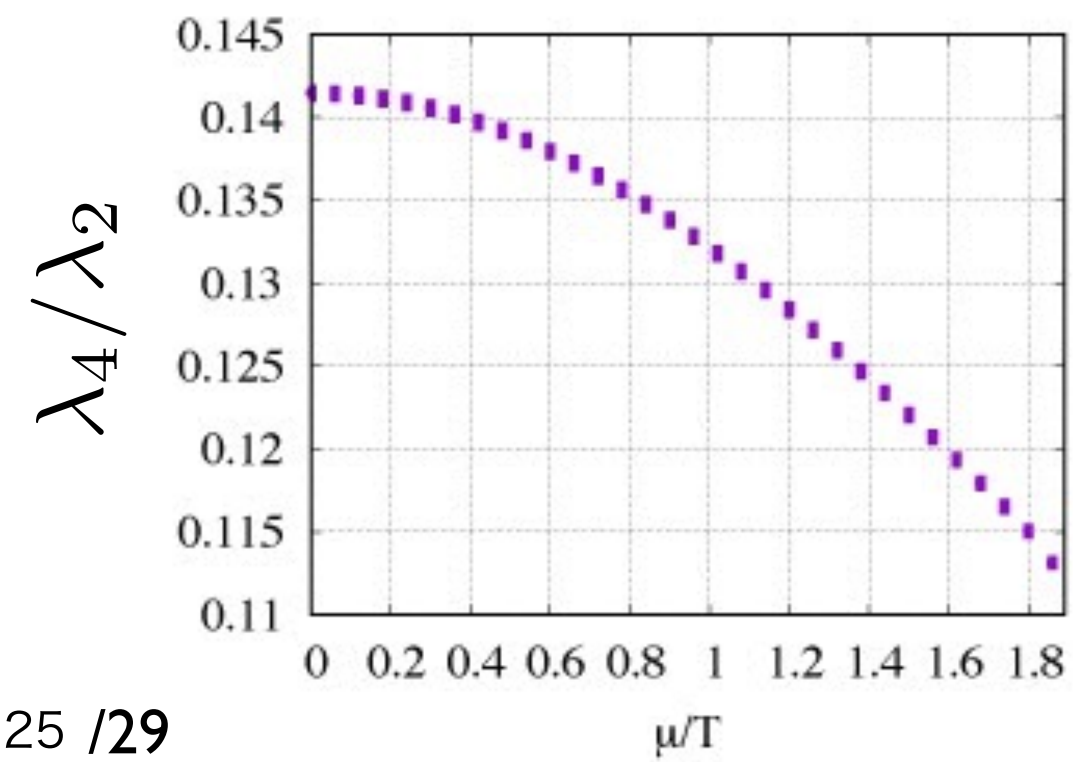




✖ \`Experiment' constructed from RHIC Star $\sqrt{s_{NN}} = 39\,(\mathrm{GeV})$

$T/T_c = 1.35$

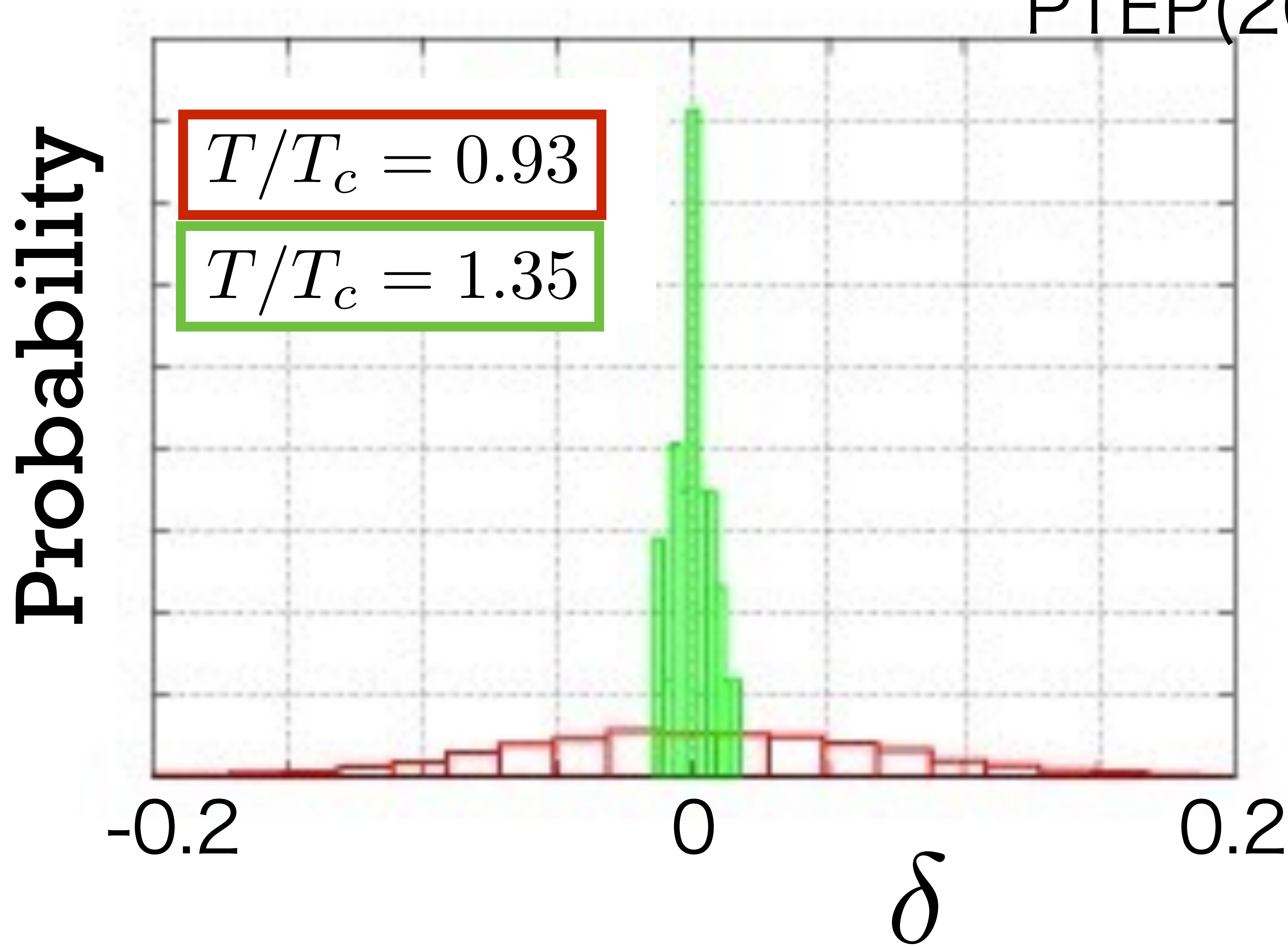




Hidden Sign Problem ?

Z_n have phase on each configuration !

V.Goy et al.,
PTEP(2017) 031D01

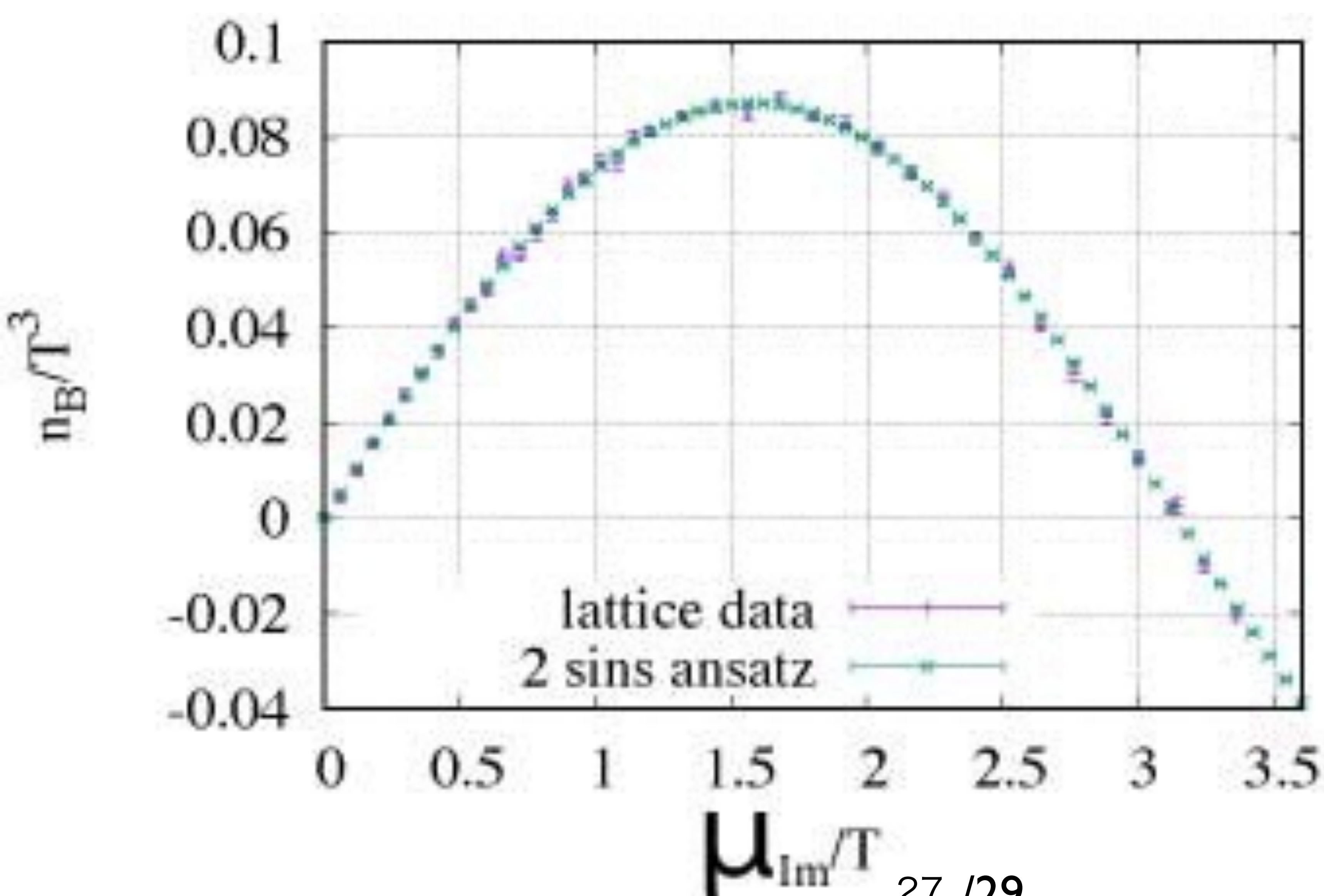


$z_n \simeq |z_n| e^{in\delta}$

$Z_n = \langle z_n \rangle$
are real positive.

A Remark of Function Form of $n_B(\mu_I)$

Preliminary

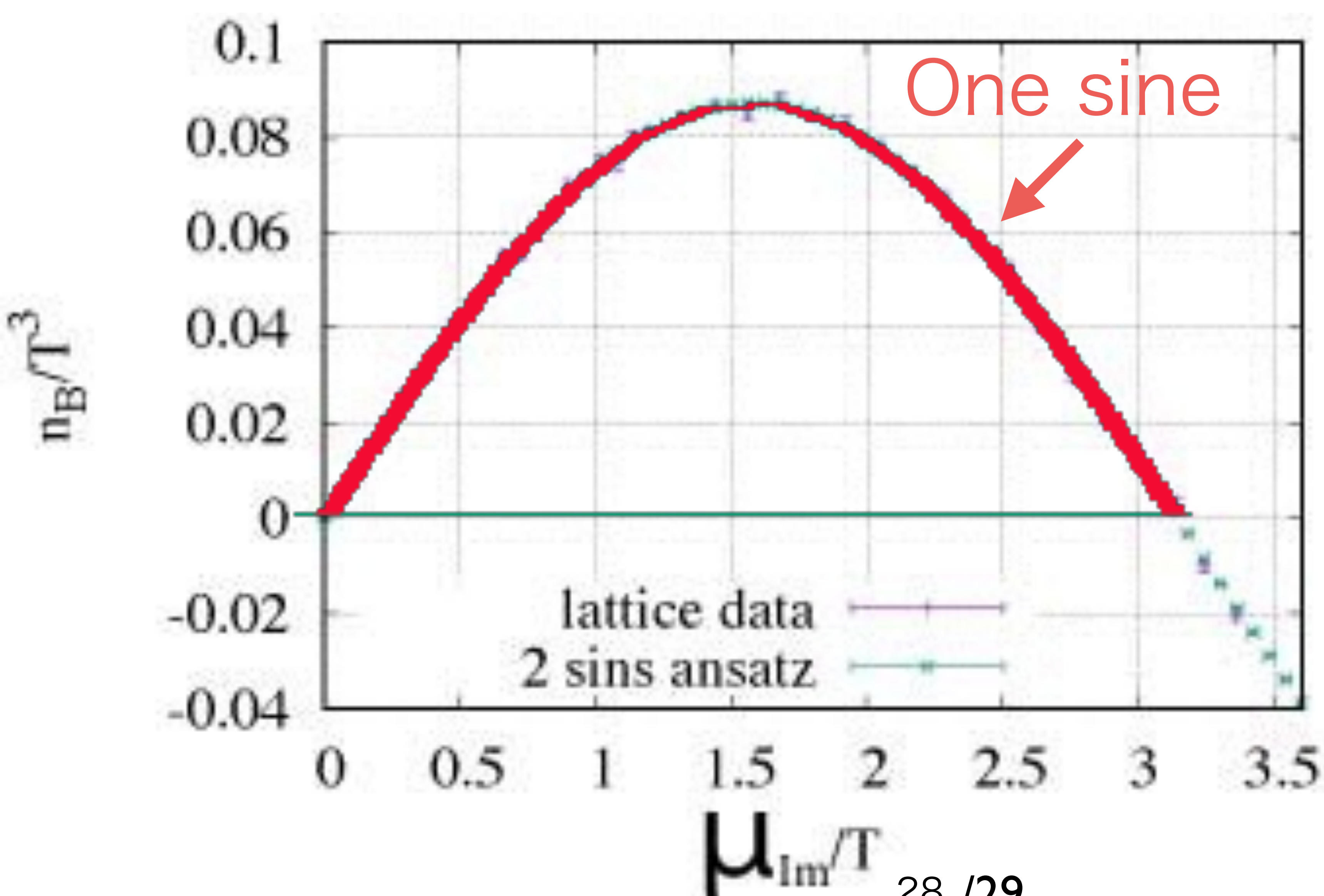


$n_B(\mu_I)$ is well approximated by sine function at $T<Tc$.

Takahashi et al. Phy. Rev. D 91 (1) (2015) 014501.
Bornyakov et al., Phys.Rev. D95, 094506 (2017)

A Remark of Function Form of $n_B(\mu_I)$

Preliminary



$n_B(\mu_I)$ is well approx-imated by sine function at $T<Tc$.

Takahashi et al. Phy. Rev. D 91 (1) (2015) 014501.
Bornyakov et al., Phys.Rev. D95, 094506 (2017)

In general,

$$n_B/T^3 = \sum_k f_{3k} \sin(k\theta_I)$$

$f_3 = 0.0871(3)$, $f_6 = -0.00032(27)$ ($\chi^2/\mathrm{dof} = 0.93$)

Lowest order,

$$n_B/T^3 \sim f_3 \sin(\theta_I)$$

$$\Rightarrow Z_n \propto I_n(f_3)$$

This is Skellam Model, which is used in Heavy Ion Collisions to describe the gross structure.

(Skellam is the difference of two independent Poisson Distributions.
f6, f9 … include the dynamics.)

Concluding Remarks

- ⭐We have developed the Canonical Approach for revealing QCD Phase Structure.
  We believe (hope) that we are in the right path.
- ⭐The canonical partition functions Z_n drop very rapidly as n goes large, and we need multi-precision calculations.
- ⭐The phase of Z_n fluctuates rapidly as n goes large in the confinement phase.
  No such problem in the deconfinement phase.
- ⭐Quark masses are heavy, because this is a test to see whether the Canonical Approach works for finite baryon density.
  We do not see any conceptual problem. So now it is time to go towards Physical Parameters.