Landau Levels in Lattice QCD in an External Magnetic Field

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Based on

F. Bruckmann, G. Endrődi, MG, S. D. Katz,
T. G. Kovács, F. Pittler, J. Wellnhofer,

arXiv:1705.10210

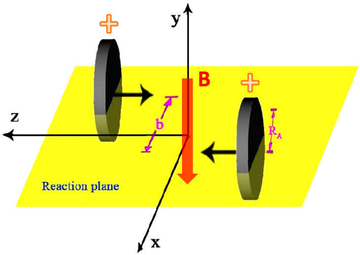
QCD in an External Magnetic Field

External magnetic field B on strongly interacting matter

[review: Andersen, Naylor, Tranberg (2016)]

Relevant to a number of problems:

- heavy ion collisions: colliding ions generate magnetic field which affects the QGP



Au+Au @ RHIC:
$\sqrt{s} = 200\ \mathrm{GeV}$, $B \approx 0.01 - 0.1\ \mathrm{GeV}^2$

Pb+Pb @ LHC:
$\sqrt{s} = 2.76\ \mathrm{TeV}$, $B \approx 1\ \mathrm{GeV}^2$

[review: Huang (2016)]

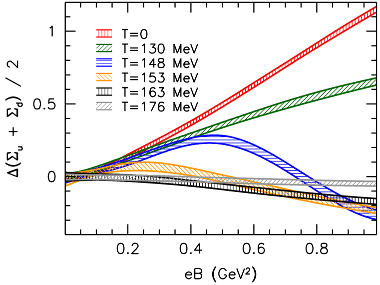
- neutron stars (magnetars)
- evolution of the early universe

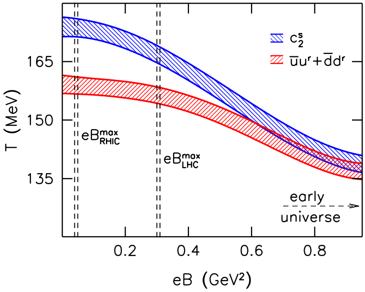
Phase Diagram from the Lattice

Zero chemical potential (no sign problem), $T_c(B=0,\mu=0) = 155\ \mathrm{MeV}$

$T \le 130\ \mathrm{MeV}$, $T \gtrsim 190\ \mathrm{MeV}$
magnetic catalysis (MC):
$\langle\bar\psi\psi\rangle$ increases with B

$130\ \mathrm{MeV} < T < 190\ \mathrm{MeV}$
inverse magnetic catalysis (IMC):
$\langle\bar\psi\psi\rangle$ decreases with B





T_c **decreases with** B

[Bali *et al.* (2012a), Bali *et al.* (2012b)]

MC only in early studies [D'Elia *et al.* (2010)]

To observe IMC physical quark masses, fine lattices required

(Inverse) Magnetic Catalysis

$$\langle\bar{\psi}\psi\rangle = \frac{\int DA\, \det(\not{D}(A,B)+m)e^{-S_g[A]}\mathrm{Tr}\frac{1}{\not{D}(A,B)+m}}{\int DA\, \det(\not{D}(A,B)+m)e^{-S_g[A]}}$$

Valence effect: the density of low Dirac modes increases with $B \Rightarrow$ MC

Sea effect: the determinant suppresses configurations with higher density of low modes $\Rightarrow$ IMC

Sea effect wins over valence only near T_c, where it is most effective
[Bruckmann, Endrődi, Kovács (2013)]

Analytic calculations in low-energy models (χPT, (P)NJL, bag model...) in general predict MC at all temperatures:

- MC mostly attributed to the behaviour of the **Lowest Landau Level (LLL)**: is that correct?
- Higher Landau Levels (HLLs) often neglected in calculations $\rightarrow$ **LLL approximation**: how good is it?

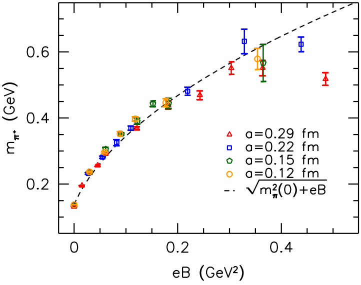
Landau Levels and Particle Spectrum

Energy levels of an otherwise free particle in a uniform magnetic field B

(minimal coupling + magnetic moment, Minkowski, relativistic, $qB \geq 0$)

$$E^2_{k,s_z}(p_z) = p_z^2 + qB(2k + 1 - 2s_z) + m^2$$

Turn on strong interactions: LLs work for weakly coupled particles



[Bali *et al.* (2012a)]

$$m^2_{\pi^\pm}(B) = E^2_{n,0}(0) = m^2_{\pi^\pm}(0) + eB$$

What about quarks?
(electrically charged
& strongly coupled)

Landau Levels for Free Fermions and Magnetic Catalysis

In the continuum, magnetic field along z: eigenvalues of $-\not{D}^2$ in 2D

$$\lambda_n^2 = qB \underbrace{[2k + 1 - 2s_z]}_{n=0,1,\ldots}$$

$$qB \geq 0, \quad k = 0, 1, \ldots, \quad s_z = \pm\tfrac{1}{2}$$

Degeneracy of λ_n^2 : $\nu_n = N_B(2 - \delta_{0n})$

$$N_B = \frac{L_x L_y qB}{2\pi} = \frac{\Phi_B}{2\pi}$$

Provides simple explanation for magnetic catalysis:

1. Switch on strong interactions, spectrum still approx. organised in LLs
2. LLL is $\lambda = 0$ with degeneracy $\propto B \Rightarrow B$ increases $\rho(0)$
3. Chiral condensate $\langle\bar{\psi}\psi\rangle \propto \rho(0)$ increases due to increased degeneracy $\Rightarrow$ magnetic catalysis (valence effect)

. . . is any of this true? Check on the lattice and curse the day you were born

Landau Levels for Free Fermions and Magnetic Catalysis

In the continuum, magnetic field along z: eigenvalues of $-\not{D}^2$ in 4D

$$\lambda^2_{np_zp_t} = qB\,\underbrace{[2k+1-2s_z]}_{n=0,1,\ldots} + p_z^2 + p_t^2$$

$qB \geq 0, \quad k = 0,1,\ldots, \quad s_z = \pm\frac{1}{2}$

Degeneracy of $\lambda^2_{np_zp_t}$: $\nu_n = 2N_B(2-\delta_{0n})$

$$N_B = \frac{L_x L_y qB}{2\pi} = \frac{\Phi_B}{2\pi}$$

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Landau Levels for Free Fermions and Magnetic Catalysis

In the continuum, magnetic field along z: eigenvalues of $-\not{D}^2$ in 4D with N_c colours

$$\lambda^2_{np_zp_t} = qB \underbrace{[2k + 1 - 2s_z]}_{n=0,1,\ldots} + p_z^2 + p_t^2$$

$$qB \geq 0, \quad k = 0, 1, \ldots, \quad s_z = \pm\tfrac{1}{2}$$

Degeneracy of $\lambda^2_{np_zp_t}$: $\nu_n = 2N_B(2 - \delta_{0n})N_c$

$$N_B = \frac{L_x L_y qB}{2\pi} = \frac{\Phi_B}{2\pi}$$

Provides simple explanation for magnetic catalysis:

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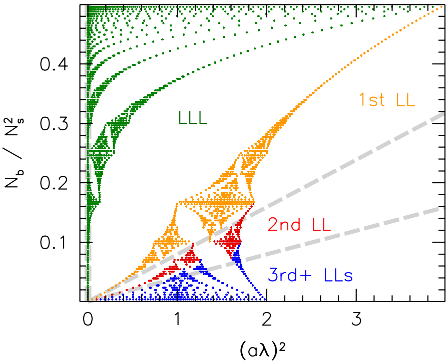
Landau Levels on a 2d Lattice

Finite lattice with periodic boundary conditions, staggered fermions

"Free" fermions (interacting only with B)

2d free spectrum



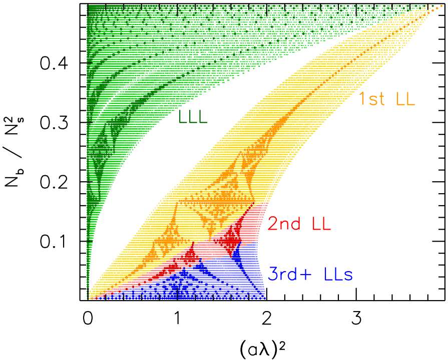
- Group eigenvalues according to LL degeneracy
- LL structure spoiled by finite spacing artefacts
- Gaps $\sim$ remnant of the continuum LL structure
- Fractal structure: Hofstadter's butterfly [Hofstadter (1976)]

Landau Levels on a 2d Lattice

Add SU(3) interaction

2d configuration = 2d slice of a 4d configuration

2d interacting spectrum



- Group eigenvalues according to LL degeneracy
- Hofstadter's butterfly washed away, gaps disappear. . .
- . . . but "lowest Landau level" (LLL) survives, wide gap
- LLL gap survives the continuum limit

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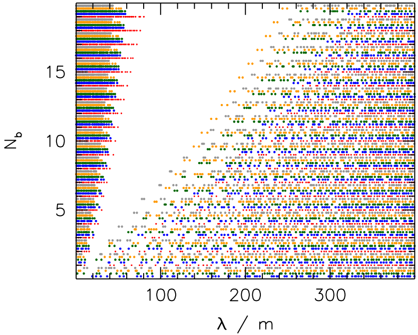

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- Hofstadter's butterfly washed away, gaps disappear...
- ...but "lowest Landau level" (LLL) survives, wide gap
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Different colours = different spacings

Use λ/m: spectrum renormalises like the quark mass

Topological Origin of 2d LLL

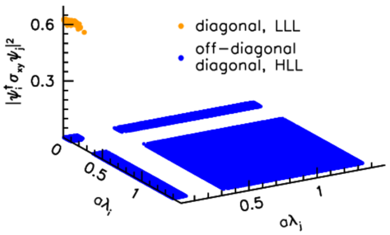
LLL has topological origin, "robust" under small deformations

In 2d index theorem $Q_{\rm top} = n_- - n_+$ + "vanishing theorem": $n_- \cdot n_+ = 0$

[Kiskis (1977); Nielsen and Schroer (1977); Ansourian (1977)]

In 2d topological charge $Q_{\rm top}$ = flux of Abelian field N_B, chirality $= 2s_z$, also in the presence of SU(3) fields

LLL (= zero modes) survives; other LLs are mixed by the SU(3) interaction



Matrix elements of σ_{xy} for 2d eigenstates $\phi_{i,j}$

$$(\sigma_{xy})_{ij} = \langle \phi_i | \sigma_{xy} | \phi_j \rangle$$

$6N_B$ almost-zero modes with almost-definite spin

Landau Levels in 4d

In 4d the $p_z^2 + p_t^2$ contribution makes it impossible to identify LL from the spectrum even in the free continuum case

Factorisation of the eigenmodes holds in the free case (also on the lattice): $\phi^{(j)}$ solution of Dirac eq. in 2D, $\tilde{\psi}^{(j)}_{p_z p_t}(x,y,z,t) = \phi^{(j)}(x,y)e^{ip_z z}e^{ip_t t}$

Projector on the LLL: sum over momenta and over j the projectors for each of the LLL modes

$$P = \sum_{j=1}^{2N_B} \sum_{p_z,p_t} \tilde{\psi}^{(j)}_{p_z p_t} \tilde{\psi}^{(j)\dagger}_{p_z p_t} = \sum_{j=1}^{2N_B} \phi^{(j)}\phi^{(j)\dagger} \otimes \mathbf{1}_z \otimes \mathbf{1}_t = \sum_{j=1}^{2N_B} \sum_{z,t} \psi^{(j)}_{zt}\psi^{(j)\dagger}_{zt}$$

$$\psi^{(j)}_{z_0 t_0}(x,y,z,t) = \phi^{(j)}(x,y)\delta_{zz_0}\delta_{tt_0}$$

Can this be exported to the case when strong interactions are switched on? LL structure washed away already in 2d, but LLL survives on each slice $\rightarrow$ project the 4d mode on the union of the 2d LLLs

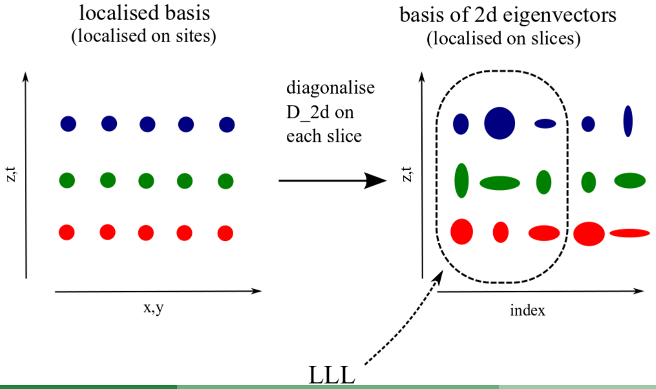
LLL Projector

$$P(B) = \sum_{j=1}^{3N_B \times 2} \sum_{z,t} \psi_{zt}^{(j)}(B)\psi_{zt}^{(j)\dagger}(B)$$

$$\psi_{z_0 t_0}^{(j)}(x, y, z, t; B) = \phi_{z_0 t_0}^{(j)}(x, y; B)\delta_{zz_0}\delta_{tt_0}$$

$\phi_{z_0 t_0}^{(j)}$ solution of Dirac eq. in 2D with B + strong interactions at z_0, t_0

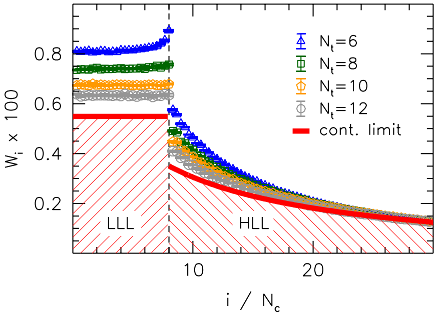
All 2d modes, all slices form complete 4d basis



Overlap with LLL

Determine the 2d eigenmodes on each slice, identify the LLL on each slice, and project the 4d mode on the union of the LLLs

$T \approx 400$ MeV, $N_B = 8$



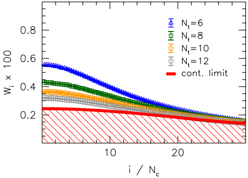
Low modes ($220 < \lambda/m < 225$)

$$W_i(\phi) = \sum_{\text{doublers}} \sum_{zt} |\psi_{zt}^{(i)\dagger}\phi|^2$$

ψ_{izt}: ith 2d mode>0 of $D_{\text{stag}}|_{z=z_0,t=t_0}$

Flat in the LLL, jump at the end (also in the continuum)

Low 4d modes have a bigger overlap with LLL than bulk modes



$B = 0$ case for comparison

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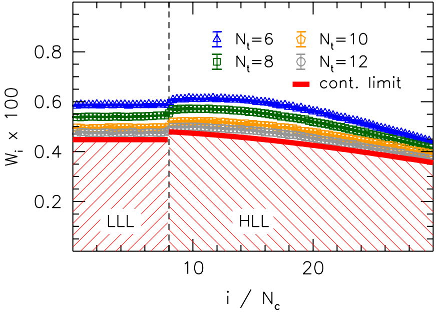

Bulk modes ($535 < \lambda/m < 545$)

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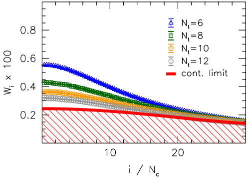

$B = 0$ case
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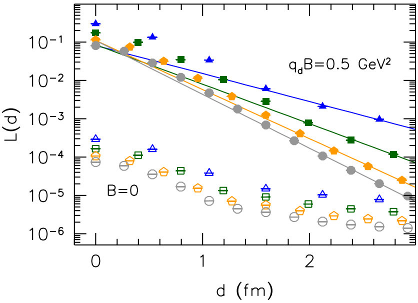
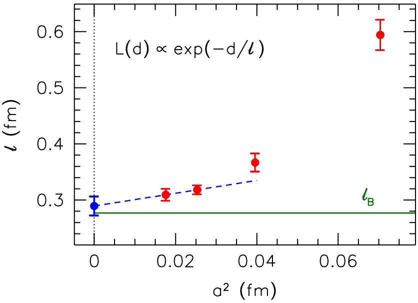
Locality

In the continuum LLL modes extend over $\ell_B = 1/\sqrt{qB}$, what about our projected modes on the lattice?

Put a source ξ at (x, y, z, t), and project it on the LLL, $\psi = P\xi$: how far does ψ extend?

$$L(d) = \langle \|\psi(x', y', z, t)\| \rangle \qquad d = \sqrt{(x - x')^2 + (y - y')^2}$$





LLL-Projected Condensate

Full quark condensate: $\langle\bar{\psi}\psi\rangle_B = \langle\mathrm{Tr}\, D_{\mathrm{stag}}^{-1}\rangle_B$

Change in the condensate $\langle\bar{\psi}\psi\rangle_B - \langle\bar{\psi}\psi\rangle_{B=0}$ is free from additive divergence

$=$ change in the contribution from all the 2d modes on all slices

Projected condensate: $\langle\bar{\psi}\psi^{\mathrm{LLL}}\rangle_B = \langle\mathrm{Tr}\, P D_{\mathrm{stag}}^{-1} P\rangle_B$

Only **valence** effect considered here

Change in the contribution from the first $6N_B$ 2d modes on all slices

$$\widetilde{P}(B) = \sum_{j=1}^{3N_B\times 2} \sum_{z,t} \psi_{zt}^{(j)}(0)\psi_{zt}^{(j)\dagger}(0)$$

$$\Delta\langle\bar{\psi}\psi^{\mathrm{LLL}}\rangle(B) = \langle\bar{\psi}P(B)\psi\rangle_B - \langle\bar{\psi}\widetilde{P}(B)\psi\rangle_0$$

Additive divergence from large modes, SU(3) interaction negligible $\sim$ "free" case
$\rightarrow$ shown explicitly in the continuum that it is free of additive divergences

Multiplicative divergence: if the LLL-overlap has a finite continuum limit (and it seems it does) $\rightarrow$ same as in the full case, cancels in ratios
Requires more studies, continuum limit **not taken** in this work

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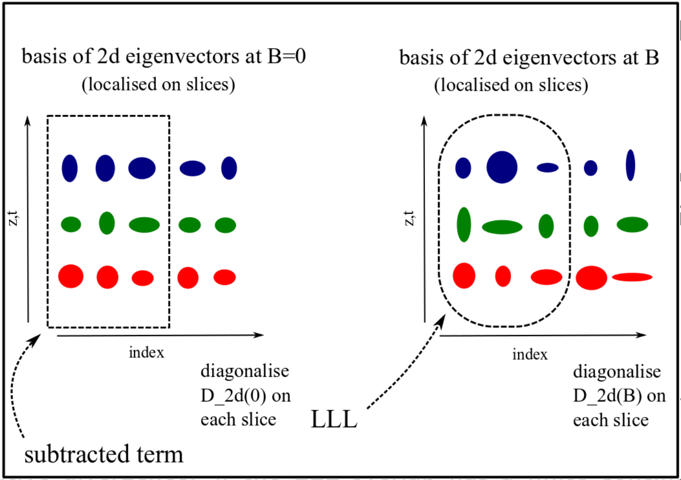
Change ir … livergence

= change …

Projected … nsidered here

Change ir … ices

… "free" case divergences



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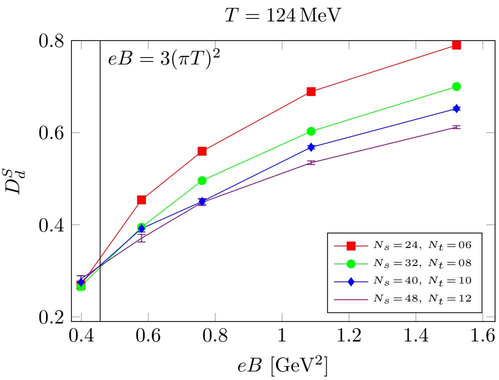
LLL Contribution to the Quark Condensate

How much of the change in the condensate comes from the LLL?

$$D^S = \frac{\Delta\langle\bar{\psi}\psi^{\mathrm{LLL}}\rangle(B)}{\Delta\langle\bar{\psi}\psi\rangle(B)} = \frac{\langle\bar{\psi}P(B)\psi\rangle_B - \langle\bar{\psi}\widetilde{P}(B)\psi\rangle_0}{\langle\bar{\psi}\psi\rangle_B - \langle\bar{\psi}\psi\rangle_0}$$

Below T_c



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- No B in the fermion determinant
- d quark condensate ($q = -\frac{1}{3}$)
- Finite continuum limit (it seems)
- Ratio increases with B
- Ratio decreases with T

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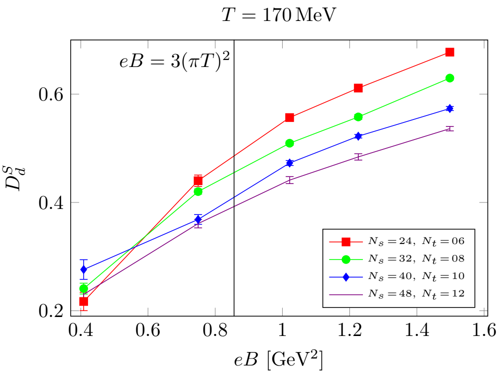

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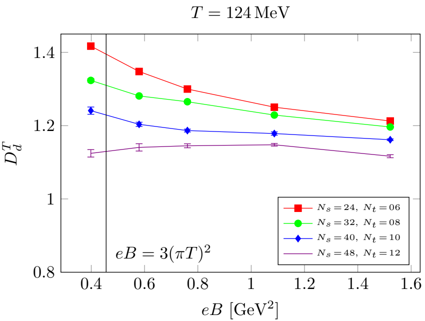
LLL Contribution to the Spin Polarisation

Spin polarisation along the magnetic field

Same additive divergence is expected in the full and LLL projected quantities (backed up by explicit calculation in the "free" case)

$$D^T = \frac{\Delta\langle\bar{\psi}\sigma_{xy}\psi^{\mathrm{LLL}}\rangle(B)}{\Delta\langle\bar{\psi}\sigma_{xy}\psi\rangle(B)} = \frac{\langle\bar{\psi}P\sigma_{xy}P\psi\rangle_B - T^{\mathrm{div}}}{\langle\bar{\psi}\sigma_{xy}\psi\rangle_B - T^{\mathrm{div}}}$$

Below T_c



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- Continuum limit seems to exist
- LLL approx overestimates spin polarisation
- Deviation from unity only $\sim 15\%$ on finest lattices

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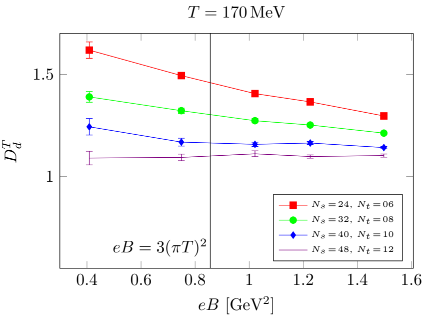

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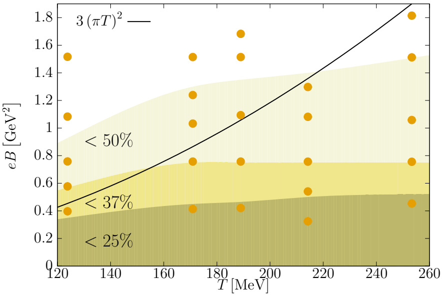
Lowest-Landau-Level Dominance?

LLL approximation underestimates the change in the quark condensate due to a magnetic field



- Dots: simulation points
- Shaded areas: $D_d^S|_{N_t=12} < X\%$ reconstructed by spline interpolation
- Solid line: $|qB| = (\pi T)^2$

Conclusions and Outlook

What happens to quark Landau levels in QCD?

- Landau level structure washed out, both in 2d and 4d, but...
- ...in 2d lowest Landau level survives (topological reasons)...
- ...and 4d observables seem to "feel" the 2d LLL: sizeable fraction of the change in the (valence) condensate comes from it...
- ...but not enough to fully explain magnetic catalysis, except for very large magnetic field ($eB \gtrsim 1\ \mathrm{GeV}^2$)

Open issues:

- Use B in the determinant
- Continuum limit?
- Does the LLL have important sea effects (e.g. inverse catalysis)?

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