Soft corrections &
ultra-Planckian scattering

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50 years of Veneziano model
GGI, Arcetri

based on work done in collaboration with
A. Addazi and G. Veneziano

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Foreword

Veneziano amplitude

$$\mathcal{A}(s,t) = g_{op}^2 \frac{\Gamma(-\alpha_0 - \alpha' s)\Gamma(-\alpha_0 - \alpha' t)}{\Gamma(-2\alpha_0 - \alpha'(s+t))}$$

Planar duality, unitarity for $D \leq 26$ and $24\alpha_0 = D - 2$

Soft behaviour at high energies $\alpha' s >> 1$

- fixed angle

$$|t|/s = \mathcal{O}(1) : \mathcal{A}(s,t) \approx g_{op}^2 e^{-\alpha'[s \log \alpha' s + t \log \alpha' t + u \log \alpha' u]}$$

  same factor as in B_0^{grav} ... see below !!!

- Regge regime (small angle, eikonal scattering)

$$|t|/s << 1 : \quad \mathcal{A}(s,t) \approx g_{op}^2 (-\alpha' s)^{\alpha_0 + \alpha' t} \Gamma(-\alpha_0 - \alpha' t)$$

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yet NO Adler zero (tachyons not π's) ...
but Gabriele is working to amend this with Yankielowicz and Onofri
... see Armoni's talk

Summary

- Soft behaviour of Scattering Amplitudes
- High-energy scattering from Veneziano to Amati, Ciafaloni, Veneziano
- Glimpses of BH formation and evaporation
- Gravitational Waves: ZFL ... and beyond

1. Soft behaviour of Scattering Amplitudes

Soft behaviour of scattering amplitudes

Massless particles as $q \to 0$
Two cases

- Mediators of 'gauge' interactions: photons, gravitons, gluons ... gravitini
  Mass protected by exact gauge invariance
- (Pseudo) Goldstone bosons: pions, 'dilatons', moduli ... goldstino
  Mass protected by partial global current conservation
  Adler zero

Focus on the first ... gravity

Soft theorems in Gravity at tree level

In Gravity [Weinberg; Gross, Jackiw; Bern, Dixon, Dunbar; ... Cachazo, Strominger; ... Laddha, Sen]

$$\mathcal{M}_{n+1} = \hat{S}^{(0)}_{grav}\mathcal{M}_n + \hat{S}^{(1)}_{grav}\mathcal{M}_n + \hat{S}^{(2)}_{grav}\mathcal{M}_n + \ldots$$

where

$$\hat{S}^{(0)}_{grav} = \sum_i \frac{p_i^\mu h_{\mu\nu} p_i^\nu}{q \cdot p_i}$$

$$\hat{S}^{(1)}_{grav} = \sum_i \frac{p_i^\mu h_{\mu\nu} J_i^{\nu\lambda} q_\lambda}{q \cdot p_i}$$

$$\hat{S}^{(2)}_{grav} = \sum_i \frac{q \cdot J_i \cdot h \cdot J_i \cdot q}{2 q \cdot p_i}$$

with $r_{\mu\nu\rho\sigma} = q_{[\mu} h_{\nu][\rho} q_{\sigma]}$
In the IR gravity is better than gauge theory, no collinear divergences

Are soft theorems universal?

Quantum corrections (loops) violate even leading behaviour in gauge theories not in gravity [Bern, Davies, Nohle; He, Huang, Wen; MB; ...]

- Loops: correct sub-leading and sub-sub-leading due to IR divergences in $D < 5$ [Laddha, Sen; Chakrabarti, Kashyap, Sahoo, Sen, Verma; ...]
- Effective field theories with F^3, R^3 (non-susy, present in bosonic string) or ϕR^2 or ϕF^2(appear in SUGRA, strings, ...) [Carrasco, Kallosh, Roiban, Tseytlin]
- Full-fledged String Theory (finite α'): tree level (disk/sphere) OK [MB, He, Huang, Wen; Bern, Di Vecchia, Nohle; MB, Guerrieri; Di Vecchia, Marotta, Mojaza; ...]
- Soft dilatons and pions, DBI, conformal DBI, A-V [Di Vecchia, Marotta, Mojaza; MB, Guerrieri, Huang, Lee, Wen; ... Kallosh] ... no corrections [Guerrieri, Huang, Li, Wen]

Why are we interested in soft limits?

- Important corrections to scattering amplitudes e.g. in Gravity

$$B_0^{grav} = \frac{G}{\pi} \log\left(\frac{E_{Max}}{E_{min}}\right) \sum_{ij} p_i p_j \log\left(\frac{p_i p_j}{\mu_{IR}^2}\right)^2$$

  exponentiation and cancellation of IR divergences between 'virtual' (-ve, $E_{min} = \mu_{IR}$), 'real' (+ve, $E_{min} = \Delta_{exp}$) [Weinberg]
- Gravitational wave production in high energy collisions, BH vs fuzzball mergers, ... [Ciafaloni, Colferai, Coraldeschi, Veneziano; Gruzinov, Veneziano; ... Addazi, MB, Veneziano; Lunin]
- Soft 'hairs' and BH entropy: soft gravitons as Goldstone bosons of broken BMSvB symmetry in $D = 4$ [Bondi, Metzner, Sachs, van der Burg; Barnich; Strominger; Hawking, Perry; Dvali, Gomez, Lüst] BUT [Porrati; Bousso]

2. High Energy string scattering: from Veneziano to Amati, Ciafaloni, Veneziano

Ultra-Planckian scattering à la Amati, Ciafaloni, Veneziano (ACV)

From Veneziano to 4-graviton amplitude [Shapiro, Virasoro; Brink, Green, Schwarz; Kawai, Lewellen, Tye; ...] ... gravity = (gauge)^2 [Bern, Dixon, Dunbar, Kosower, ...]

$$\mathcal{M}_4^{tree}(s,t,u) = \frac{G\mathcal{R}^4}{stu}\frac{\Gamma(1-\alpha' s/4)\Gamma(1-\alpha' t/4)\Gamma(1-\alpha' u/4)}{\Gamma(1+\alpha' s/4)\Gamma(1+\alpha' t/4)\Gamma(1+\alpha' u/4)}$$

Regge regime $-u \approx s = E^2_{CM} >> |t| = |\vec{q}|^2 = E^2_{CM}\sin^2(\theta_{CM}/2)$:
neglecting 'kinematical factor' $\mathcal{R}^4 = \mathcal{F}^4_L \mathcal{F}^4_R$

$$\mathcal{M}_4^{tree}(s,t) \simeq -Gs\frac{\Gamma(-\alpha' t/4)}{\Gamma(1+\alpha' t/4)}\left(\alpha' s/4\right)^{\alpha' t/4} e^{-i\pi\alpha' t/4}$$

growth with energy, violation of unitarity $|\mathcal{A}_J(s)| \leq 1$, ... loops:
1-loop [Brink, Green, Schwarz; ...], 2-loops [D'Hoker, Phong; ...], L-loops, ...
ladder diagrams dominate, Fourier transform to impact parameter space $b = J/\sqrt{s}$, ... exponentiation [Amati, Ciafaloni, Veneziano]

$$S(b,s) = \exp 2i\delta(b,s)$$

Leading eikonal approximation

Saddle point in $D = 4 + d$, $\mathrm{Im}\delta \neq 0$ (set $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$)

$$\delta(b,s) = \hat{\delta}(b,s)\Big|_{\hat{X}=0} \approx \left(\frac{b_E}{b}\right)^d + i\,\frac{G_D s}{\ell_s^d (\log \alpha' s)^{d/2+1}}\, e^{-b^2/\ell_s^2 \log \alpha' s}\,,$$

where G_D is the D-dimensional Newton's constant and

$$b_E^d = \frac{s}{8\pi\Omega_d M_D^{d+2}}$$

- Very large b regime, $b > b_E$: eikonal scattering, massless graviton pole dominates (distorted by Coulomb phase in $D = 4$)
- Intermediate regime $b_E > b > b_B, b_I$: opening of inelastic channels at $b_I = \ell_s\sqrt{\log \alpha' s}$ and threshold for bremsstrahlung at $b_B^{d+2} = b_E^d R_S^2$
- $b_B > b > b_I$: classical corrections, capture and gravitational brems-strahlung
- Inelastic regime $b_I > b > \ell_s > \ell_P$, inelastic channels of both classical and string absorption

ACV meet AGK

Unitarity, forward scattering, AGK cutting rules [Abramowicz, Gribov, Kanceli]
Average number of cut gravi-Reggeon (cGR)

$$\langle N \rangle = 4\mathrm{Im}\delta = 4\,\frac{G_D s}{\ell_s^d (\log \alpha' s)^{d/2+1}} e^{-b^2/\ell_s^2 \log \alpha' s}$$

Average energy per cGR $\sim$ Hawking temperature

$$\langle E \rangle = \frac{\sqrt{s}}{\langle N \rangle} = \frac{\log \alpha' s}{R_S}$$

but Poissòn rather than thermal spectrum

$$\frac{d\sigma(2 \to N)}{d^2 b} = \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle}$$

for $b < \ell_s \sqrt{\log \alpha' s}$

3. Glimpses of BH formation and evaporation

Classicalization à la DGILS

BH's as Bose-Einstein condensates of a large number of gravitons ($N \approx M_{BH}^2/M_{Pl}^2$) that, at a critical value of $\alpha_G = Gs$, behave as Bogoliubov modes [Dvali, Gomez, Isermann, Lüst, Stieberger]

At criticality $\alpha_G^{\rm crit} \approx N$, processes with high multiplicity dominate, DGILS estimate

$$\sigma(2 \to N{-}2) \sim N! \left(\frac{\ell_P^2 s}{N^2}\right)^N \sim \frac{N!}{N^N} \sim e^{-N}$$

using helicity spinors, MHV amplitudes [Parke, Taylor], KLT relations [Kawai, Lewellen, Tye] and CHY scattering equations [Cachazo, He, Yuan]

Violation of unitarity! Including BH's with

$$d_{BH}(N) \sim e^N$$

restore unitarity ... ?

Puzzling features: specific helicity configuration, loop corrections neglected, exponential sensitivity to the choice of $\alpha_G^{\rm crit}$, 'truly' soft gravitons not included ...

Soft gravitons à la Weinberg: *"Because I can."*

Virtual soft gravitons, from loop corrections

$$|S_{2\to N}|^2 = |S^0_{2\to N}|^2 \exp\mathrm{Re}\int_\lambda^\Lambda d^4k B(k) = |S^0_{2\to N}|^2 \left(\frac{\Lambda}{\lambda}\right)^{-B_0}$$

where ($\eta = \pm$ for out/in)

$$B(k) = \frac{-8\pi i G}{(2\pi)^4(k^2 - i\epsilon)} \sum_{i,j} \frac{\eta_i\eta_j[(p_i\cdot p_j)^2 - \frac{1}{2}m_i^2 m_j^2]}{(p_i\cdot k - i\eta_i\epsilon)(-p_j\cdot k - i\eta_j\epsilon)}$$

in the massless limit

$$B_0 = -\frac{G}{2\pi}\sum_{i,j} 2p_i p_j \log\left(\frac{p_i p_j}{\mu^2}\right)^2 \geq 0!!!$$

Real soft gravitons, from leading soft term $|\hat{S}_0|^2$

$$B_0 \log\frac{\Delta_E}{\lambda} = \int \frac{d^3q}{2|q|(2\pi)^3}\sum_{s=\pm 2}\left|\sum_{i=1}^N \frac{p_i h_s p_i}{\eta_i q p_i}\right|^2$$

same B_0 !!! [Weinberg; ... Addazi, MB, Veneziano]

Reinterpretation, resolution and reconciliation

Treat N produced gravitons as 'hard-ish' and include truly soft ones

$$\frac{d\sigma^{tree}}{d\omega_1 \ldots d\omega_N} \sim N! \left(c\, e^2 \, \frac{Gs}{N^2} \right)^N \frac{1}{\omega_1 \ldots \omega_N} \sim \frac{1}{N!} (c\, Gs)^N \frac{1}{\omega_1 \ldots \omega_N} \,,$$

where c is some $\mathcal{O}(1)$ constant
IR-safe observables in analogy with N-jet cross sections in QCD

$$\sigma \left(2 \to N(E_i \geq \bar{E}) + M(E_{soft} \leq \Delta) \right) \quad ; \qquad \bar{E} N < \sqrt{s}$$

generating function(al) for IR-safe cross sections:

$$\Sigma \left(z(\omega), \bar{E}, \Delta \right) = \sum_{N,M} \int_{\bar{E}} d\omega_1 \ldots d\omega_N z(\omega_1) \ldots z(\omega_N)$$

$$\int_{\lambda}^{\bar{E}} d\epsilon_1 \ldots d\epsilon_M \theta(\Delta - \sum_j \epsilon_j) \frac{d^{N+M}\sigma}{d\omega_1 \ldots d\omega_N d\epsilon_1 \ldots d\epsilon_M}$$

Soft corrections

$$\frac{\partial \Sigma}{\partial \Delta}\left(z(\omega), \bar{E}, \Delta\right) = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv \exp\left(-iu\sqrt{s} - iv\Delta\right) \times$$

$$\exp\left(c\, Gs \int_{\bar{E}}^{\sqrt{s}} \frac{d\omega}{\omega} z(\omega) e^{i\omega u} + \tilde{c}\, Gs \int_{\lambda}^{\bar{E}} \frac{d\epsilon}{\epsilon} e^{i\epsilon(u+v)} - \tilde{c}\, Gs \int_{\lambda}^{\Lambda} \frac{d\epsilon}{\epsilon}\right)$$

complex saddle points for $u^* = ix/\sqrt{s}$, $v^* = iy/\Delta$
Using $\bar{E}, \Delta \ll \sqrt{s}$ one gets, $T_H = 1/R_S$

$$x = \frac{\sqrt{s}}{\bar{E}} W_0\left(\frac{\bar{E}}{T_H}\right),$$

$W_0(z) = \sum_{n=1}^{\infty}(-n)^{n-1} z^n/n! = z - z^2 + \ldots$ (first branch of the) Lambert (or product-log) function and

$$\hat{y} \equiv \frac{y}{Gs} = 1 - e^{-\beta\hat{y}} = 1 + \frac{W_0(-\beta e^{-\beta})}{\beta} \quad ; \quad \beta \equiv \frac{\bar{E}}{T_H}\frac{\sqrt{s}}{\Delta}$$

Two regimes

Upper cutoff on virtual gravitons > lower cutoff on real 'hard' gravitons: $\Lambda > \bar{\omega} = \bar{E}/W_0(\bar{E}/T_H)$. Set

$$\Phi \equiv \frac{\log \Sigma(z=1)}{Gs} = -\log\left(\frac{\Lambda\sqrt{s}}{T_H \Delta}\right) + \mathcal{E}_1(W_0(\bar{E}/T_H))$$

where $\mathcal{E}_1(v) = \int_v^\infty du u^{-1} e^{-u}$

- $\mathbf{\bar{E} \ll T_H}$: $\quad \Phi < -\log\left(\frac{\sqrt{s}}{\Delta}\right) - \log\left(\frac{\bar{E}}{T_H}\right) = -\log\beta \leq 0$

$$\frac{1}{\sigma}\frac{d\sigma}{d\omega} = \frac{Gs}{\omega} e^{-\frac{\omega}{T_H}} ,$$

- $\mathbf{\bar{E} \gg T_H}$:

$$\Phi < -\log\left[\frac{\bar{E}}{T_H \log(\frac{\bar{E}}{T_H})}\frac{\sqrt{s}}{\Delta}\right] - \frac{T_H/\bar{E}}{\log(\bar{E}/T_H)} < -\log\left[\frac{\bar{E}}{T_H \log(\frac{\bar{E}}{T_H})}\right] < 0$$

$$\frac{1}{\sigma}\frac{d\sigma}{d\omega} = \frac{Gs}{\omega} e^{-\frac{\omega}{\bar{E}} W_0(\bar{E}/T_H)} ,$$

Summary

- Bremsstrahlung at small ω, hard-graviton spectrum cut-off at large ω
- For $\bar{E} \gg T_H$ multi-jet cross section exponentially suppressed as $\exp(-Gs \log \bar{E}/T_H)$,
  hard-graviton cut off at $\bar{\omega} \sim \bar{E}/\log(\bar{E}/T_H) > T_H$.
- For $\bar{E} \ll T_H$ multi-jet cross section can be $\mathcal{O}(1)$,
  hard-graviton cutoff is T_H (Boltzmann) independent of $\bar{E}$.
- Taking $\Lambda \sim T_H$ and $\beta - 1 = \mathcal{O}(1)$, no exponential suppression. Very different from QCD. At $\bar{E} = T_H$ fraction of energy in quanta below T_H is $\mathcal{O}(1)$, similar to black-hole evaporation.
- Cutoff energy for soft gravitons: Δ/Gs.

4. Gravitational waves: ZFL ... and beyond

Zero Frequency Limit (ZFL)

IR divergent integral of $|\hat{S}_0|^2$ in $D = 4$ leads to bremsstrahlung

$$\frac{dN_0}{d\omega} = -\frac{2G}{\pi\hbar\omega}\sum_{i,j}(p_i p_j)\log\frac{|p_i p_j|}{\mu^2} \qquad : \qquad B_0 = \int_\lambda^\Lambda d\omega \frac{dN_0}{d\omega}$$

and to GW energy spectrum:

$$\frac{dE_0^{GW}}{d\omega} = \hbar\omega\frac{dN_0}{d\omega} = -\frac{2G}{\pi}\sum_{i,j}(p_i p_j)\log\frac{|p_i p_j|}{\mu^2} \ ,$$

constant ZFL of gravitational radiation [Weinberg]. At 4-points

$$\frac{dE^{GW}}{d\omega}(\omega = 0) = \frac{4G}{\pi}\left[s\log Gs + t\log(-Gt) + u\log(-Gu)\right]$$

In the small-t (deflection angle θ_s) limit

$$\frac{dE^{GW}}{d\omega}(\omega = 0) \to \frac{Gs}{\pi}\theta_s^2\log(4e\theta_s^{-2}) \ , \quad \theta_s = 4G\sqrt{s}/b \ ,$$

in agreement with recent classical and quantum calculations [Gruzinov, Veneziano; Ciafaloni, Colferai; Coraldeschi; ...].

Sub-leading corrections

Neglecting log-correction to $\hat{S}_1$ [Bern, Davies, Nohle; Laddha, Sen], $\hat{S}_0\hat{S}_1^* + h.c.$:

$$B_1 = 8\pi G \int \frac{d^3q}{2|q|(2\pi)^3} \sum_{i,j} \sum_{s=\pm 2} \left[\frac{(p_i h^s p_i)(p_j h^{(-s)} J_j q)}{qp_i qp_j} + (i \leftrightarrow j) \right] .$$

sum over h polarisations

$$B_1 = 8\pi G \int \frac{d^3q}{2|q|(2\pi)^3} \sum_{i,j} \frac{p_i p_j}{qp_i qp_j} [p_i \overrightarrow{J}_j + p_j \overleftarrow{J}_i] q$$

shift q to avoid collinear divergences for fixed i, j, set $P = p_1 + p_2$

$$K_{ij}^{\mu}(P, \Lambda) = \int \frac{d^4q}{(2\pi)^3} \frac{\delta_+(q^2)\delta((qP/\Lambda^2)+1)}{(qp_i)(qp_j)} [(p_i p_j)q - (qp_j)p_i - (qp_i)p_j]^{\mu}$$

finally yields soft-graviton distribution in the CM frame ($s = -P^2$)

$$\frac{dB_1}{d\omega} = -2\frac{G\sqrt{s}}{\pi} \sum_{ij} \frac{1}{\tilde{s}_{ij}} \log\left[\frac{-s(p_i p_j)}{2(Pp_i)(Pp_j)}\right] \Pi_{ij}^{\mu}[p_i \overrightarrow{J}_j + p_j \overleftarrow{J}_i]_{\mu} = \frac{dN_1}{d\omega}$$

where $\Pi_{ij}^{\mu} \equiv \left(P^{\mu} - \frac{Pp_j}{p_i p_j} p_i^{\mu} - \frac{Pp_i}{p_i p_j} p_j^{\mu} \right)$ and $\tilde{s}_{ij} = P^2 - 2\frac{(Pp_i)(Pp_j)}{p_i p_j}$

GW Energy spectrum at $O(\omega)$ in CM frame

$$\frac{dE_1^{GW}}{d\omega} = -2\frac{G\sqrt{s}\omega}{\pi}\sum_{ij}\frac{(p_i p_j)}{\tilde{s}_{ij}}\log\left[\frac{-s(p_i p_j)}{2(Pp_i)(Pp_j)}\right]\Pi^{\mu}_{ij}\left(\frac{\overleftarrow{\partial}}{\partial p_i}+\frac{\overrightarrow{\partial}}{\partial p_j}\right)_{\mu}$$

Consider 4-point scattering of massless scalars, for simplicity
Using Breit (brick-wall) frame and/or covariant prescription

$$s \to -\Delta_s^2 \quad \text{with} \quad \Delta_s = \tfrac{1}{2}(p_1 + p_2 - p_3 - p_4)$$

and similarly for t and u, get

$$\partial_1^{\mu} = -\Delta_s^{\mu}\partial_s - \Delta_t^{\mu}\partial_t - \Delta_u^{\mu}\partial_u$$

and similar for $\partial^{\mu}_{2,3,4}$ so that $p_i\partial_i = \frac{1}{2}(s\partial_s + t\partial_t + u\partial_u)$ for all i
and finally get

$$\frac{dB_1}{d\omega} = \frac{dE_1^{GW}}{d\omega} = 0$$

so far neglecting log corrections [Laddha, Sen] ... stay tuned

The $O(\omega^2)$ correction to the GW spectrum

$$B_2 = B_{02}+B_{20}+B_{11} = 8\pi G \int \frac{d^3 q}{2(2\pi)^3|q|} \sum_s \sum_{ij} [\hat{S}_0 \hat{S}_2^* + \hat{S}_2 \hat{S}_0^* + \hat{S}_1 \hat{S}_1^*]_{ij}\,,$$

where, using 'naive'/tree-level soft operators

$$[\hat{S}_0 \hat{S}_2^* + \hat{S}_2 \hat{S}_0^* + \hat{S}_1 \hat{S}_1^*]_{ij} = \frac{q^\mu q^\nu \mathcal{W}^{ij}_{\mu\nu}}{qp_i qp_j}$$

$$\mathcal{W}^{ij}_{\mu\nu} = -\frac{1}{2}[(\overrightarrow{J}_j p_i)_\mu (p_i \overrightarrow{J}_j)_\nu + (\overleftarrow{J}_i p_j)_\mu (p_j \overleftarrow{J}_i)_\nu] + \frac{1}{2}(\overleftarrow{J}_i p_i)_{(\mu}(p_j \overrightarrow{J}_j)_{\nu)} - (p_i p_j)(\overleftarrow{J}_i \overrightarrow{J}_j)_{(\mu\nu)} - (\overrightarrow{J}_j p_i)_{(\mu}(p_j \overleftarrow{J}_i)_{\nu)}$$

'shifted' integrals to avoid collinear divergences for fixed i, j

$$\widetilde{M}^{\mu\nu}_{ij} = \int \frac{d^3 q}{|q|} \sum_{i,j} \frac{\tilde{q}^\mu \tilde{q}^\nu}{qp_i qp_j} \delta\Big(\frac{qP}{\Lambda^2}+1\Big) = AP^{\mu\nu}_{ij} + BH^{\mu\nu}_{ij}$$

$$A = \frac{8\pi\omega_0^2}{p_i p_j \tilde{s}_{ij}} \left(1 + \frac{P^2}{\tilde{s}_{ij}} \log \frac{P^2 p_i p_j}{2Pp_i Pp_j}\right) \quad \textit{and} \quad B = \frac{4\pi\omega_0^2 P^2}{p_i p_j \tilde{s}_{ij}} \log \frac{P^2 p_i p_j}{2Pp_i Pp_j}$$

Un-improved B_2 for 4-pt massless scalar amplitudes

After tedious manipulations and cross-checks, get final expression

$$
\begin{aligned}
B_2 &= \frac{G\omega^2}{\pi}\sum_{ij}\left(C_1^{ij}+C_2^{ij}+C_3^{ij}\right)\\
C_1^{ij} &= 8\overleftarrow{D}_i\overrightarrow{D}_i-4(\overleftarrow{D}_i\overleftarrow{D}_i+\overrightarrow{D}_i\overrightarrow{D}_i)-3\overleftarrow{D}_i\overrightarrow{D}_j\\
C_2^{ij} &= \frac{P^2}{\tilde{s}_{ij}}\log\frac{P^2 p_i p_j}{2Pp_i Pp_j}[p_i p_j(\overleftrightarrow{\partial_{ij}})^2-2p_i(\overleftrightarrow{\partial_{ij}})p_j(\overleftrightarrow{\partial_{ij}})]\\
C_3^{ij} &= \frac{2}{p_i p_j \tilde{s}_{ij}}\left[1+\frac{P^2}{\tilde{s}_{ij}}\log\frac{P^2 p_i p_j}{2Pp_i Pp_j}\right](p_i p_j)^2\left(\Pi_{ij}^{\mu}(\overleftrightarrow{\partial_{ij}})_{\mu}\right)^2
\end{aligned}
$$

resulting in an $\mathcal{O}(\omega^2)$ correction to the GW energy spectrum

$$
\frac{dE_2^{GW}}{d\omega}=\hbar\omega\frac{dB_2}{d\omega}=\frac{4G\omega^2}{\pi|\mathcal{M}|^2}\times
$$

$$
\mathcal{M}^*\left\{\overleftarrow{D}^2+\overrightarrow{D}^2+[st+us\log\left(-\frac{u}{s}\right)]\overleftrightarrow{\Delta}^2_{st}+[su+ts\log\left(-\frac{t}{s}\right)]\overleftrightarrow{\Delta}^2_{su}\right\}\mathcal{M}
$$

Outlook

- Soft corrections to scattering amplitudes are interesting and important
- Gravity is better behaved than gauge theory in the IR: no collinear divergences
- Leading soft term $\hat{S}_0$ exact to all orders in a finite quantum theory of gravity as string theory. Sub- and sub-sub-leading more involved.
- Glimpses of 'BH/fuzzball' formation and evaporation ... but not yet there
- Sub-leading corrections to eikonal scattering (computed by ACV) ... the in-famous H-diagram ...
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and all started with Veneziano amplitude!