

Hagedorn spectrum and the thermodynamics of SU(N) Yang-Mills theories.¹

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M. Caselle, L. Castagnini, A Feo, F. Gliozzi, M.Panero, arXiv:1105.0359 <JHEP 06 (2011) 142

Summary:

- 1 Introduction and motivations
- 2 Lattice Thermodynamics
- 3 Glueball gas and closed string model
- 4 Results for $SU(2)$ and $SU(3)$ YM theories in $(3+1)$ dimensions
- 5 $SU(N)$ YM theories in $(2+1)$ dimensions
- 6 Conclusions

Introduction and motivations

One of the main features of $SU(N)$ non-abelian gauge theories is the existence of a deconfinement phase transition, i.e. a temperature above which gluons are “deconfined”, like the quark-gluon plasma (QGP) in Quantum Chromodynamics. Our goal is to study the thermodynamics of pure gauge theories in the confining phase when approaching the deconfinement transition

We choose to study the pure gauge sector of the theory, because it retains most of the non trivial features of the full theory, without the problems that the regularization of fermions on the lattice induces. This choice allows much faster and precise Montecarlo simulations and more importantly **In the confining phase the only degrees of freedom of the theory without quarks are the so-called “glueballs”**. Looking at the thermodynamics of the theory in the confining phase we have a tool to **explore the glueball spectrum of the theory**.

Introduction and motivations

Our main result is that the thermodynamics of the model can only be described assuming a string-like description of glueballs (and thus a Hagedorn spectrum)

The fine details of the spectrum agree remarkably well with the predictions of the Nambu-Goto effective string **This turns out to be an highly non trivial test of the effective string picture of confinement.**

This analysis was performed in the 3+1 dimensional SU(3) model in 2009 in the pioneering work of Meyer¹. Now, using the high precision lattice data for SU(3) of ² and a new set of data on (3+1) SU(2) that we obtained in ³, we are in the position to refine the effective string analysis and test its predictive power.

The present results confirm our previous findings⁴ for (2+1) dimensional SU(N) theories (with $N = 2, 3, 4, 5, 6$).

¹H. Meyer, *High-Precision Thermodynamics and Hagedorn Density of States*, 2009

²Sz. Borsanyi *et al.*, *Precision SU(3) lattice thermodynamics for a large temperature range*, 2012

³M. Caselle, A.Nada, M. Panero, *Hagedorn spectrum and thermodynamics of SU(N) Yang-Mills theories*, arXiv:1505.01106

⁴M. Caselle *et al.*, *Thermodynamics of SU(N) Yang-Mills theories in 2+1 dimensions I - The confining phase*, 2011

Lattice regularization

Only a truly non-perturbative approach such as lattice regularization can describe the deconfinement transition and the confined phase of non-abelian gauge theories.

For $SU(N)$ pure gauge theories on the lattice the dynamics are described by the standard Wilson action

$$S_W = \beta \sum_{p=sp, tp} \left(1 - \frac{1}{N} \text{ReTr} U_p\right)$$

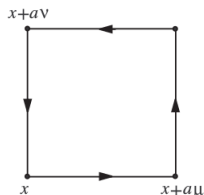
where U_P is the product of four U_μ $SU(N)$ variables on the space-like or time-like plaquette P and $\beta = \frac{2N}{g^2}$.

The partition function is

$$Z = \int \prod_{x, \mu} dU_\mu(x) e^{-S_W}$$

and the expectation value of an observable A

$$\langle A \rangle = \frac{1}{Z} \int \prod_{n, \mu} dU_\mu(n) A(U_\mu(n)) e^{-S_W}$$



Thermodynamic quantities

On a $N_t \times N_s^3$ lattice the volume is $V = (aN_s)^3$ (where a is the lattice spacing), while the temperature is determined by the inverse of the temporal extent (with periodic boundary conditions): $T = (aN_t)^{-1}$.



The thermodynamic quantities taken into account will be:

- the pressure p , that in the thermodynamic limit (i.e. for large and homogenous systems) can be written as

$$p \simeq \frac{T}{V} \log Z(T, V)$$

- the trace of the energy-momentum tensor Δ , that in units of T^4 is

$$\frac{\Delta}{T^4} = \frac{\epsilon - 3p}{T^4} = T \frac{\partial}{\partial T} \left(\frac{p}{T^4} \right)$$

Energy density $\epsilon = \Delta + 3p$ and entropy density $s = \frac{\epsilon + p}{T} = \frac{\Delta + 4p}{T}$ can be easily calculated.

Thermodynamics on the lattice

The pressure can be estimated by the means of the so-called “integral method”¹:

$$p(T) \simeq \frac{T}{V} \log Z(T, V) = \frac{1}{a^4} \frac{1}{N_t N_s^3} \int_0^{\beta(T)} d\beta' \frac{\partial \log Z}{\partial \beta'}.$$

It can be written (relative to its $T = 0$ vacuum contribution) as

$$\frac{p(T)}{T^4} = -N_t^4 \int_0^{\beta} d\beta' [3(P_\sigma + P_\tau) - 6P_0]$$

where P_σ and P_τ are the expectation values of spacelike and timelike plaquettes respectively and P_0 is the expectation value at zero T .

The trace of energy-momentum tensor is simply

$$\frac{\Delta(T)}{T^4} = T \frac{\partial}{\partial T} \left(\frac{p}{T^4} \right) = -N_t^4 T \frac{\partial \beta}{\partial T} [3(P_\sigma + P_\tau) - 6P_0].$$

ϵ and s can be obtained indirectly as linear combinations.

¹J. Engels *et al.*, *Nonperturbative thermodynamics of SU(N) gauge theories*, 1990

Ideal glueball gas

The behaviour of the system is supposed to be dominated by a gas of non-interacting glueballs.

The prediction of an ideal relativistic Bose gas can be used to describe the thermodynamics of such gas. Its partition function for 3 spatial dimensions is

$$\log Z = (2J + 1) \frac{2V}{T} \left(\frac{m^2}{2\pi} \right)^2 \sum_{k=1}^{\infty} \left(\frac{T}{km} \right)^2 K_2 \left(k \frac{m}{T} \right)$$

where m is the mass of the glueball, J is its spin and K_2 is the modified Bessel function of the second kind of index 2.

Observables such as Δ and p thus can be easily derived:

$$p = \frac{T}{V} \log Z = 2(2J + 1) \left(\frac{m^2}{2\pi} \right)^2 \sum_{k=1}^{\infty} \left(\frac{T}{km} \right)^2 K_2 \left(k \frac{m}{T} \right)$$

$$\Delta = \epsilon - 3p = 2(2J + 1) \left(\frac{m^2}{2\pi} \right)^2 \sum_{k=1}^{\infty} \left(\frac{T}{km} \right)^2 K_1 \left(k \frac{m}{T} \right)$$

Test with the SU(2) model

The SU(2) model is a perfect laboratory to test these results.

- It is easy to simulate: very precise results may be obtained with a reasonable amount of computing power
- The deconfinement transition is of second order and thus it is expected to coincide with the Hagedorn temperature
- The masses of several states of the glueball spectrum are known with remarkable accuracy
- The infrared physics of the model is very similar to that of the SU(3) theory, with one important difference: the gauge group is real and thus only C=1 glueball exist. The glueball spectrum contains only half of the states with respect to SU(3).

Lattice setup

N_s^4 at $T = 0$	$N_s^3 \times N_t$ at $T \neq 0$	n_β	β -range	n_{conf}
32^4	$60^3 \times 5$	17	[2.25, 2.3725]	1.5×10^5
40^4	$72^3 \times 6$	25	[2.3059, 2.431]	1.5×10^5
40^4	$72^3 \times 8$	12	[2.439, 2.5124]	10^5

Table : *

Setup of our simulations. The first two columns show the lattice sizes (in units of the lattice spacing a) for the $T = 0$ and finite-temperature simulations, respectively. In the third column, n_β denotes the number of β -values simulated within the β -range indicated in the fourth column. Finally, in the fifth column we report the cardinality n_{conf} of the configuration set for the $T = 0$ and finite- T simulations.

Scale setting

The SU(2) scale setting is fixed by calculating the string tension via the computation of Polyakov loop correlators with the multilevel algorithm.

The range of the parameter β which has been considered ($\beta \in [2.27, 2.6]$) covers approximately the temperature region analyzed in the finite temperature simulations.

The string tension is obtained with a two-parameter fit of potential

$$V = -\frac{1}{N_t} \log \langle PP \rangle$$

with the first order effective string prediction for the potential

$$V = \sigma r + V_0 - \frac{\pi}{12r}$$

Higher order effective string corrections turned out to be negligible within the precision of our data.

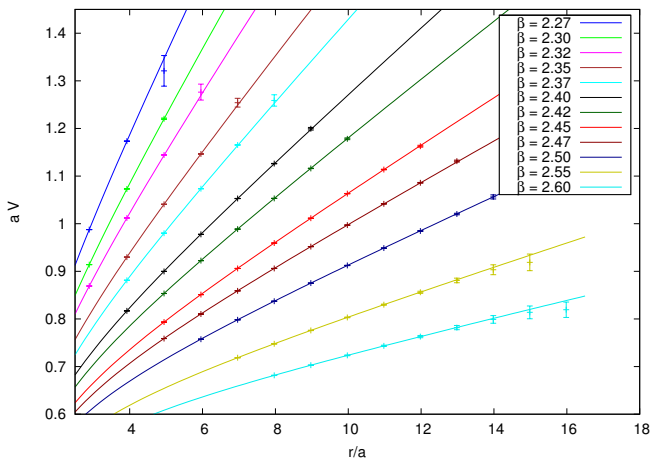
Scale setting

β	r_{\min}/a	σa^2	aV_0	χ_{red}^2
2.27	2.889	0.157(8)	0.626(14)	0.6
2.30	2.889	0.131(4)	0.627(30)	0.1
2.32	3.922	0.115(6)	0.627(32)	2.3
2.35	3.922	0.095(3)	0.623(20)	0.2
2.37	3.922	0.083(3)	0.621(18)	1.0
2.40	4.942	0.068(1)	0.617(10)	1.4
2.42	4.942	0.0593(4)	0.613(5)	0.1
2.45	4.942	0.0482(2)	0.608(4)	0.4
2.47	4.942	0.0420(4)	0.604(5)	0.3
2.50	5.954	0.0341(2)	0.599(2)	0.1
2.55	6.963	0.0243(13)	0.587(11)	0.2
2.60	7.967	0.0175(16)	0.575(16)	0.3

Table : *

Results for the string tension in units of the inverse squared lattice spacing at different values of the Wilson action parameter β (first column), calculated by fitting the potential V as a function of the tree-level improved interquark distance r to the Cornell form. V was extracted from Polyakov loop correlators on lattices of temporal extent $L_t = 32a$.

Scale setting

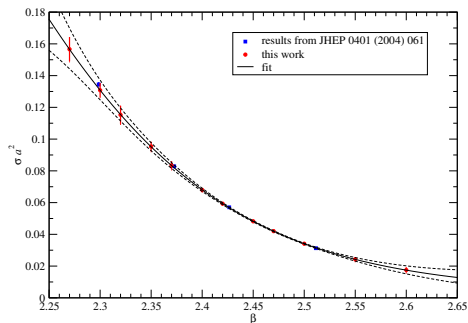


Scale setting

The values of the string tension are interpolated by a fit to

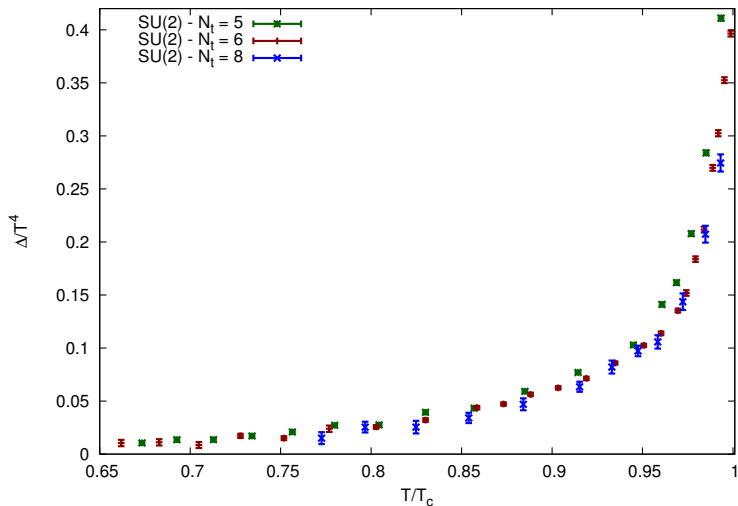
$$\log(\sigma a^2) = \sum_{j=0}^{n_{\text{par}}-1} a_j (\beta - \beta_0)^j \quad \text{with } \beta_0 = 2.35$$

which yields a χ_{red}^2 of 0.01. It is presented below along with older data¹.



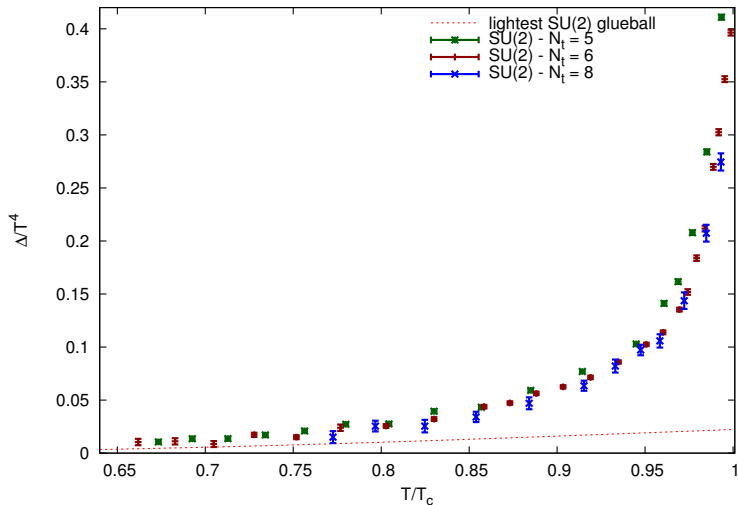
¹B. Lucini, M. Teper, U. Wenger, *The high temperature phase transition in $SU(N)$ gauge theories*, 2003

SU(2): trace of energy-momentum tensor



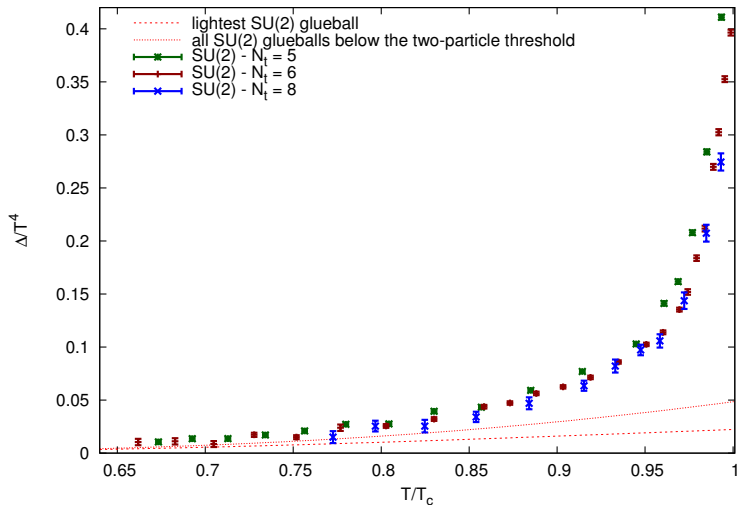
Despite the small values of N_t the data scale reasonably well.

SU(2): trace of energy-momentum tensor



Plot of the contribution of lowest glueball state $m_{0^{++}}$ compared with the data .

SU(2): trace of energy-momentum tensor



The contribution of all glueball states with mass $m < 2m_{0^{++}}$.

A few important observations

Usually the thermodynamics of the system is saturated by the first state (or, in some cases, the few lowest states) of the spectrum due to the exponential dependence on the mass.

The large gap between the $m_{0^{++}}$ and the $m < 2m_{0^{++}}$ curves and those between them and the data show that the spectrum must be of the **Hagedorn** type, i.e. that the number of states **increases exponentially** with the mass.

A Hagedorn spectrum is typically the signature of a string like origin of the spectrum.

The thermal behaviour of the model in the confining phase is thus a perfect laboratory to study **the nature of this spectrum and of the underlying string model**.

Effective string theory suggests that, with a very good approximation, this model should be a **Nambu-Goto** string. Let us see the consequences of this assumption.

Glueballs as rings of glue

A closed string model for the full glueball spectrum that follows the original work of Isgur and Paton¹² can be introduced to account for the values of thermodynamic quantities near the transition. In the closed-string approach glueballs are described in the limit of large masses as “**rings of glue**”, that is closed tubes of flux modelled by closed bosonic string states.

The **mass spectrum** of a closed strings gas in D spacetime dimensions is given by

$$m^2 = 4\pi\sigma \left(n_L + n_R - \frac{D-2}{12} \right)$$

where $n_L = n_R = n$ are the total contribution of left- and right-moving phonons on the string.

Every glueball state corresponds to a given phonon configuration, but associated to each fixed n there are multiple different states whose number is given by $\pi(n)$, i.e. the **partitions** of n .

¹N. Isgur and J. Paton, *A Flux Tube Model for Hadrons in QCD*, 1985

²R. Johnson and M. Teper, *String models of glueballs and the spectrum of $SU(N)$ gauge theories in $(2+1)$ -dimensions*, 2002

The **density of states** $\rho(n)$ is expressed through the square of $\pi(n)$:

$$\rho(n) = \pi(n_L)\pi(n_R) = \pi(n)^2 \simeq 12(D-2)^{\frac{D-1}{2}} \left(\frac{1}{24n}\right)^{\frac{D+1}{2}} \exp\left(2\pi\sqrt{\frac{2(D-2)n}{3}}\right)$$

in D spacetime dimensions.

Spectral density

The **Hagedorn temperature**¹ is defined as

$$T_H = \sqrt{\frac{3\sigma}{\pi(D-2)}}$$

The spectral density as a function of the mass (i.e. $\hat{\rho}(m)dm = \rho(n)dn$) can be expressed as

$$\hat{\rho}(m) = \frac{(D-2)^{D-1}}{m} \left(\frac{\pi T_H}{3m}\right)^{D-1} e^{m/T_H}$$

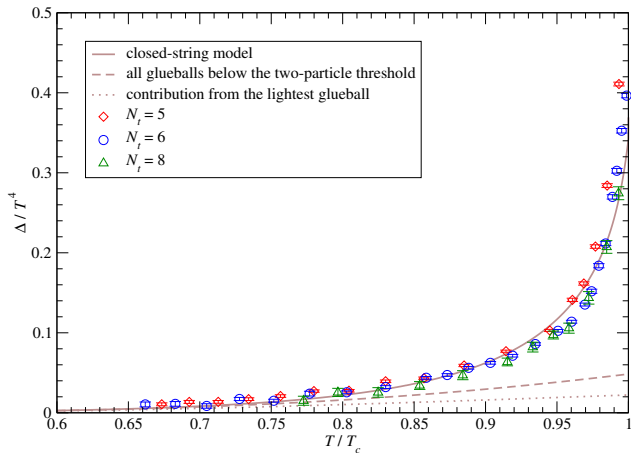
where the characteristic Hagedorn-like exponential spectrum appears and can be used to describe the glueball spectrum above an arbitrary mass threshold.

The trace of the energy-stress tensor can be integrated on masses bigger than $2m_{0++}$ with the degeneracy $\hat{\rho}(m)$ and summed to the contribution of the mass states computed on the lattice.

$$\Delta = \sum_{m < 2m_{0++}} (2J+1)\Delta(m, T) + \int_{2m_{0++}}^{\infty} dm \hat{\rho}(m) \Delta(m, T)$$

¹R. Hagedorn, Nuovo Cim. Suppl. 3, 147 (1965)

SU(2): trace of energy-momentum tensor



SU(2) vs. SU(3)

The SU(3) case was studied for the first time in 2009 in the pioneering work of Meyer¹. Now, using the high precision lattice data for SU(3) of ² we are in the position to test the Hagedorn behaviour in a very stringent way.

There are two main differences between SU(2) and SU(3):

- SU(3) has a first order deconfining transition, so $T_c < T_H$.
- SU(3) has complex representations, thus glueballs have an additional quantum number C and the glueball spectrum contains twice the number of glueballs than in the SU(2) case

In principle we could consider in this case T_H as a free parameter, but in the effective string framework we may safely fix it at the expected Nambu-Goto value

$T_H = \sqrt{3\sigma/2\pi} = 0.691 \dots \sqrt{\sigma}$. Lorentz invariance of the effective string tells us that this should be a very good approximation of the exact result and that we should expect only small deviations from this value.

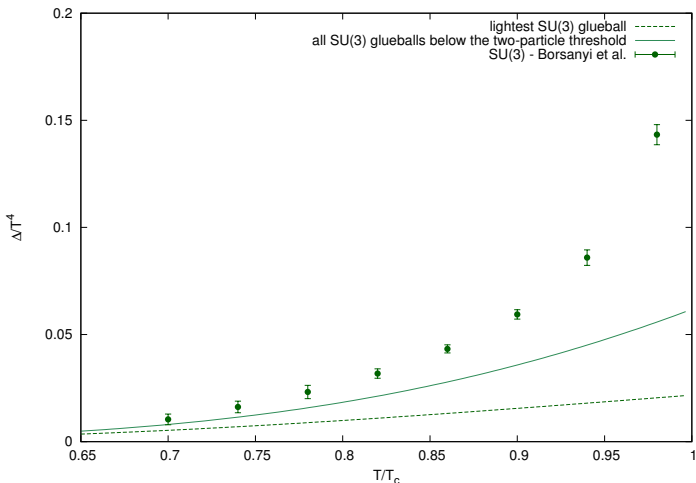
The relation between T_H and T_c is:

$$\frac{T_H}{T_c} = 1.098$$

¹H. Meyer, *High-Precision Thermodynamics and Hagedorn Density of States*, 2009

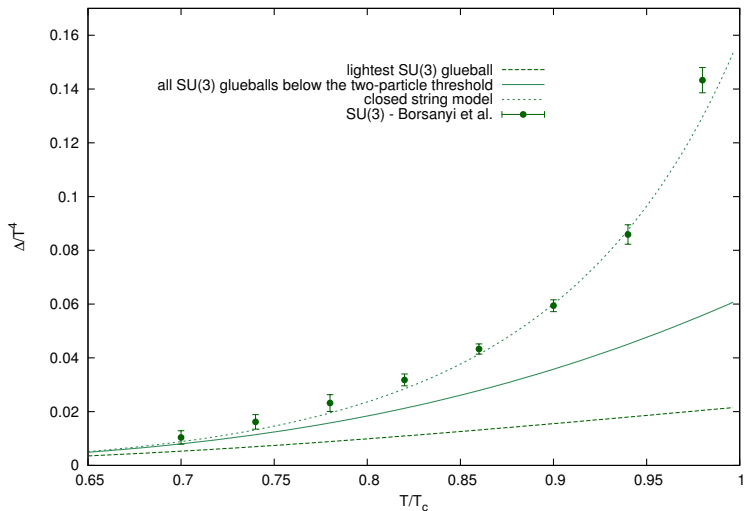
²Sz. Borsanyi et al., *Precision SU(3) lattice thermodynamics for a large temperature range*, 2012

SU(3): trace of energy-momentum tensor



Also in this case the $m < 2m_{0^{++}}$ sector of the glueball spectrum is not enough to fit the behaviour of Δ/T^4 .

SU(3): trace of energy-momentum tensor



While including the whole Hagedorn spectrum we find again a remarkable agreement with no free parameter!

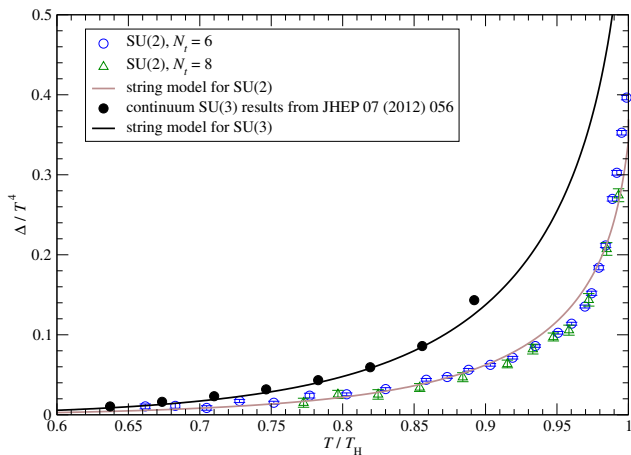
SU(2) vs. SU(3)

It is instructive to compare the SU(2) and SU(3) data

For $N = 3$ the closed flux tube has two possible orientations that account for the $\mathcal{C} = +1 / -1$ sectors. Thus a further twofold degeneracy must be introduced in the string spectrum.

This doubling of the Hagedorn spectrum is clearly visible in the data

SU(2) vs. SU(3): results for trace of energy-momentum tensor



SU(N) Yang-Mills theories in (2 + 1) dimensions

The same picture is confirmed by a study we performed a few years ago¹ in (2+1) dimensional SU(N) Yang-Mills theories for $N = 2, 3, 4, 6$.

Also in (2+1) dimensions we found that:

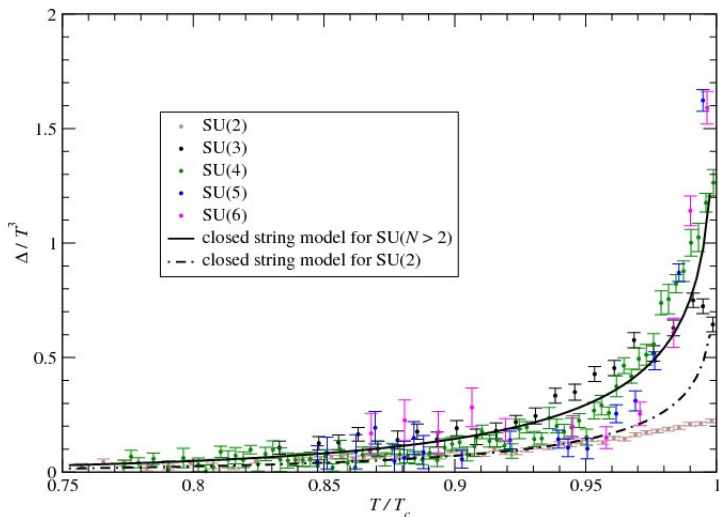
- a Hagedorn spectrum was mandatory to fit the thermodynamic data
- there was a jump between the SU(2) and the SU($N > 2$) case due to the doubling of the spectrum
- we had to fix the Hagedorn temperature to the Nambu-Goto value which, due to the different number of transverse degrees of freedom is different from the (3+1) dimensional one: $T_H = \sqrt{3\sigma/\pi} = 0.977.. \sqrt{\sigma}$

Moreover we found that in the vicinity of the critical point there was an excess of Δ/T^4 with respect to our predictions for $N = 4, 5$ and 6 and that this excess increases with N . This could be understood as due to the k -string glueballs

¹M. Caselle et al., *Thermodynamics of SU(N) Yang-Mills theories in 2+1 dimensions I - The confining phase*, 2011

SU(N) Yang-Mills theories in (2 + 1) dimensions

Trace of the energy-momentum tensor and string model



Universality of the effective string action.

Why the Nambu-Goto string works so well?

- The main reason is that the Effective String action is strongly constrained by Lorentz invariance. **The first few orders of the action are universal and coincide with those of the Nambu-Goto action.**
- The terms with only first derivatives coincide with the Nambu-Goto action to all orders in the derivative expansion.
- The first allowed correction to the Nambu-Goto appears at a very large order and, in $SU(N)$ gauge models gives a contribution to the interquark potential of the order $1/R^5$ in $d = 3 + 1$ and $1/R^7$ in $d = (2 + 1)$ which are almost negligible^{1 2 3}.
- **The effective string action is much more predictive than typical effective models in particle physics!**
- The fact that the first deviations from the Nambu-Goto string are of such high order explains why it works so well both in describing the interquark potential and the **glueball spectrum**

¹M. Luscher and P. Weisz JHEP07(2004)014

²H. B. Meyer JHEP05(2006)066

³O. Aharony and M. Field JHEP01(2011)065

Conclusions

- The thermodynamics of $SU(2)$ and $SU(3)$ Yang-Mills theories in $d = (3 + 1)$ is well described by a gas of non-interacting glueballs
- The agreement is obtained only assuming a Hagedorn spectrum for the glueballs
- The fine details of the spectrum, in particular the Hagedorn temperature, agree well with the predictions of the Nambu-Goto effective string.
- The special role played by the Nambu-Goto string can be understood in the framework of the effective string approach to the infrared regime of confining gauge theories and is a direct consequence of the non-linear realization of the Lorentz invariance of these theories.
- The results agree with previous findings in $d = (2 + 1)$ $SU(N)$ Yang Mills theories with $N = 2, 3, 4, 5, 6$
- As N increases the data suggest the presence of extra states in the spectrum which could be k-glueballs states, which could be described by a k-string spectrum
- Overall the behaviour of thermodynamic observables in the confining regime of Yang-Mills theories turns out to be an highly non trivial test of the effective string picture of confinement.

Acknowledgements

Collaborators:

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