Characterising Exotic Matter Driving Wormholes[1]

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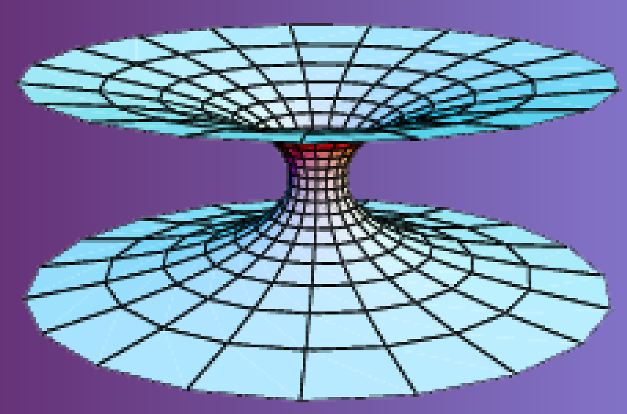
Abstract

We develop an iterative approach to span the whole set of exotic matter models able to drive a traversable wormhole. The method, based on a Taylor expansion of metric and stress-energy tensor components in a neighbourhood of the wormhole throat, reduces the Einstein equation to an infinite set of algebraic conditions, which can be satisfied order by order. The approach easily allows the implementation of further conditions linking the stress-energy tensor components among each other, like symmetry conditions or equations of state. The method is then applied to some relevant examples of exotic matter characterized by a constant energy density and that also show an isotropic behaviour in the stress-energy tensor or obeying to a quintessence-like equation of state.

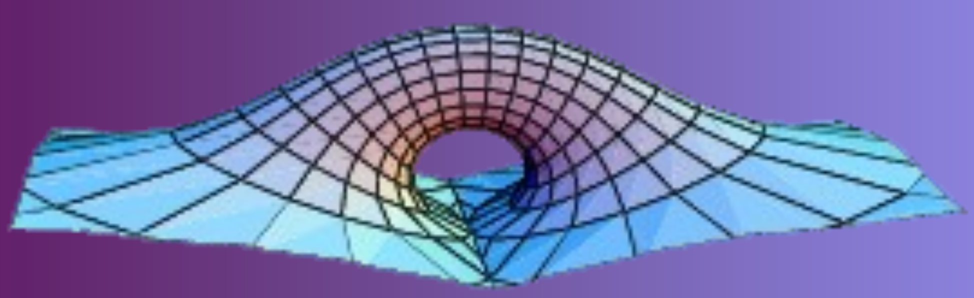
Introduction

Wormholes

Wormhole solutions of Einstein equations are handles connecting two regions of the space-time manifold. They can be seen as short-cuts linking together two distant places in the same universe or even a bridge between two different universes



Inter-Universe Wormhole[2]



Intra-Universe Wormhole[2]

Traversable Wormhole[3]

G.R. Compatible <=> Einstein's Equations

$$G_{\hat{\mu}\hat{\nu}} = 8\pi T_{\hat{\mu}\hat{\nu}}.$$

Einstein's

$G_{\mu\nu}$ =Einstein's Tensor

$T_{\mu\nu}$ =Stress-Energy Tensor

Two-Way Travel <=> Horizon Free

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\Phi(r)}dt^2 + \left(1 - \frac{b(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Horizon-free, static and spherically symmetric metric

$\phi(r)$ =Redshift Function

$b(r)$ =Redshift Function

$$b'(r) < \frac{b(r)}{r}$$

$$\tau(r) > \rho(r)$$

Bridge

Flaring Out
(Metric Condition)
Einstein Tensor

Exotic Matter
(NEC violation)
Stress-Energy

Our Method

- **Series Expansion of Relevant Quantities**
- **Algebraic Conditions Order by Order**
- **General Source Candidate Compatible**

Implementation

Exotic Matter near the Wormhole Throat

We adopt the "*proper reference frame*" associated with an observer at rest in the Schwarzschild frame (the coordinates so introduced are denoted by an hat). In this frame $G_{\mu\nu}$ and $T_{\mu\nu}$ are both diagonal.

$$G_{\hat{t}\hat{t}} = \frac{b'}{r^2},$$
$$G_{\hat{r}\hat{r}} = -\frac{b}{r^3} + 2\left(1 - \frac{b}{r}\right)\frac{\Phi'}{r},$$
$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \left(1 - \frac{b}{r}\right)\left[\Phi'' - \frac{b'r - b}{2r(r-b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r - b}{2r^2(r-b)}\right].$$

Nonvanishing components of Einstein Tensor

We perform a series expansion in a neighbourhood of the wormhole's throat, where $b(r)=r$. Then from Einstein's equations we obtain, order by order in the relative distance from the throat, a set of algebraic condition

$$\bar{b}_1 = \bar{\rho}_0 < 1,$$
$$\bar{\tau}_0 = 1,$$
$$\bar{\Phi}_1 = \frac{2\bar{p}_0}{1-\bar{\rho}_0} - 1.$$

Zero-th Order

$$\bar{b}_2 = 2\bar{\rho}_0 + \bar{\rho}_1,$$
$$\bar{\tau}_1 = -(4\bar{p}_0 + \bar{\rho}_0 + 1),$$
$$\bar{\Phi}_2 = \frac{6\bar{p}_0 + 2\bar{p}_1}{3(1-\bar{\rho}_0)} + \frac{2\bar{p}_0(2\bar{\rho}_0 + \bar{\rho}_1)}{3(1-\bar{\rho}_0)^2} - \frac{1}{3}\left(\frac{2\bar{p}_0}{1-\bar{\rho}_0} - 1\right)\left(\frac{4\bar{p}_0}{1-\bar{\rho}_0} + 1\right).$$

First Order Conditions

$$\bar{b}_3 = 2\bar{\rho}_0 + 4\bar{\rho}_1 + \bar{\rho}_2,$$
$$\bar{\tau}_2 = -4(1-\bar{\rho}_0)(\bar{\Phi}_1 + \bar{\Phi}_2) + \bar{b}_2(1 + 2\bar{\Phi}_1) - 6(1 + \bar{\tau}_1),$$
$$\bar{\Phi}_3 = \left[2\bar{p}_2 + \bar{b}_2(2\bar{\Phi}_1^2 + 5\bar{\Phi}_1 + 4\bar{\Phi}_2 + 1) + \bar{b}_3(1 + \bar{\Phi}_1) + 12(\bar{p}_0 + \bar{p}_1) + - (1-\bar{\rho}_0)(8\bar{\Phi}_1^2 + 8\bar{\Phi}_1\bar{\Phi}_2 + 4\bar{\Phi}_1 + 14\bar{\Phi}_2)\right]\frac{1}{5(1-\bar{\rho}_0)}$$

Second Order Conditions

The set of equations further simplifies if one considers additional constraints linking each other the components of the stress-energy tensor; for instance a generic equation $F(\rho; p; \tau)=0$. This is the case of symmetry conditions or equations of state.

$$F_0 = F(\bar{\rho}_0, \bar{p}_0, \bar{\tau}_0) = 0.$$

Zero-th Order

$$F_1 = \frac{1}{\bar{r}_0^2}\left[\left.\frac{\partial F}{\partial \bar{\rho}}\right|_{\bar{r}_0}\bar{\rho}_1 + \left.\frac{\partial F}{\partial \bar{p}}\right|_{\bar{r}_0}\bar{p}_1 + \left.\frac{\partial F}{\partial \bar{\tau}}\right|_{\bar{r}_0}\bar{\tau}_1\right] = 0$$

First Order Constraint

$$F_2 = \frac{1}{\bar{r}_0^4}\left[\left.\frac{\partial^2 F}{\partial \bar{\rho}^2}\right|_{\bar{r}_0}\bar{\rho}_1^2 + \left.\frac{\partial^2 F}{\partial \bar{p}^2}\right|_{\bar{r}_0}\bar{p}_1^2 + \left.\frac{\partial^2 F}{\partial \bar{\tau}^2}\right|_{\bar{r}_0}\bar{\tau}_1^2 + +2\left(\left.\frac{\partial^2 F}{\partial \bar{\rho}\,\partial \bar{p}}\right|_{\bar{r}_0}\bar{\rho}_1\bar{p}_1 + \left.\frac{\partial^2 F}{\partial \bar{\rho}\,\partial \bar{\tau}}\right|_{\bar{r}_0}\bar{\rho}_1\bar{\tau}_1 + \left.\frac{\partial F}{\partial \bar{p}\,\partial \bar{\tau}}\right|_{\bar{r}_0}\bar{p}_1\bar{\tau}_1\right) + + \left.\frac{\partial F}{\partial \bar{\rho}}\right|_{\bar{r}_0}\bar{\rho}_2 + \left.\frac{\partial F}{\partial \bar{p}}\right|_{\bar{r}_0}\bar{p}_2 + \left.\frac{\partial F}{\partial \bar{\tau}}\right|_{\bar{r}_0}\bar{\tau}_2\right] = 0.$$

Second Order Constraint

Application

Asymptotic Vacuum and Transition Layer

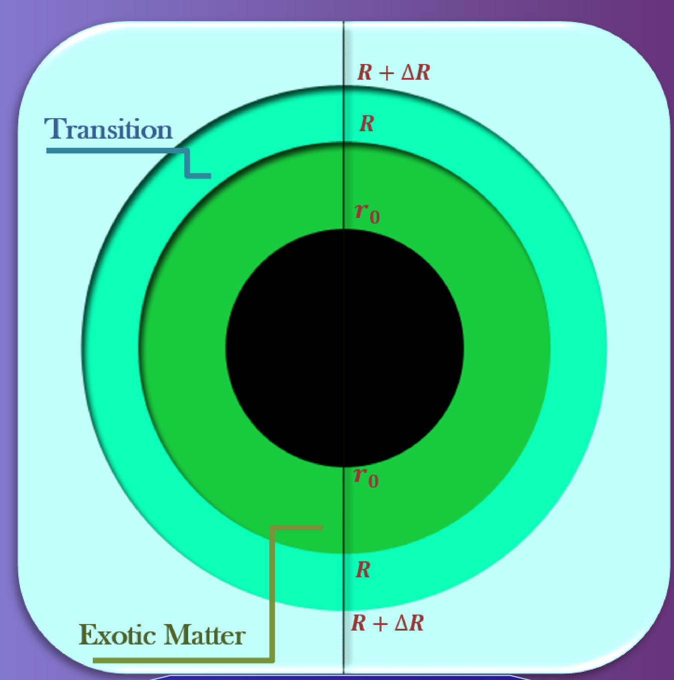
Far away from the wormhole throat, the metric must approach the Schwarzschild's vacuum solution. We bound the exotic matter into a finite spherical region of radius R.

Junction Shell of Non-Exotic Material

- Spherically Symmetric
- Radius R
- Thickness ΔR
- Constant energy density ρ
- Constant transverse pressure p
- Linear decreasing τ

$$\bar{b}(\bar{r}) = \bar{b}(\bar{R}) + \left(\frac{\bar{r} - \bar{R}}{\Delta\bar{R}}\right)\bar{b}(\bar{R}),$$
$$\bar{\rho}(\bar{r}) = \left(\frac{\bar{R}}{\Delta\bar{R}}\right)\bar{\tau}(\bar{R}),$$
$$\bar{p}(\bar{r}) = \left(\frac{\bar{R}}{2\Delta\bar{R}}\right)\bar{\tau}(\bar{R}).$$
$$\bar{\phi}'(\bar{r}) = \left(\frac{\bar{r} - \bar{R}}{\Delta\bar{R}}\right)\frac{\bar{b}(\bar{R})}{\bar{R}^2},$$
$$\bar{\tau}(\bar{r}) = \bar{\tau}(\bar{R}) - \left(\frac{\bar{r} - \bar{R}}{\Delta\bar{R}}\right)\bar{\tau}(\bar{R}),$$

Transition Layer



Matter Distribution

$$\bar{M} = \int_{\bar{r}_0}^{\bar{R}}\bar{\rho}(\bar{r})\bar{r}^2 d\bar{r} + \bar{r}_0.$$

Schwartzchild's Mass

$$\bar{\Phi}_0 = \frac{1}{2}\ln\left(1 - \frac{2\bar{M}}{\bar{R} + \Delta\bar{R}}\right) - \frac{\Delta\bar{R}}{2}\frac{\bar{b}(\bar{R})}{\bar{R}^2} - \sum_{n=1}^{\infty}\frac{\bar{\Phi}_n}{n!}\left(\frac{\bar{R} - \bar{r}_0}{\bar{r}_0}\right)^n.$$

Redshift at the Wormhole Throat

Sphere of Isotropic Exotic Material with Constant Energy Density

$$\bar{b}(\bar{r}) = \bar{r}_0\left\{1 + \frac{\bar{\rho}_0}{3}\left[\left(\frac{\bar{r}}{\bar{r}_0}\right)^3 - 1\right]\right\},$$
$$\bar{\Phi}(\bar{r}) = \bar{\Phi}_0 - \frac{3-\bar{\rho}_0}{1-\bar{\rho}_0}\left(\frac{\bar{r}-\bar{r}_0}{\bar{r}_0}\right) - \frac{(7-\bar{\rho}_0)(3-\bar{\rho}_0)}{6(1-\bar{\rho}_0)^2}\left(\frac{\bar{r}-\bar{r}_0}{\bar{r}_0}\right)^2 + -\frac{(3-\bar{\rho}_0)(7-4\bar{\rho}_0+\bar{\rho}_0^2)}{3(1-\bar{\rho}_0)^3}\left(\frac{\bar{r}-\bar{r}_0}{\bar{r}_0}\right)^3 + \mathcal{O}\left(\left(\frac{\bar{r}-\bar{r}_0}{\bar{r}_0}\right)^4\right),$$
$$\bar{\tau}(\bar{r}) = \frac{1}{\bar{r}_0^2}\left[1 + (3-\bar{\rho}_0)\left(\frac{\bar{r}-\bar{r}_0}{\bar{r}_0}\right) + \frac{2(12-7\bar{\rho}_0+\bar{\rho}_0^2)}{3(1-\bar{\rho}_0)}\left(\frac{\bar{r}-\bar{r}_0}{\bar{r}_0}\right)^2 + \mathcal{O}\left(\left(\frac{\bar{r}-\bar{r}_0}{\bar{r}_0}\right)^3\right)\right],$$
$$\bar{p}(\bar{r}) = -\bar{\tau}(\bar{r}).$$

Sphere of Isotropic Exotic Matter with Constant Mass

$$\bar{M} = \bar{r}_0\left\{1 + \frac{\bar{\rho}_0}{3}\left[\left(\frac{\bar{R}}{\bar{r}_0}\right)^3 - 1\right]\right\}$$

Mass

$$\bar{M}_{\max} = \bar{r}_0\left\{1 + \frac{\bar{\rho}_0}{3}\left[\frac{1}{8}\left(\sqrt{\frac{12}{\bar{\rho}_0} - 3} - 1\right)^3 - 1\right]\right\} \quad \text{for} \quad 0 < \bar{\rho}_0 < 1$$

Maximum Mass

[1] M.Chianese, E. Di Grezia, M. Manfredonia and G. Miele, "*Characterising exotic matter driving wormholes*", arXiv:1701.08770

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[3] M. S. Morris and K. S. Thorne, "*Wormholes in space-time and their use for interstellar travel: A tool for teaching generalrelativity*", Am. J. Phys., vol. 56, pp. 395-412, 1988.

[1] [2] [3]