Extracting the light quark masses from $\eta \rightarrow 3\pi$: A dispersive approach

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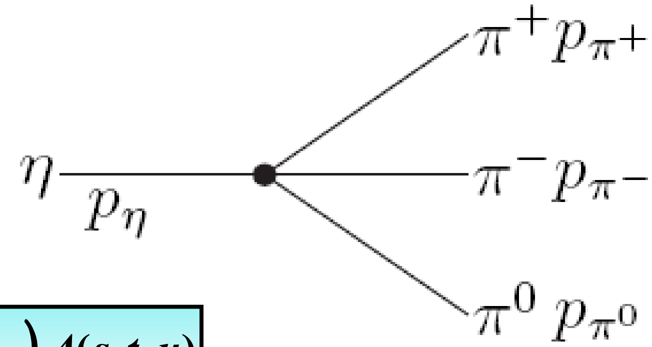
Outline :

1. $\eta \rightarrow 3\pi$ decays
2. Dispersive analysis
3. Preliminary results
4. Conclusion and outlook

1. $\eta \to 3\pi$ decays

1.1 Definitions



- η decay: $\eta \to \pi^+ \pi^- \pi^0$

$$\boxed{\langle \pi^+ \pi^- \pi^0{}_{out} | \eta \rangle = i(2\pi)^4 \delta^4\left(p_\eta - p_{\pi^+} - p_{\pi^-} - p_{\pi^0}\right) A(s,t,u)}$$

- Mandelstam variables $s = \left(p_{\pi^+} + p_{\pi^-}\right)^2, \;\; t = \left(p_{\pi^-} + p_{\pi^0}\right)^2, \;\; u = \left(p_{\pi^0} + p_{\pi^+}\right)^2$

$$s + t + u = M_\eta^2 + M_{\pi^0}^2 + 2M_{\pi^+}^2 \equiv 3s_0$$

$\Rightarrow$ only two independent variables

- Neutral channel: $\eta \to \pi^0 \pi^0 \pi^0$:

$$\boxed{\overline{A}(s,t,u) = A(s,t,u) + A(t,u,s) + A(u,s,t)}$$

1.2 Why is it interesting to study $\eta \to 3\pi$?

- Decay forbidden by isospin symmetry

$$\Rightarrow \boxed{A = (m_u - m_d) A_1 + \alpha_{em} A_2}$$

- α_{em} effects are small

Sutherland'66, Bell & Sutherland'68
Baur, Kambor, Wyler'96, Ditsche, Kubis, Meissner'09

- Decay rate measures the size of isospin breaking ($m_u - m_d$) in the SM:

$$L_{QCD} \to \boxed{L_{IB} = -\frac{m_u - m_d}{2}\left(\bar{u}u - \bar{d}d\right)}$$

$\Rightarrow$ Clean access to ($m_u - m_d$)

1.3 Quark mass ratios

- Instead of (m_u-m_d) extract Q: $\boxed{Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}}$

  Does not receive any correction at NLO!

- Mass formulae to second chiral order *Gasser & Leutwyler'85*

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s + \hat{m}}{2\hat{m}}\left[1 + \Delta_M + \mathcal{O}(m^2)\right]$$

$$\frac{M_{K^0}^2 - M_{K^+}^2}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - \hat{m}}\left[1 + \Delta_M + \mathcal{O}(m^2)\right] \qquad \left[\hat{m} \equiv \frac{m_d + m_u}{2}\right]$$

$$\text{with} \quad \Delta_M = \frac{8(M_K^2 - M_\pi^2)}{F_\pi^2}(2L_8 - L_5) + \chi\text{-logs}$$

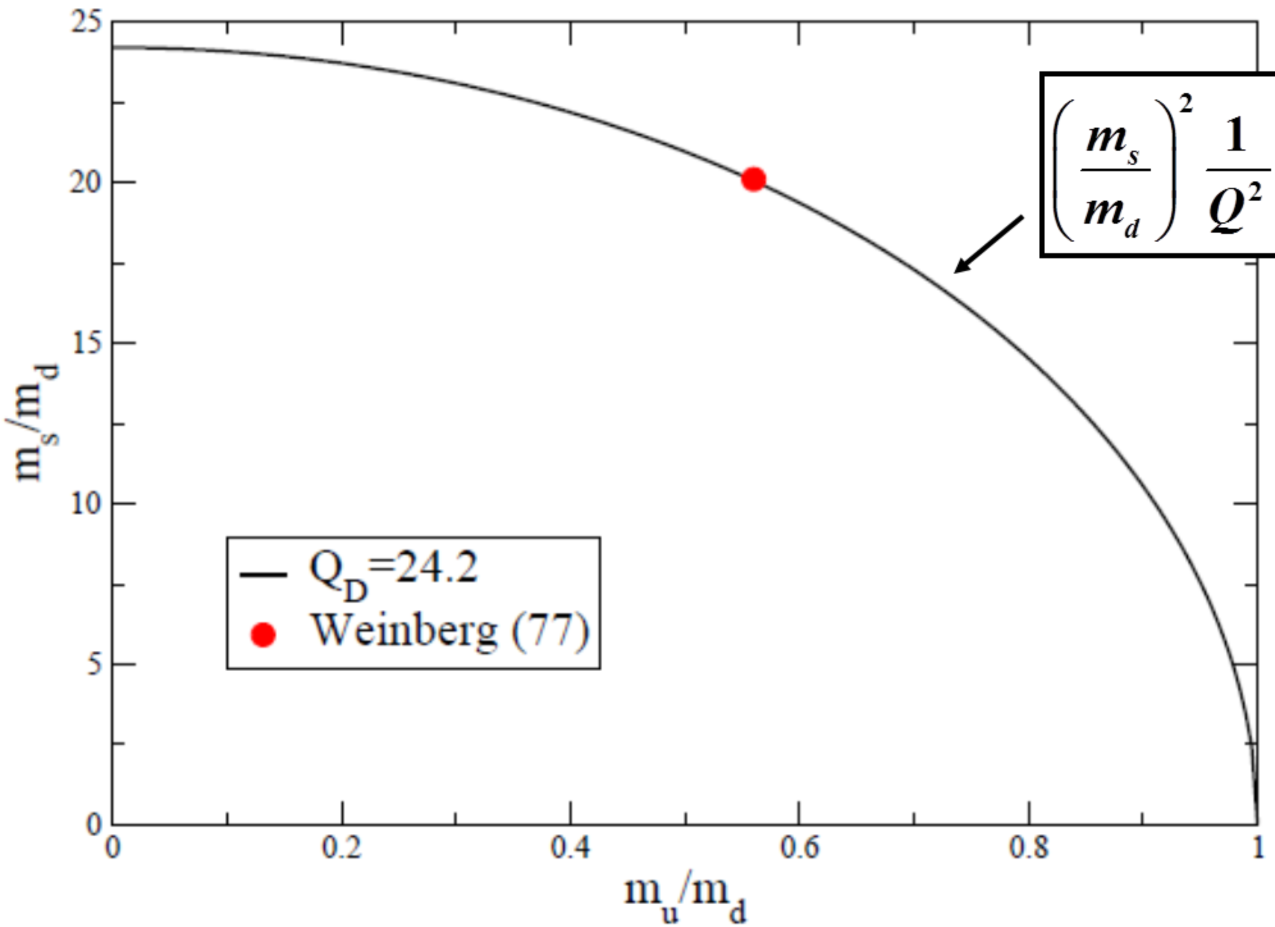
- The same O(m) correction appears in both ratios

  ⇨ Take the double ratio

$$\boxed{Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}} = \frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{\left(M_{K^0}^2 - M_{K^+}^2\right)_{QCD}}\left[1 + O(m_q^2, e^2)\right]$$

1.3 Quark mass ratios

- From Q ⇨ Ellipse in the plane m_s/m_d, m_u/m_d *Leutwyler's ellipse*



$$\left(\frac{m_s}{m_d}\right)^2 \frac{1}{Q^2} + \left(\frac{m_u}{m_d}\right)^2 = 1$$

1.3 Quark mass ratios

- Use Q to determine m_u and m_d from lattice determinations of m_s and $\hat{m}$

$$\boxed{m_u = \hat{m} - \frac{m_s^2 - \hat{m}^2}{4\hat{m}Q^2}} \quad \text{and} \quad \boxed{m_d = \hat{m} + \frac{m_s^2 - \hat{m}^2}{4\hat{m}Q^2}}$$

- From lattice determinations of m_s and $\hat{m}$ + Q

⇨ *Light quark masses: m_u, m_d, m_s*

1.4 Q from $\eta \to 3\pi$

- $$A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2}M(s,t,u)$$

$$\Rightarrow \quad \Gamma_{\eta\to 3\pi} \propto \int \left| A(s,t,u) \right|^2 \propto Q^{-4}$$

- In the following, compute the normalized amplitude M(s,t,u) with the best accuracy ⇒ *extraction of Q*

1.5 Dispersive approach

- Slow convergence of the chiral series

$$\boxed{\Gamma_{\eta\to3\pi} = (\underbrace{66}_{\text{LO}} + \underbrace{94}_{\text{NLO}} + \underbrace{\dots}_{\text{NNLO}})\,\text{eV} = (295 \pm 20)\,\text{eV}}$$

PDG'12

LO: *Osborn, Wallace'70*

NLO: *Gasser & Leutwyler'85*

NNLO: *Bijnens & Ghorbani'07*

- Large $\pi\pi$ final state interactions

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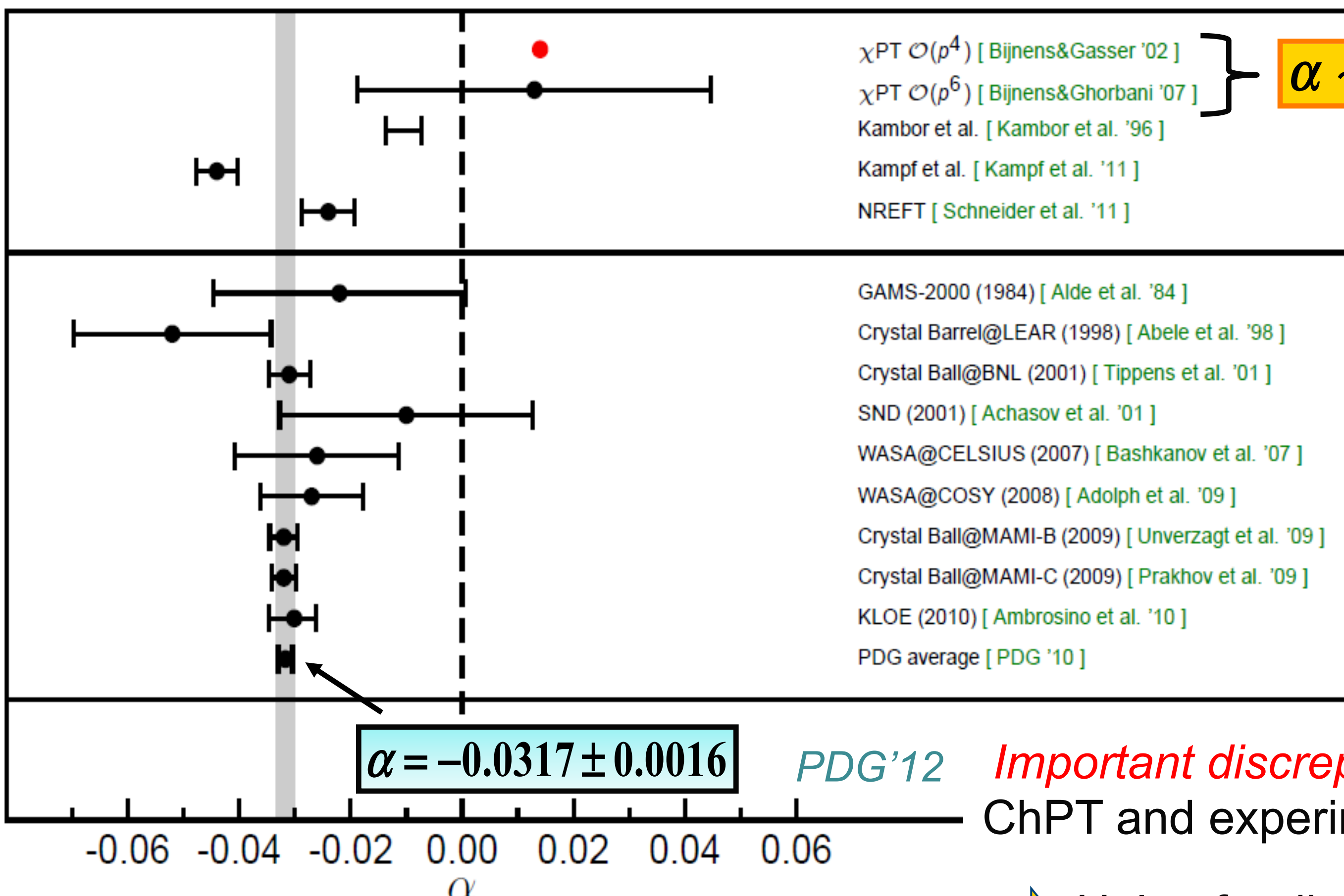
- Large $\pi\pi$ final state interactions

- Important discrepancy between ChPT and experiment in the neutral channel

Neutral Channel : $\eta \rightarrow \pi^0 \pi^0 \pi^0$

- Decay amplitude $\boxed{\Gamma_{\eta\to3\pi} \propto |\bar{A}|^2 \propto 1 + 2\alpha Z}$ with $Z = \frac{2}{3}\sum_{i=1}^{3}\left(\frac{3T_i}{Q_n} - 1\right)^2$

$Q_n \equiv M_\eta - 3M_{\pi^0}$



$\boxed{\alpha \sim 0.015}$

PDG'12

Important discrepancy between ChPT and experiment!

➡ Help of a dispersive treatment?

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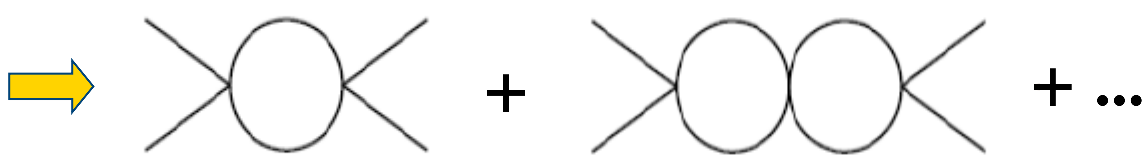
- Use of dispersion relations :
  - ➢ analyticity, unitarity and crossing symmetry
  - ➢ Take into account all the rescattering effects

Kambor, Wiesendanger & Wyler'96
Anisovich & Leutwyler'96



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- New dispersive analysis:
  - New inputs available: extraction $\pi\pi$ phase shifts has improved

    Ananthanarayan et al'01, Colangelo et al'01
    Descotes-Genon et al'01
    Kaminsky et al'01, Garcia-Martin et al'09

  - New experimental programs, precise Dalitz plot measurements

    CBall-Brookhaven, CLAS (JLab), KLOE (Frascati)
    TAPS/CBall-MAMI (Mainz), WASA-Celsius (Uppsala), WASA-Cosy (Juelich)

  - Possible combination with NNLO calculation *Bijnens & Ghorbani'07*

  - Electromagnetic effects: complete analysis of $O(e^2m)$ effects

    Ditsche, Kubis, Meissner'09

  - Isospin breaking effects: new techniques ⇨ NREFT

    Gullstrom, Kupsc, Rusetsky'09, Schneider, Kubis, Ditsche'11

2. Dispersive Analysis of $\eta \rightarrow \pi^{+} \pi^{-} \pi^{0}$ decays

2.1 Method: Representation of the amplitude

- Dispersion relations

$$\mathcal{A}^n_{\eta\to 3\pi} = \text{subtraction polynomial} + \int \text{disc}\, \mathcal{A}^n_{\eta\to 3\pi}$$

- From the discontinuity, reconstruct the amplitude everywhere in the complex plane $\Rightarrow$ need the discontinuity

$$\text{disc}\, \mathcal{A}^n_{\eta\to 3\pi} = \frac{1}{2}\sum_{n'} (2\pi)^4 \delta(p_n - p'_n)\, \mathcal{A}^{n'}_{\eta\to 3\pi} \left(\mathcal{T}^{n'n}_{3\pi\to 3\pi}\right)^*$$

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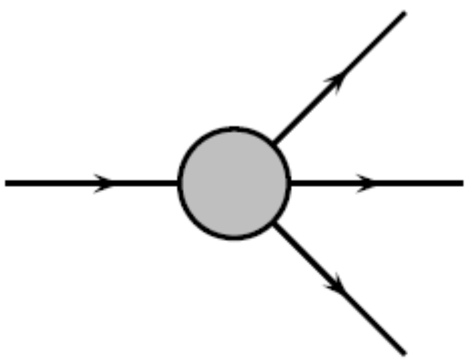
disc 

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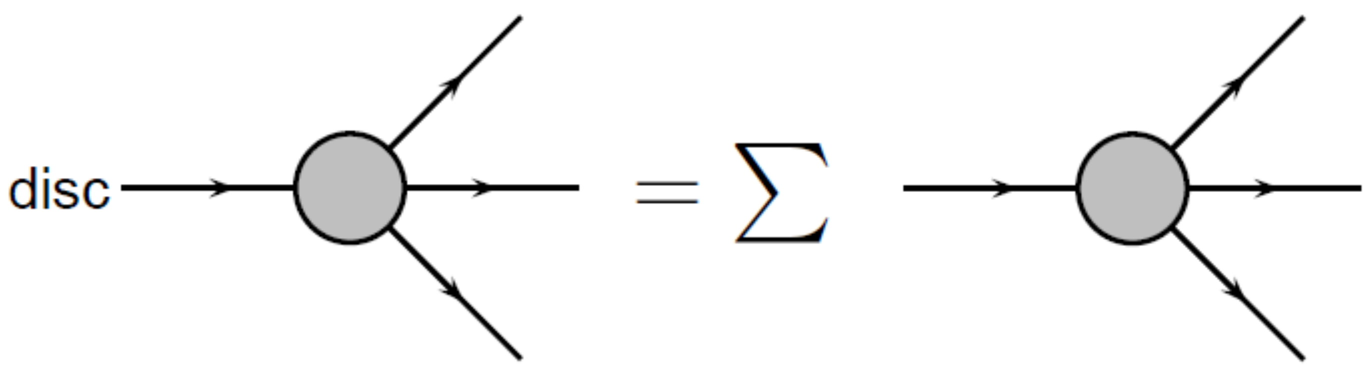

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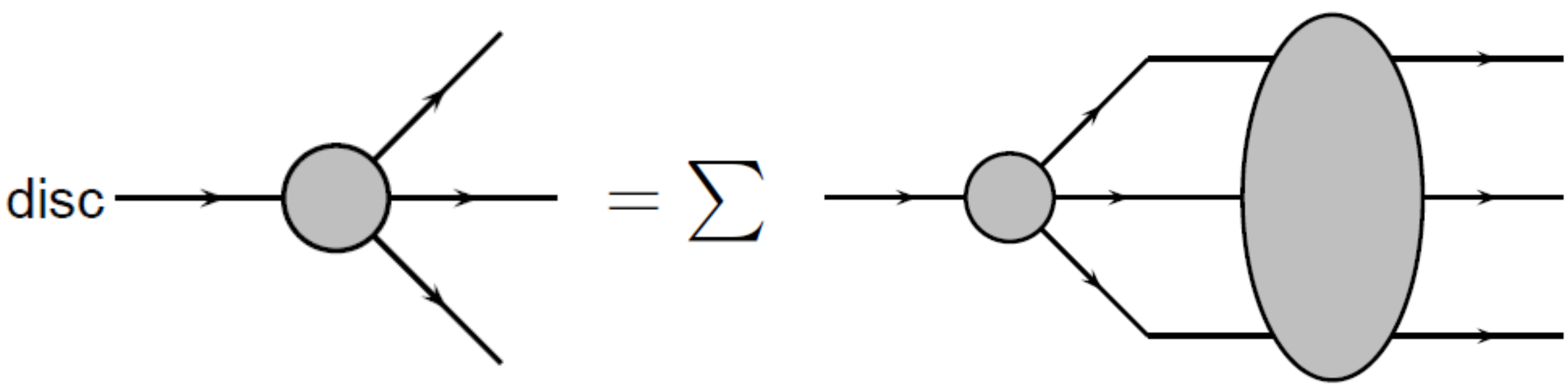

2.1 Method: Representation of the amplitude

- Decomposition of the amplitude as a function of isospin states

$$\boxed{M(s,t,u) = M_0(s) + \left(s-u\right)M_1(t) + \left(s-t\right)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)}$$

Fuchs, Sazdjian & Stern'93
Anisovich & Leutwyler'96

- ➢ M_I isospin *I* rescattering in two particles
- ➢ Amplitude in terms of S and P waves ⇨ exact up to NNLO ($\mathcal{O}(p^6)$)
- ➢ Main two body rescattering corrections inside M_I

- Dispersion relation for the M_I's

$$\boxed{M_I(s) = \Omega_I(s)\left(P_I(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^n}\frac{\sin\delta_I(s')\,\hat{M}_I(s')}{\left|\Omega_I(s')\right|\left(s'-s-i\varepsilon\right)}\right)}$$

$$\left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}ds'\frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

Omnès function

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- Inputs needed : S and P-wave phase shifts of $\pi\pi$ scattering

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Omnès function

- $\hat{M}_I(s)$: singularities in the t and u channels, depend on the other $M_I(s)$
  ⇒ subtract $M_I(s)$ from the partial wave projection of $M(s,t,u)$
  Angular averages of the other functions ⇒ Coupled equations

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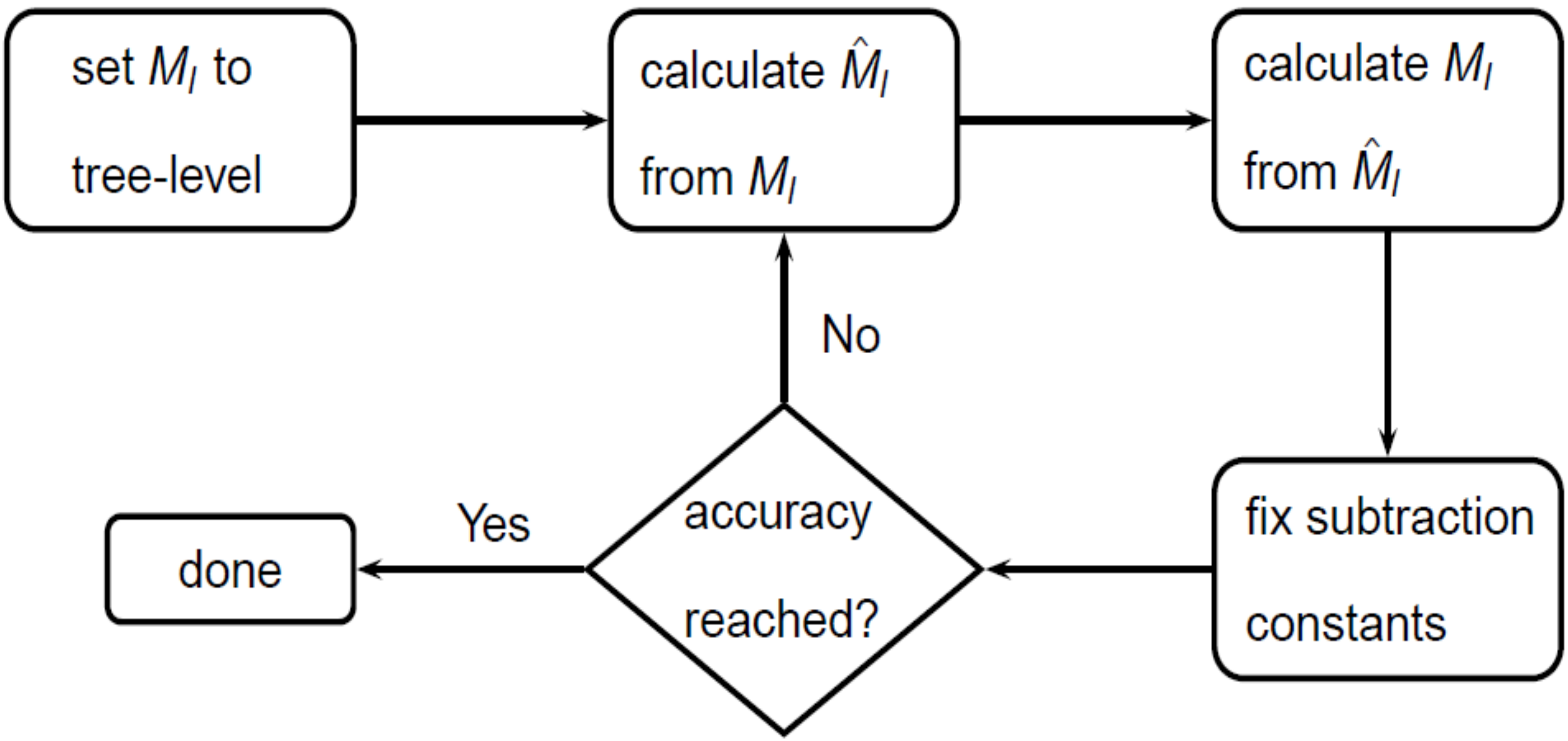
$$\left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta_I(s')}{s'(s'-s-i\varepsilon)}\right)\right]$$

- Solution depends on *subtraction constants* only ⇨ solve by iterative procedure

2.2 Iterative Procedure



2.3 Subtraction constants

- Extension of the numbers of parameters compared to *Anisovich & Leutwyler'96*

$$P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3$$
$$P_1(s) = \alpha_1 + \beta_1 s + \gamma_1 s^2$$
$$P_2(s) = \alpha_2 + \beta_2 s + \gamma_2 s^2$$

- In the work of *Anisovich & Leutwyler'96* matching to one loop ChPT
  Use of the SU(2) x SU(2) chiral theorem
  ⇨ The amplitude has an *Adler zero* along the line s=u

- Now data on the Dalitz plot exist from KLOE, WASA and MAMI

  ⇨ Use the data to directly fit the subtraction constants

- Solution *linear* in the *subtraction constants* *Anisovich & Leutwyler'96*

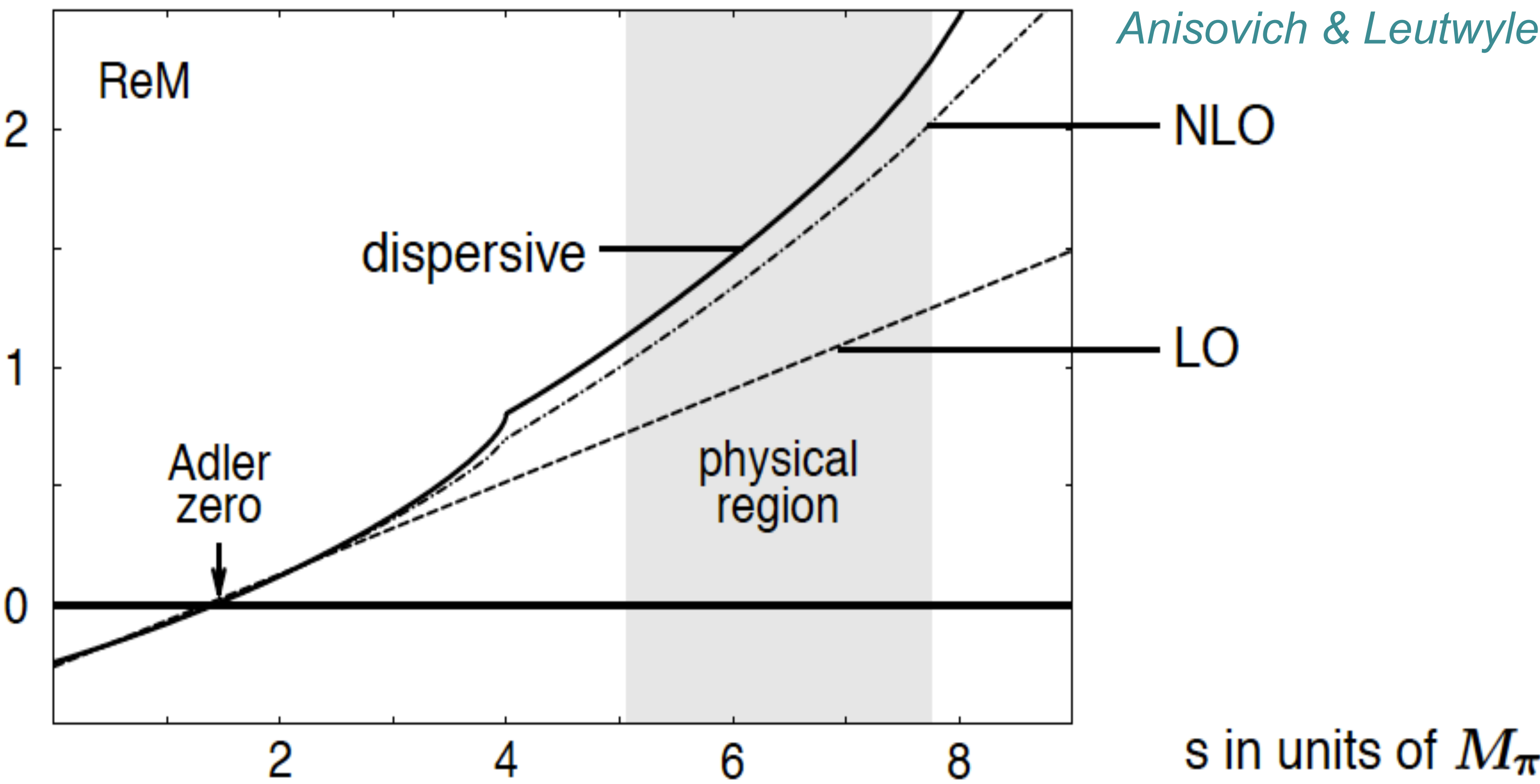
$$M(s,t,u) = \alpha_0 M_{\alpha_0}(s,t,u) + \beta_0 M_{\beta_0}(s,t,u) + \ldots$$

  ⇨ makes the fit much easier

2.3 Subtraction constants

- Adler zero: the real part of the amplitude along the line s=u has a zero

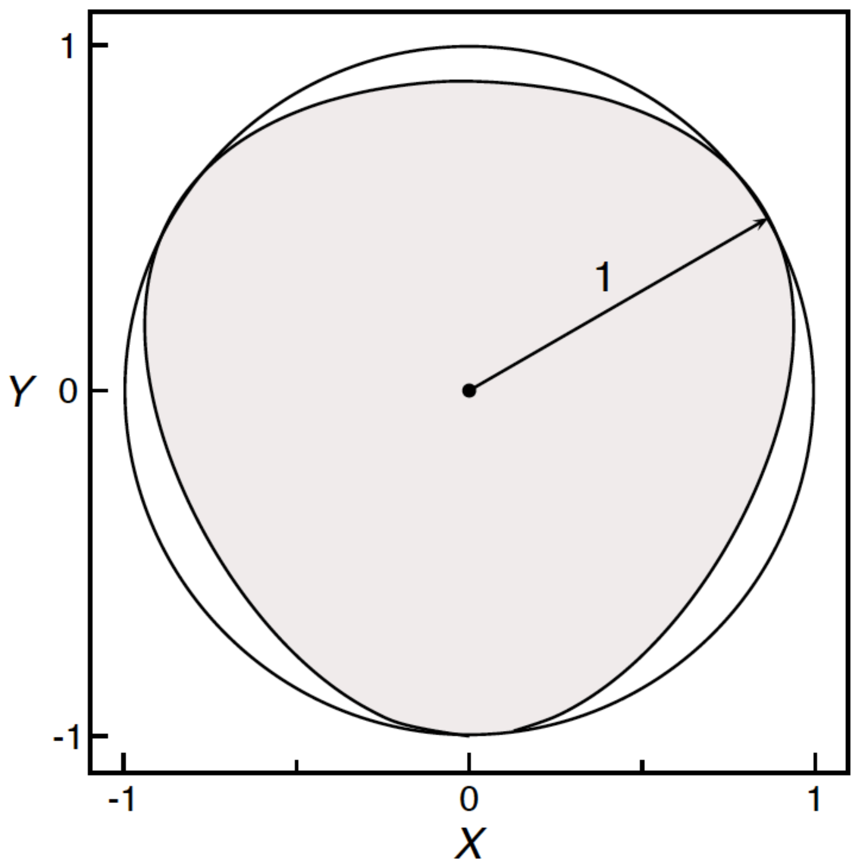
Anisovich & Leutwyler'96



2.4 Experimental measurements

- Dalitz plot measurement : Amplitude expanded in X and Y around X=Y=0

$$\left|A(s,t,u)\right|^2 = \Gamma(X,Y) = N\left(1 + aY + bY^2 + dX^2 + fY^3\right)$$



$$X = \frac{\sqrt{3}\ (T_+ - T_-)}{Q_c} = \frac{\sqrt{3}}{2M_\eta Q_c}(u-t)$$

$$Y = \frac{3T_0}{Q_c} - 1 = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$

with T_i: kinetic energy of π^i in the η rest frame

and $Q_c \equiv T_0 - T_+ - T_- = M_\eta - 2M_{\pi^+} - M_{\pi^0}$

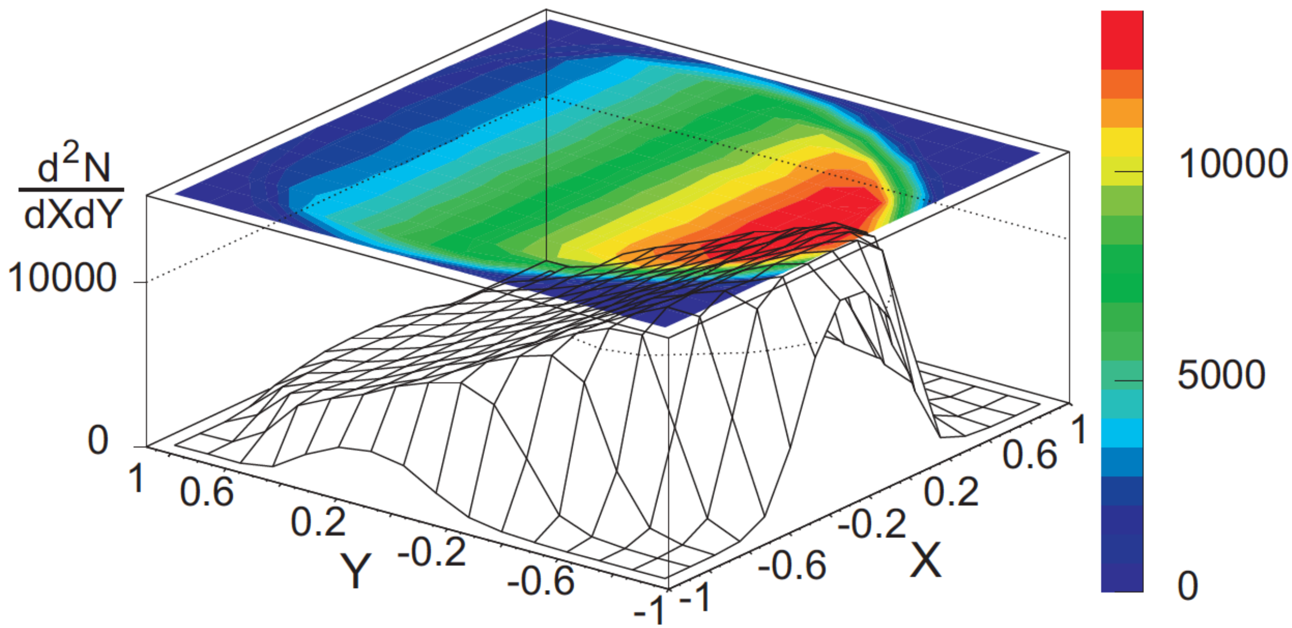
2.4 Experimental measurements : Charged channel

- Charged channel measurements with high statistics from *KLOE* and *WASA*

  e.g. *KLOE:* ~1.3 x 10^6 $\eta \to \pi^+ \pi^- \pi^0$ events from $e^+e^- \to \varphi \to \eta\gamma$

$$\left|A_c(s,t,u)\right|^2 = N\left(1 + aY + bY^2 + dX^2 + fY^3\right)$$

KLOE'08



$$Y = \frac{3}{2M_\eta Q_c}\left(\left(M_\eta - M_{\pi^0}\right)^2 - s\right) - 1$$

$$X = \frac{\sqrt{3}}{2M_\eta Q_c}(u - t)$$

2.4 Experimental measurements : Neutral channel

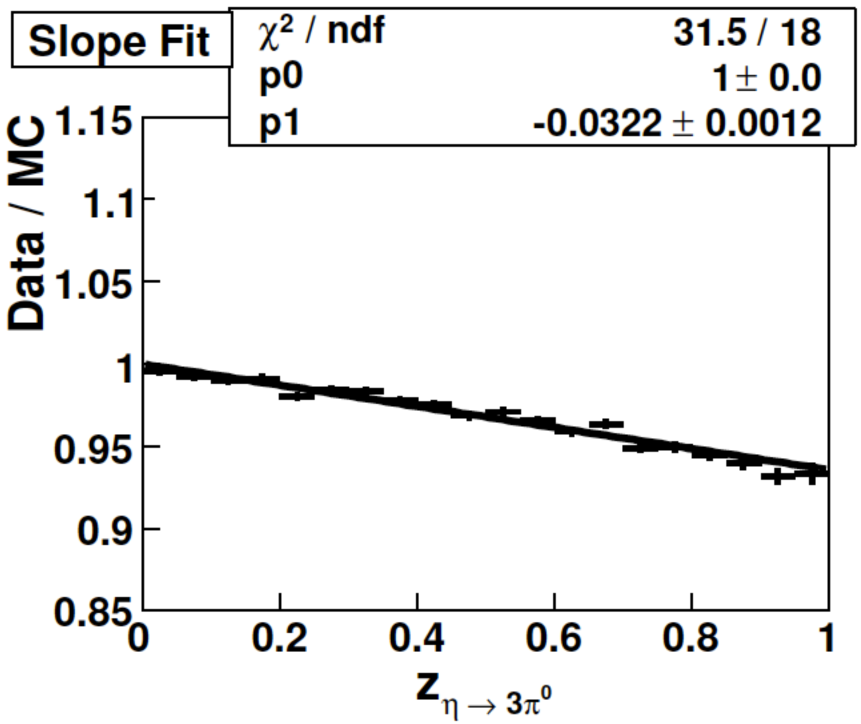
- Neutral channel measurements with high statistics from *MAMI-B*, *MAMI-C* and *WASA* e.g. *MAMI-C:* ~3 x 10^6 $\eta \rightarrow 3\pi^0$ events from $\gamma p \rightarrow \eta p$

$$\left|A_n(s,t,u)\right|^2 = N\left(1 + 2\alpha Z + 6\beta Y\left(X^2 - \frac{Y^2}{3}\right) + 2\gamma Z^2\right)$$

$$Z = \frac{2}{3}\sum_{i=1}^{3}\left(\frac{3T_i}{Q_n} - 1\right)^2 = X^2 + Y^2$$

$$Q_n \equiv M_\eta - 3M_{\pi^0}$$

⇨ Extraction of the slope :



MAMI-C'09

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2.5 Subtraction constants

- As we have seen, only Dalitz plots are measured, *unknown normalization!*

$$\boxed{A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2}M(s,t,u)}$$

To determine Q, one needs to know the normalization

⇨ For the normalization one needs to use ChPT

- The subtraction constants are

$$P_0(s) = \alpha_0 + \beta_0 s + \gamma_0 s^2 + \delta_0 s^3$$

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Only *6 coefficients* are of physical relevance

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  Only *6 coefficients* are of physical relevance

- They are determined from
  - Matching to one loop ChPT ⇒ $\delta_0 = \gamma_1 = 0$
  - Combine ChPT with fit to the data ⇒ δ_0 and γ_1 are determined from the data

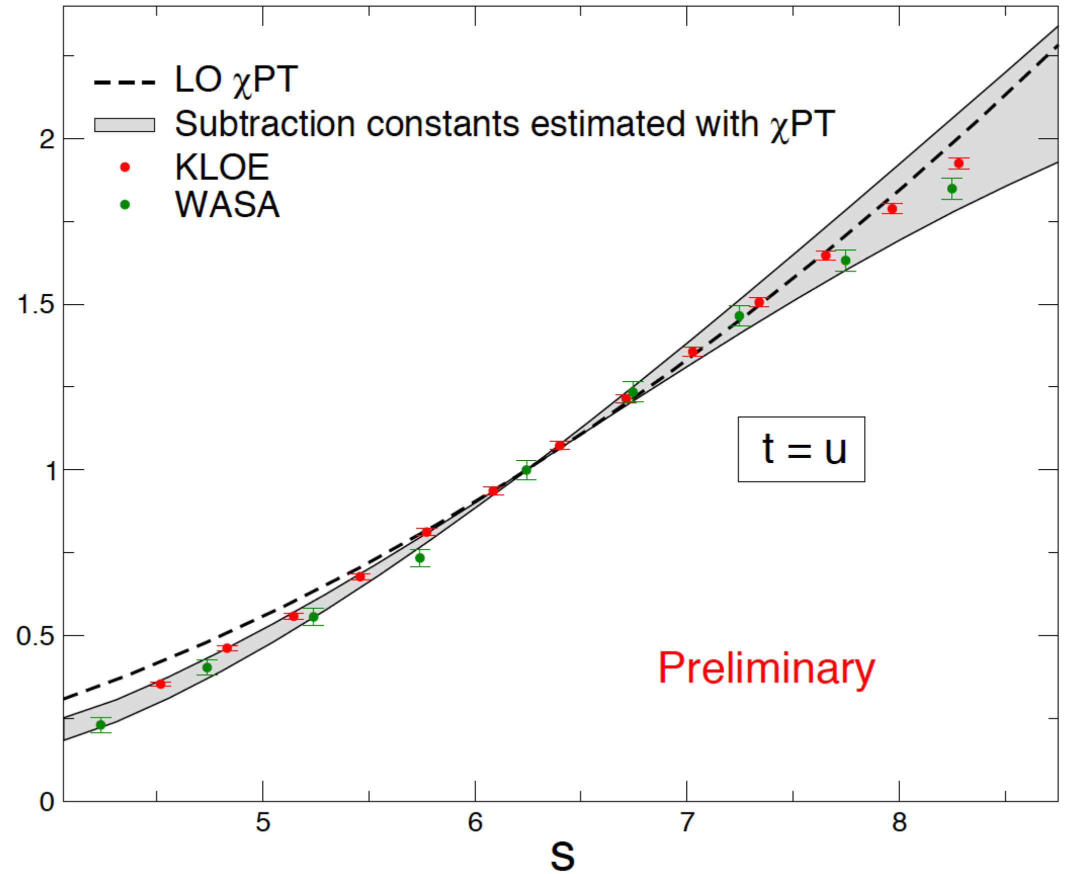
- Matching to one loop ChPT : Taylor expand the dispersive M_I
  Subtraction constants ⇔ Taylor coefficients

- Important : Adler zero should be reproduced! ⇒ Can be used to constrain the fit

3. Preliminary Results

3.1 Dalitz plot distribution of $\eta \to \pi^+ \pi^- \pi^0$ decays

- The amplitude squared along the line t = u :

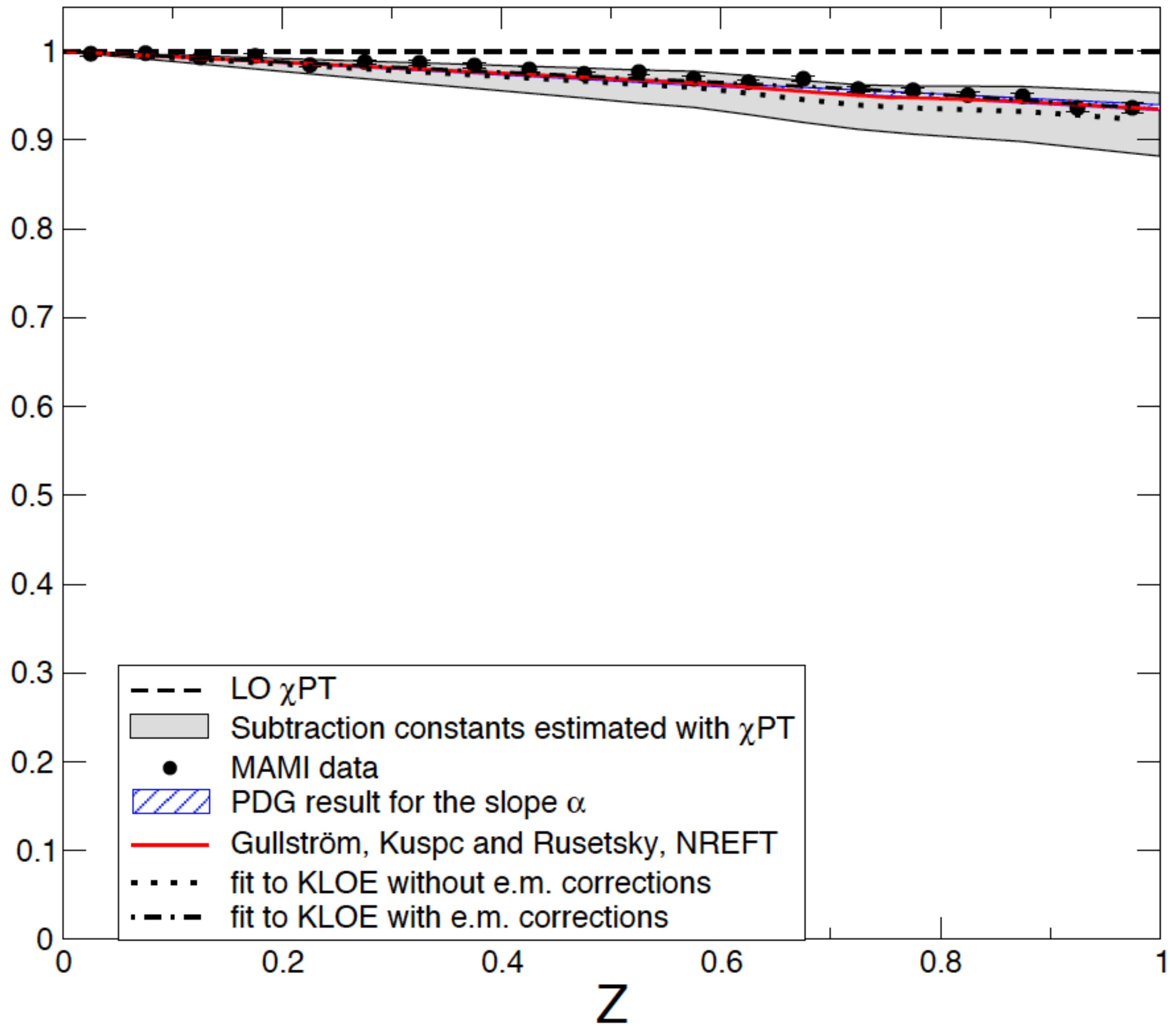


$\left[M_\pi^2\right]$

- Good agreement between theory and experiment
- The theoretical error bars are large ⇨ fit the subtraction constants to the data to reduce the uncertainties

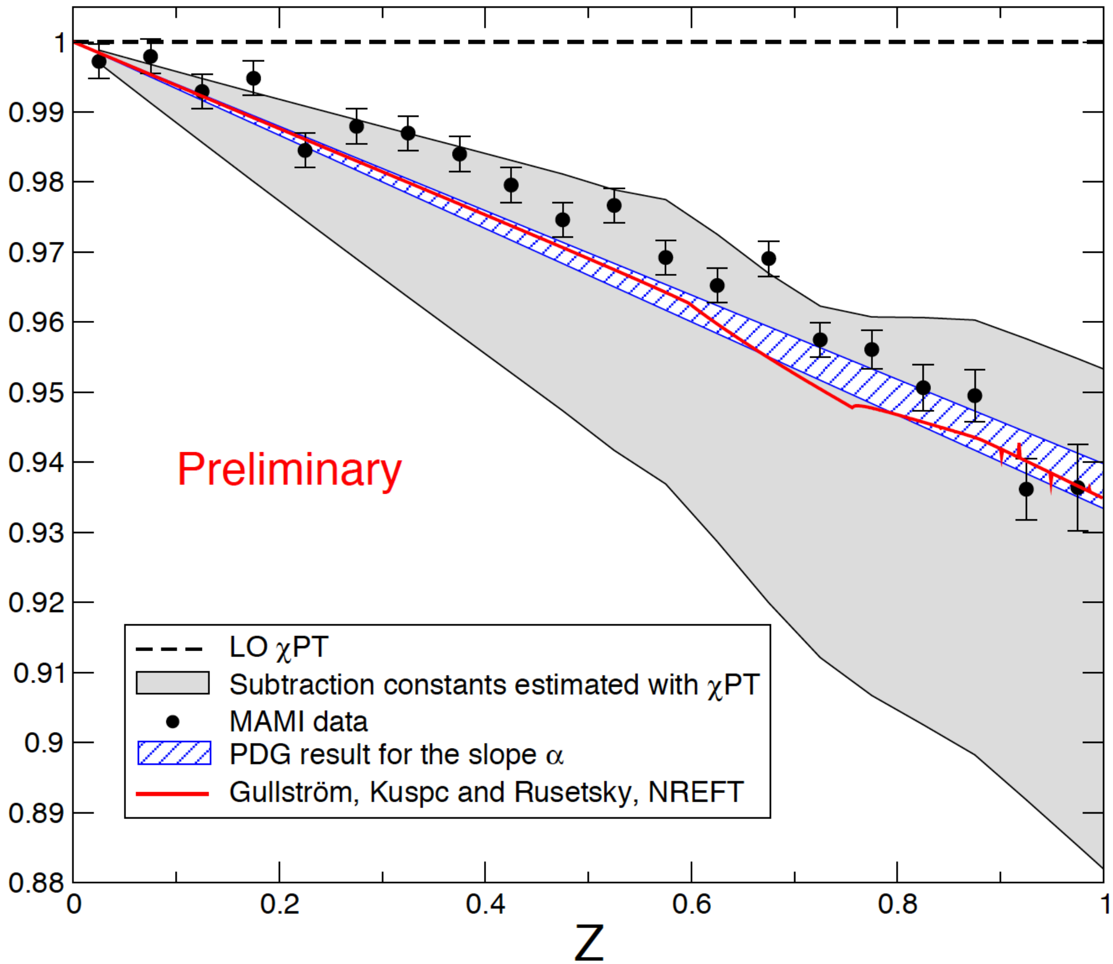
3.2 Z distribution for $\eta \to \pi^0 \pi^0 \pi^0$ decays

- The amplitude squared in the neutral channel is



Here also the agreement looks very good but ⇨

3.2 Z distribution for $\eta \to \pi^0 \pi^0 \pi^0$ decays



NRFT in η decays

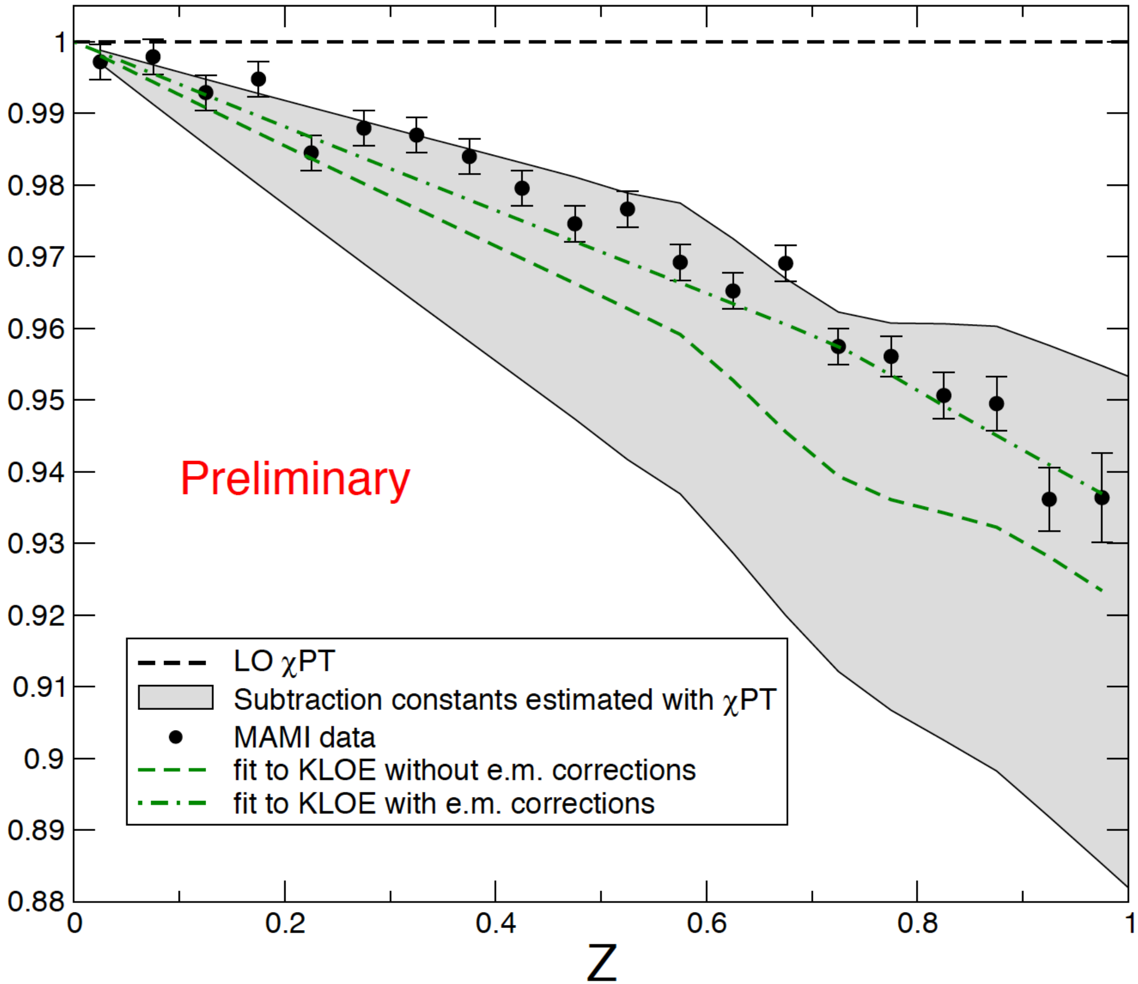
Gullstrom, Kupsc, Rusetsky'09

Schneider, Kubis, Ditsche'11

- The uncertainties coming from the matching with ChPT are very large
  ⇨ there is room for improvement using the data

3.2 Z distribution for $\eta \to \pi^0 \pi^0 \pi^0$ decays

- If one wants to fit the data, at this level of precision the e.m. corrections matter
  ⇨ use the one loop e.m. calculations from *Ditsche, Kubis and Meissner'09*



3.3 Qualitative results of our analysis

- Determination of Q from the dispersive approach :

$$\boxed{\Gamma_{\eta\to\pi^+\pi^-\pi^0} = \frac{1}{Q^4}\frac{M_K^4}{M_\pi^4}\frac{\left(M_K^2 - M_\pi^2\right)^2}{6912\pi^3 F_\pi^4 M_\eta^3}\int_{s_{\min}}^{s_{\max}} ds \int_{u_-(s)}^{u_+(s)} du \, \left|M(s,t,u)\right|^2}$$

$$\Gamma_{\eta\to3\pi} = 295 \pm 20 \text{ eV}$$ PDG'12

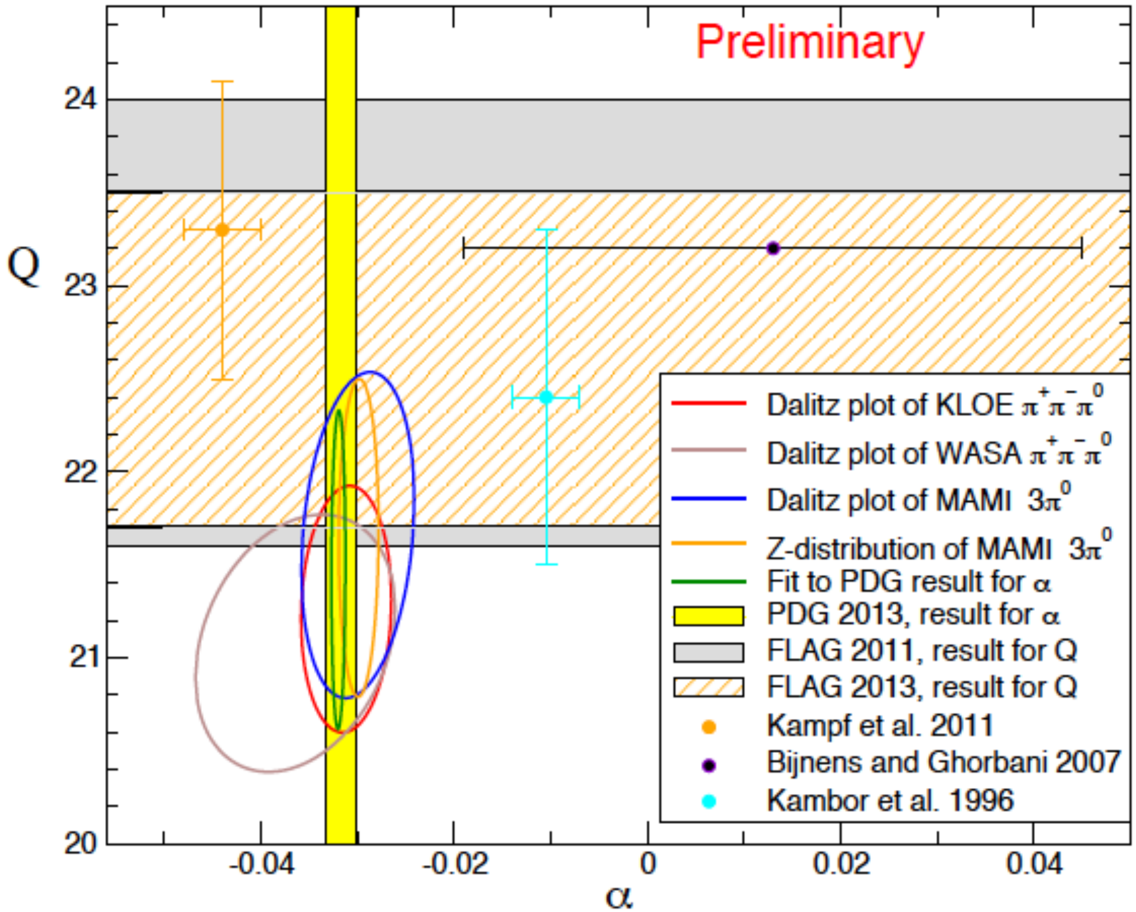
$$\left(Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2} \right)$$

- Determination of α

$$\boxed{\left|A_n\left(s,t,u\right)\right|^2 = N\left(1 + 2\alpha Z\right)}$$

3.3 Qualitative results of our analysis

- Plot of Q versus α :



NB: Isospin breaking has not been accounted for

From kaon mass spliting :

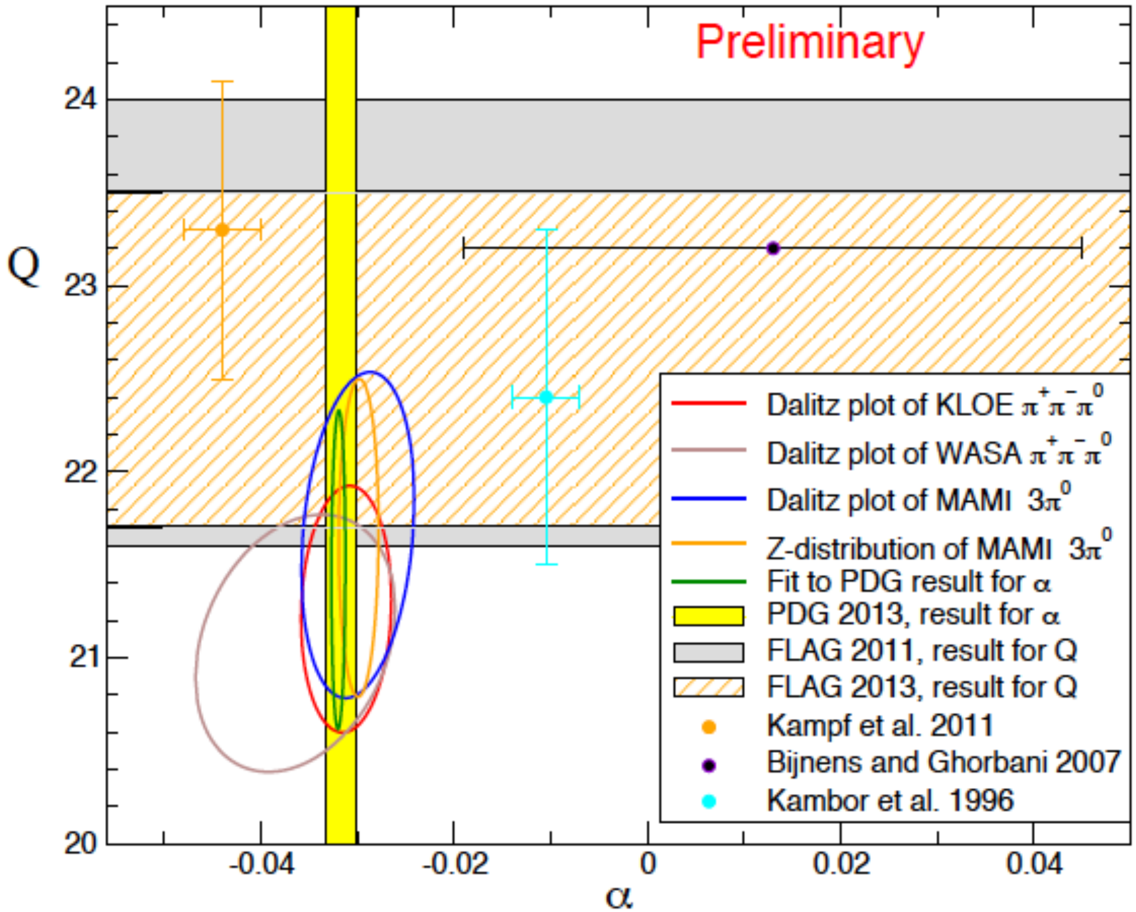
$$\boxed{Q = 20.7 \pm 1.2}$$

Kastner & Neufeld'08

- All the data give consistent results. The preliminary outcome for Q is intermediate between the lattice result and the one of Kastner and Neufeld.

3.3 Qualitative results of our analysis

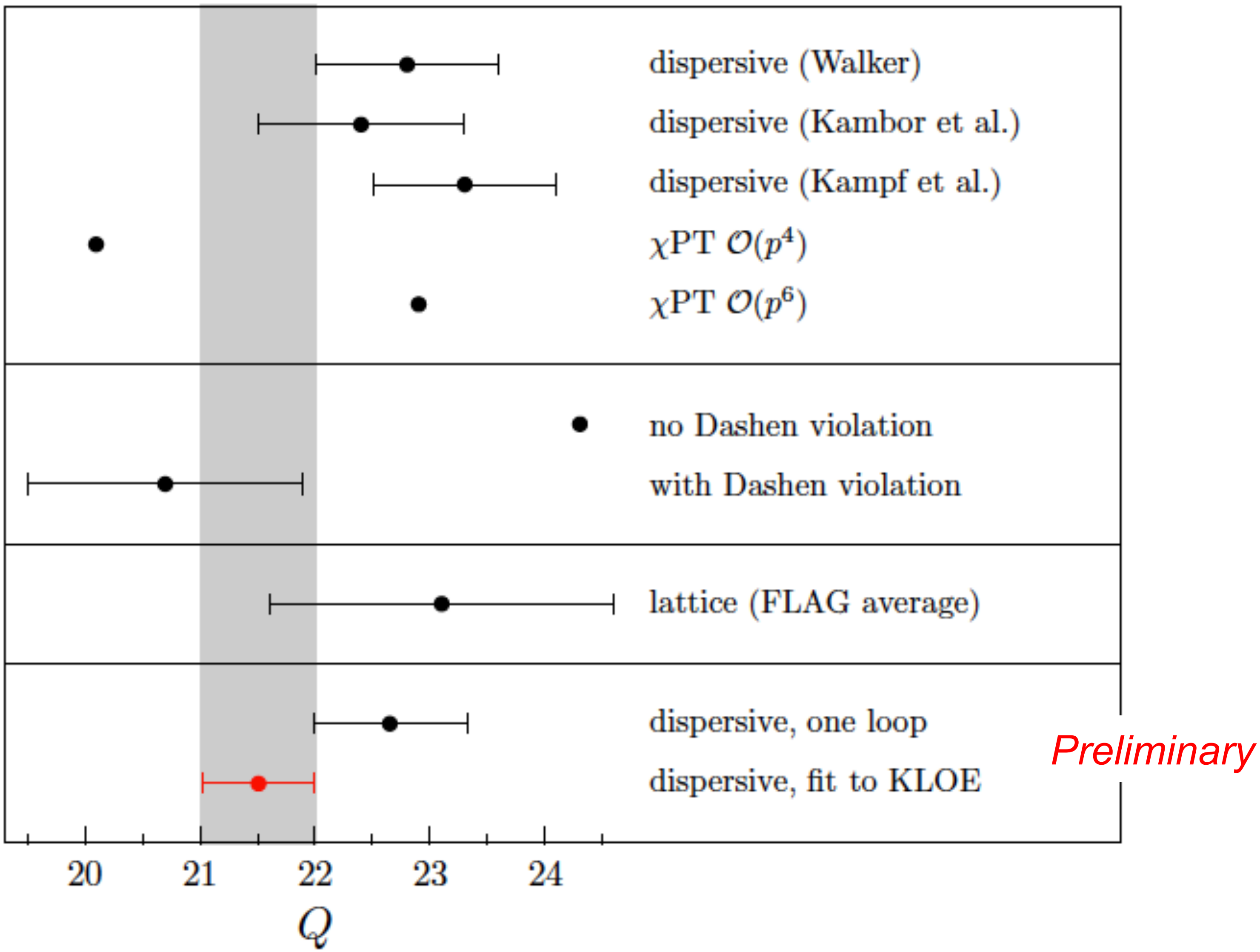
- Plot of Q versus α :



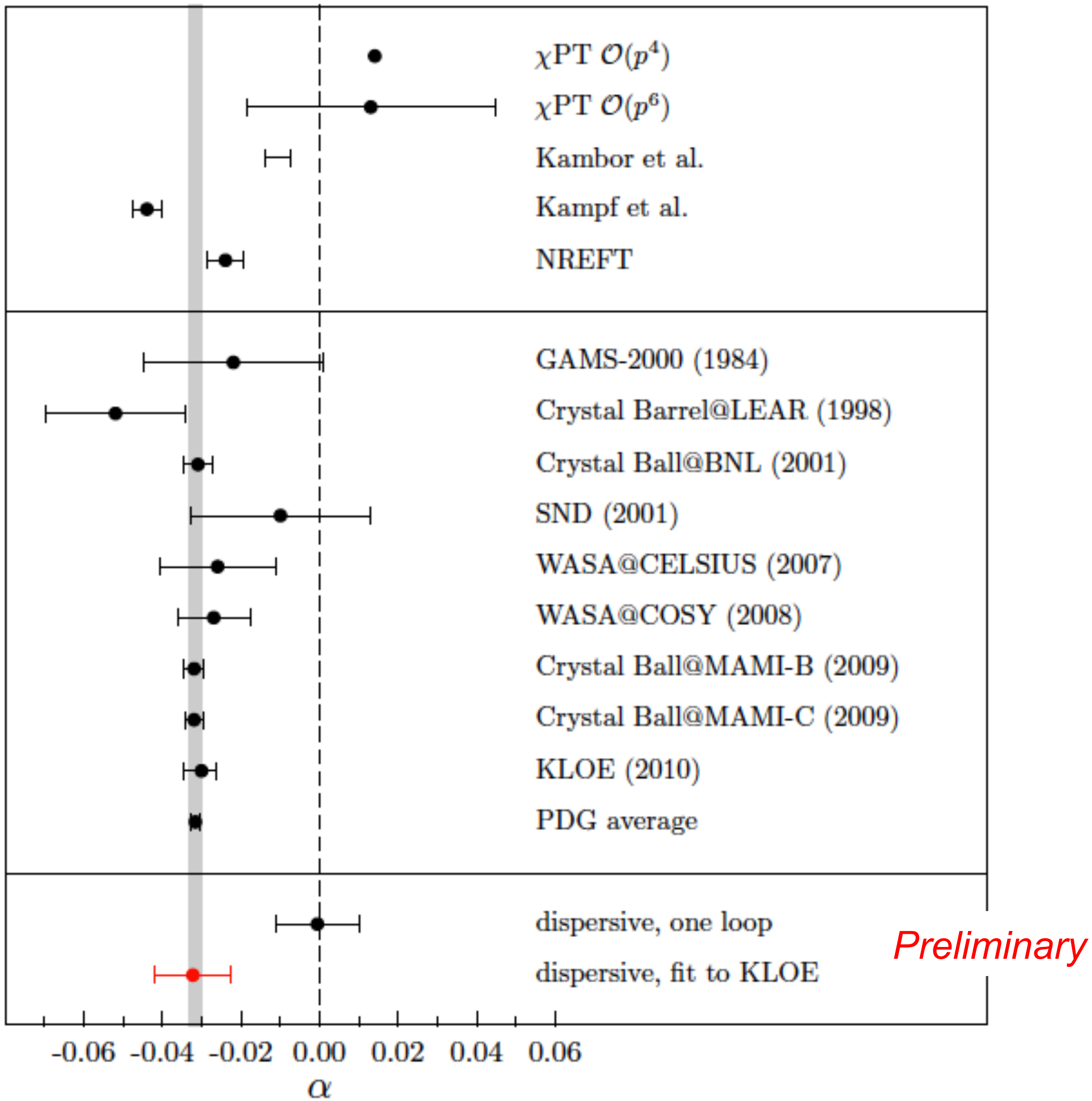
NB: Isospin breaking has not been accounted for

- All our preliminary results give a negative value for α. In particular the result using KLOE data for $\eta \rightarrow \pi^+ \pi^- \pi^0$ is in perfect agreement with the PDG value!

3.4 Comparison of results for Q

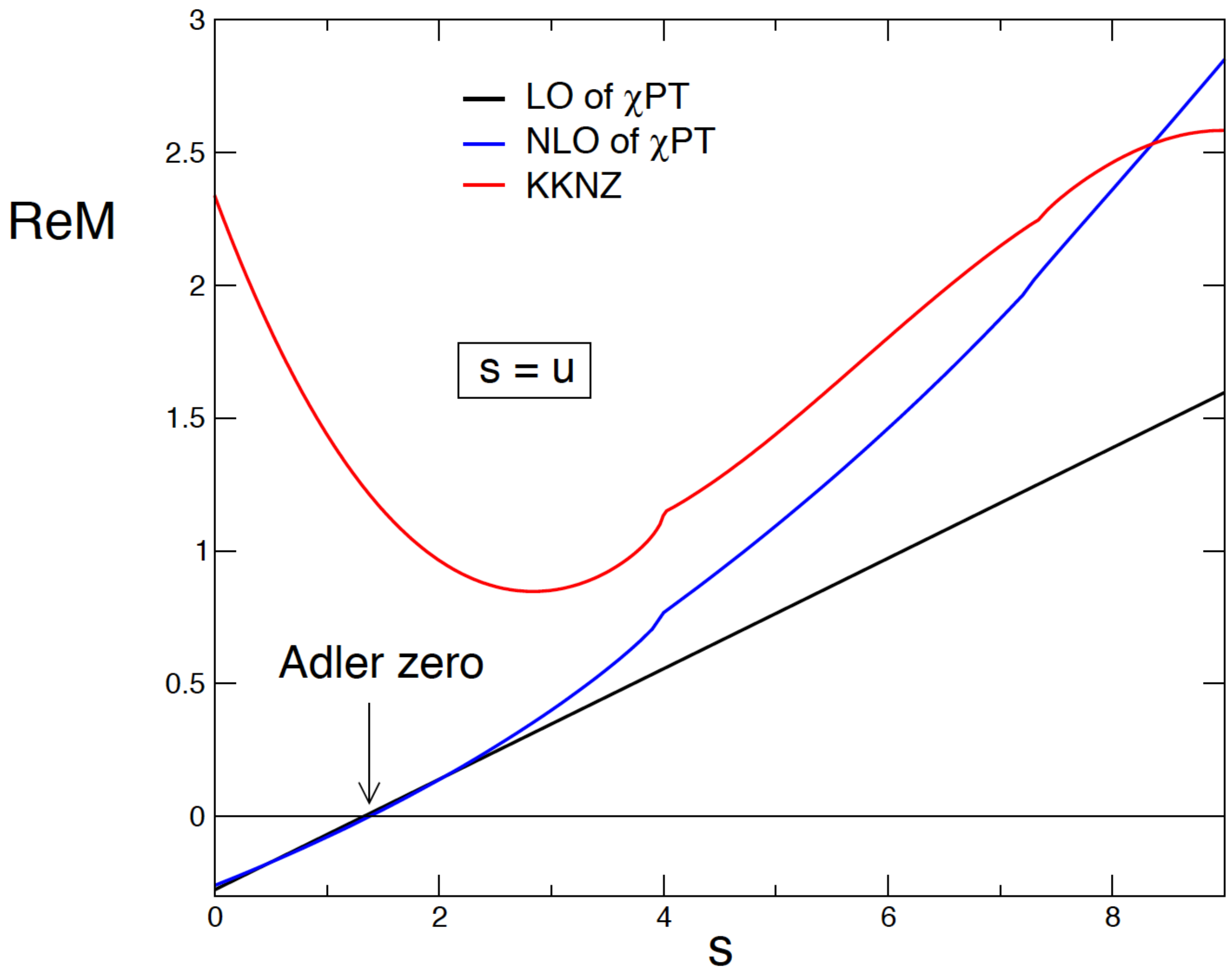


3.5 Comparison of results for α



3.6 Comparison with KKNZ

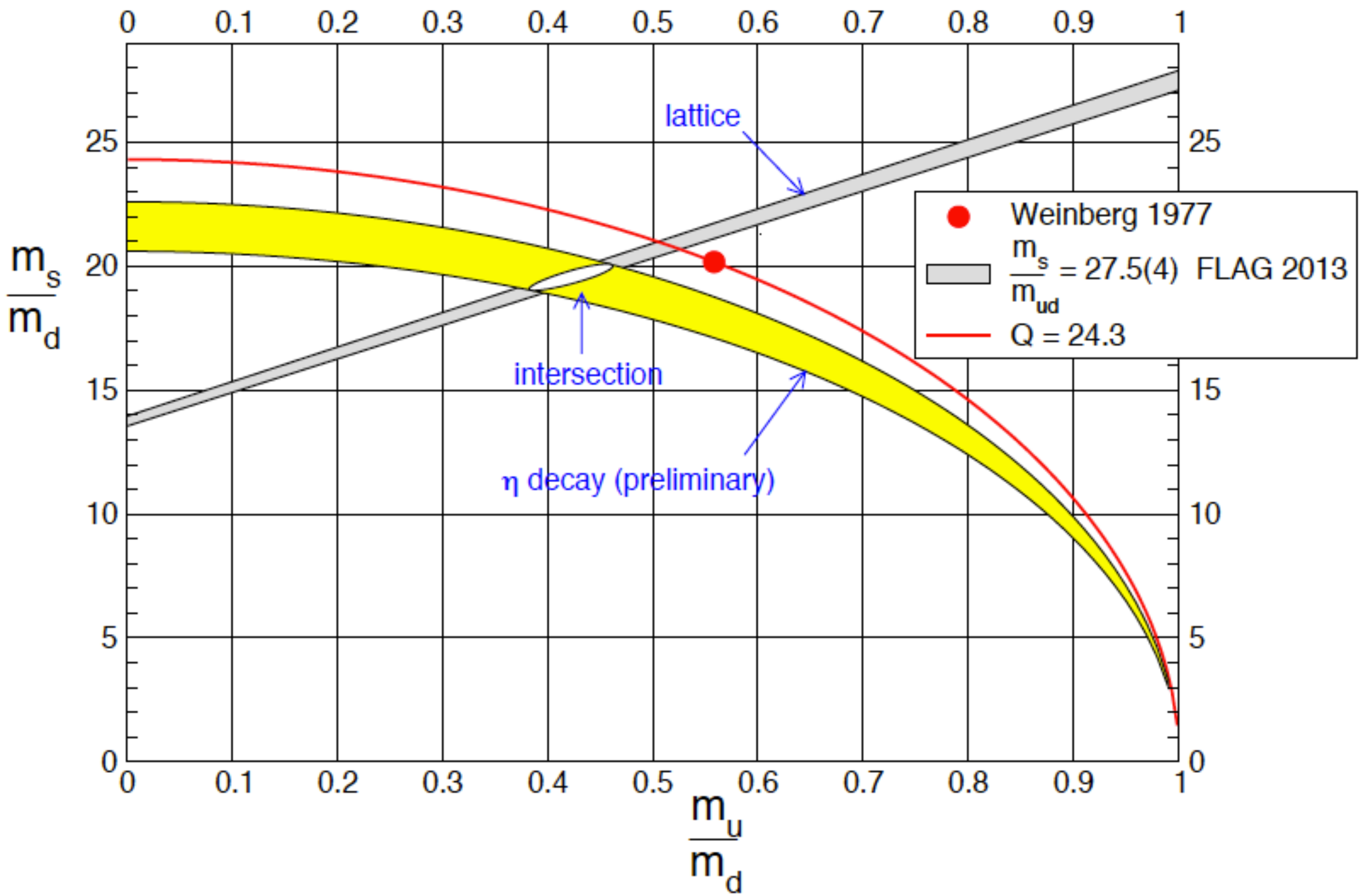
- Amplitude along the line s=u



- Adler zero not reproduced!

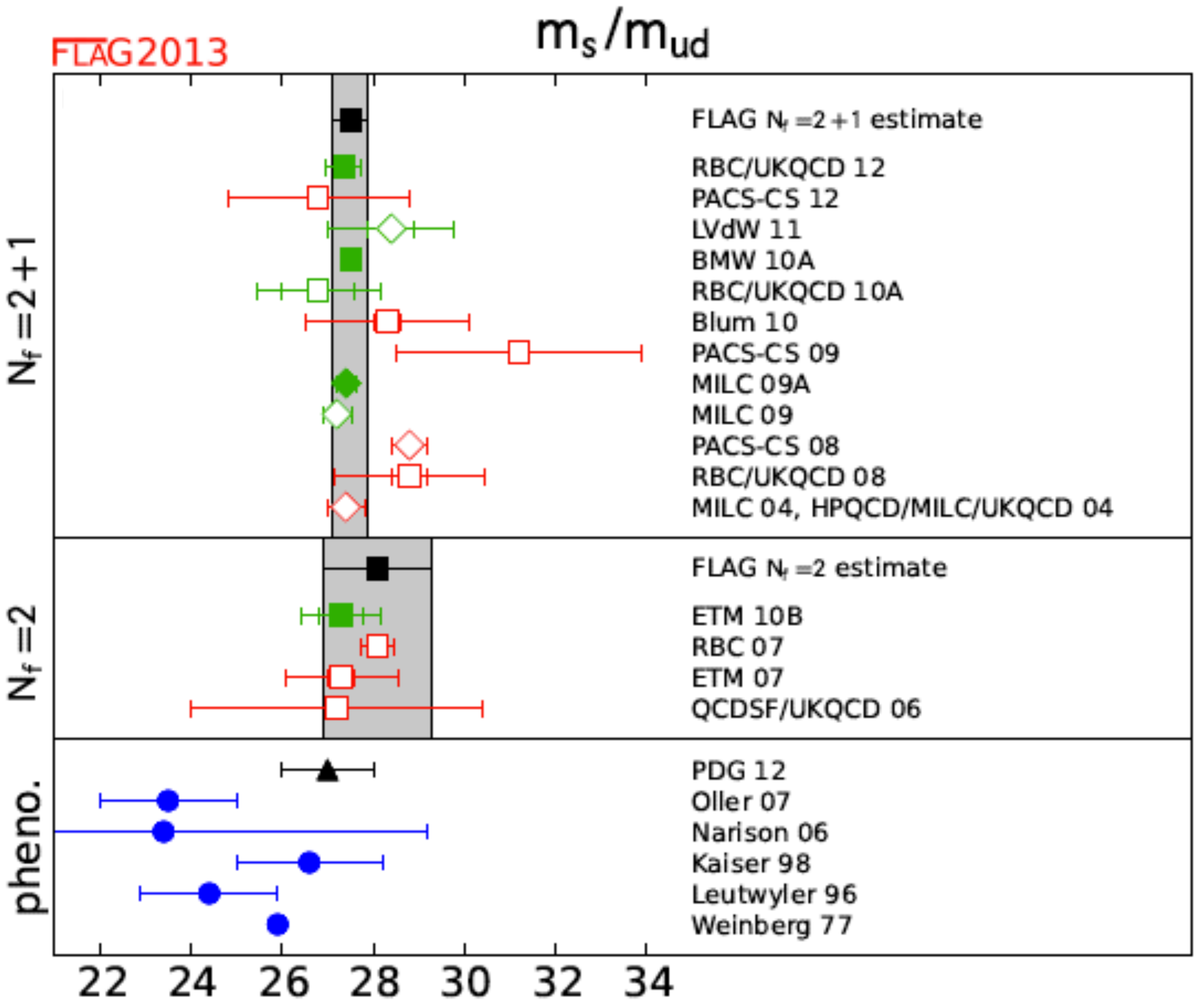
3.7 Light quark masses

H. Leutwyler



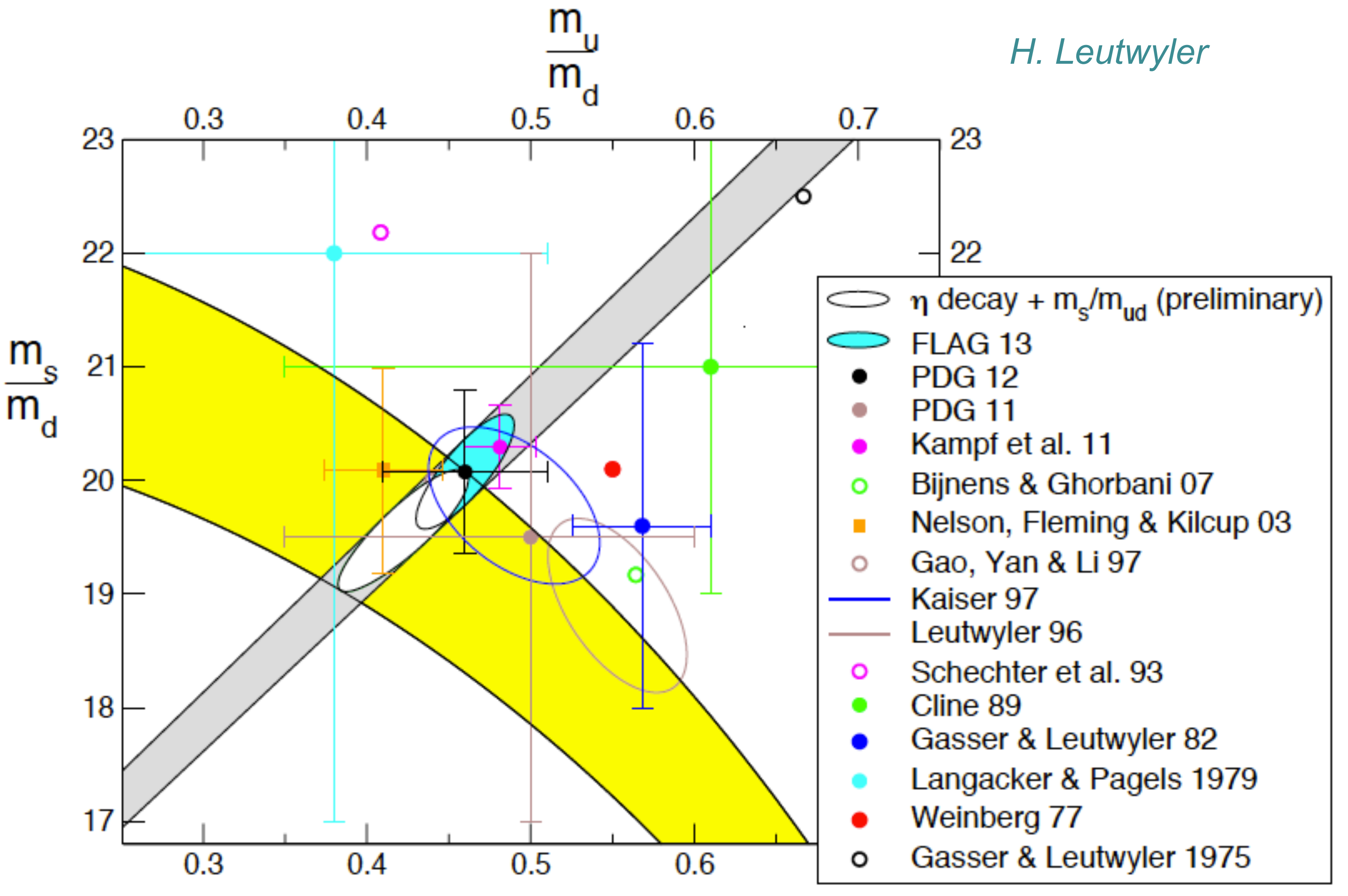
- Smaller values for Q ⇨ smaller values for ms/md and mu/md than LO ChPT

3.7 Light quark masses



3.7 Light quark masses

H. Leutwyler



4. Conclusion and outlook

4.1 Conclusion

- $\eta \to 3\pi$ decays represent a very clean source of information on the quark mass ratio *Q*

- A reliable extraction of Q requires having the strong rescattering effects in the final state under control

- This is possible thanks to dispersion relations
  ⇒ need to determine unknown subtraction constants

- This was done up to now relying exclusively on ChPT but precise measurements have become available

  - In the charged channel: *KLOE* and *WASA*
  - In the neutral channel: *MAMI-B*, *MAMI-C*, *WASA*
  - More results are expected: *KLOE, CLAS, GlueX, JEF…*

  ⇒ will allow to reduce the uncertainties in a significant way
  seems to push the value for Q towards low results

4.2 Outlook

- Analysis still in progress :
  - Determination of the subtraction constants :

    ⇨ combine ChPT and the data in the optimal way

  - Take into account the e.m. corrections

    ⇨ implementation of the one loop e.m. corrections from *Ditsche, Kubis and Meissner'09* to be able to fit to the data charged and neutral channel

  - Matching to NNLO ChPT

    ⇨ Constraints from experiment: possible insights on C_i values

  - Careful estimate of all uncertainties

  - Inelasticities

- Our preliminary results give a consistent picture between
  - all experimental measurements: Dalitz plot measurements from *both charged* and *neutral* channels

  - theoretical requirements: e.g. *Adler zero*

5. Back-up

3.4 Subtraction constants

- Matching to one loop ChPT : Taylor expand the dispersive M_I
  Subtraction constants $\Leftrightarrow$ Taylor coefficients

$$M_0(s) = a_0 + b_0 s + c_0 s^2 + d_0 s^3 + \ldots$$
$$M_1(s) = a_1 + b_1 s + c_1 s^2 + \ldots$$
$$M_2(s) = a_2 + b_2 s + c_2 s^2$$

  - gauge freedom $\Rightarrow$ a_0, b_0, a_1, a_2 tree level ChPT values
  - fix the remaining ones with one-loop ChPT c_0, b_1, b_2, c_2
  - matching to one loop : $d_0 = c_1 = 0$ or fit : d_0 and c_1 from the *data*

- Problem : this identification assumes there is not significant contributions from higher orders of the chiral expansion $\Rightarrow$ not well-justified for the *s^3 terms!*

- Solution: Match the SU(2) x SU(2) expansion of the dispersive representation with the one of the one loop representation *In progress*

- Important : Adler zero should be reproduced! $\Rightarrow$ Can be used to constrain the fit

1.2 Meson masses from ChPT

- $m_{u,d,s} \ll \Lambda_{QCD}$: masses treated as small perturbations

  ⇨ *expansion in powers of m_q*

- *Gell-Mann-Oakes-Renner relations*:

  (meson mass)^2 = (spontaneous ChSB) x (explicit ChSB)

  $\langle \bar{q}q \rangle$ $\quad\quad$ m_q

- From LO ChPT without e.m effects:

$$M^2_{\pi^+} = (m_u + m_d)\, B_0 + O(m^2)$$
$$M^2_{K^+} = (m_u + m_s)\, B_0 + O(m^2)$$
$$M^2_{K^0} = (m_d + m_s)\, B_0 + O(m^2)$$

- Electromagnetic effects: *Dashen's theorem*

$$\boxed{\left(M^2_{K^+} - M^2_{K^0}\right)_{em} - \left(M^2_{\pi^+} - M^2_{\pi^0}\right)_{em} = O\left(e^2 m\right)}$$

Dashen' 69

$$\boxed{\begin{aligned} M^2_{\pi^0} &= B_0\left(m_u + m_d\right) \\ M^2_{\pi^+} &= B_0\left(m_u + m_d\right) + \Delta_{em} \\ M^2_{K^0} &= B_0\left(m_d + m_s\right) \\ M^2_{K^+} &= B_0\left(m_u + m_s\right) + \Delta_{em} \end{aligned}}$$

2 unknowns B_0 and Δ_{em}

1.2 Meson masses from ChPT

➡ Quark mass ratios *Weinberg'77*

$$\frac{m_u}{m_d} \overset{\mathrm{LO}}{=} \frac{M_{K^+}^2 - M_{K^0}^2 + 2M_{\pi^0}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} = 0.56\,,$$

$$\frac{m_s}{m_d} \overset{\mathrm{LO}}{=} \frac{M_{K^+}^2 + M_{K^0}^2 - M_{\pi^+}^2}{M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2} = 20.2$$

1.5 Quark mass ratios

- The same O(m) correction appears in both ratios

  ⇨ Take the double ratio

$$\boxed{Q^2 \equiv \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}} = \frac{M_K^2}{M_\pi^2} \frac{M_K^2 - M_\pi^2}{\left(M_{K^0}^2 - M_{K^+}^2\right)_{QCD}} \left[1 + O(m_q^2, e^2)\right]$$

  Very Interesting quantity to determine since Q^2 does not receive any correction at NLO!

- Using Dashen's theorem and inserting Weinberg LO values

$$Q_D^2 \equiv \frac{(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 + M_{\pi^0}^2)(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 - M_{\pi^0}^2)}{4M_{\pi^0}^2(M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2)}$$

⇨ $\boxed{Q_D = 24.2}$

1.5 Quark mass ratios

- Estimate of Q: $$B_0\left(m_u - m_d\right) = \frac{1}{Q^2}\frac{M_K^2\left(M_K^2 - M_\pi^2\right)}{M_\pi^2} + O(M^3)$$

  - From corrections to the Dashen's theorem

    $$\Rightarrow \quad B_0\left(m_d - m_u\right) = \left(M_{K^+}^2 - M_{K^0}^2\right) - \left(M_{\pi^+}^2 - M_{\pi^0}^2\right) + O\left(e^2 m\right)$$

    The corrections can be large due to $e^2 m_s$ corrections, difficult to estimate due to LECs

  - From $\eta \to \pi^+ \pi^- \pi^0$: $$A(s,t,u) = -\frac{1}{Q^2}\frac{M_K^2}{M_\pi^2}\frac{M_K^2 - M_\pi^2}{3\sqrt{3}F_\pi^2}M(s,t,u)$$

    $$\Rightarrow \quad \Gamma_{\eta\to 3\pi} \propto \int \left|A(s,t,u)\right|^2 \propto Q^{-4}$$

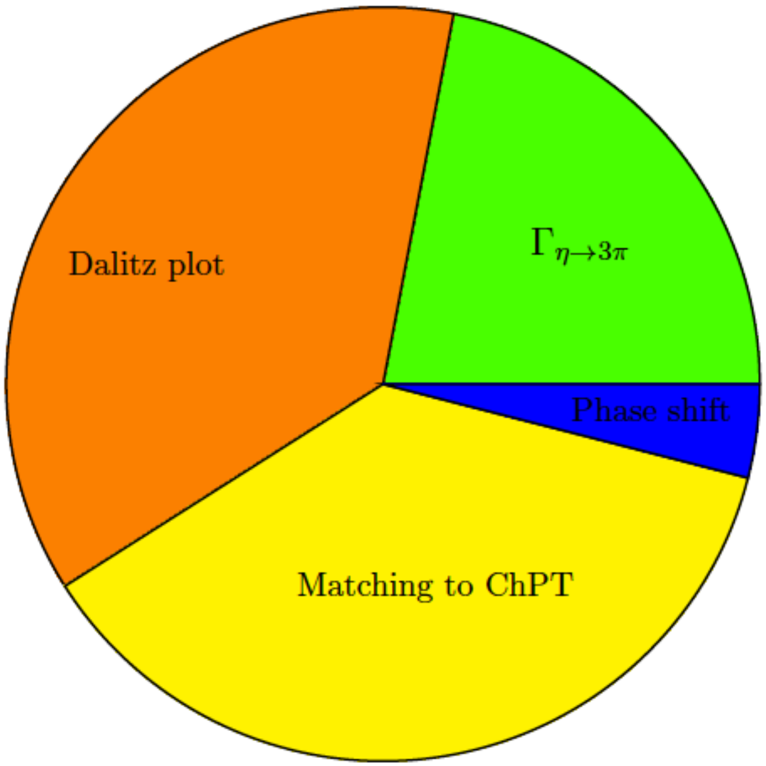
- In the following, compute the normalized amplitude M(s,t,u) with the best accuracy $\Rightarrow$ *extraction of Q*

6. Prospects at JLab

6.1 Introduction

- Attempt to quantify roughly the uncertainties

Preliminary

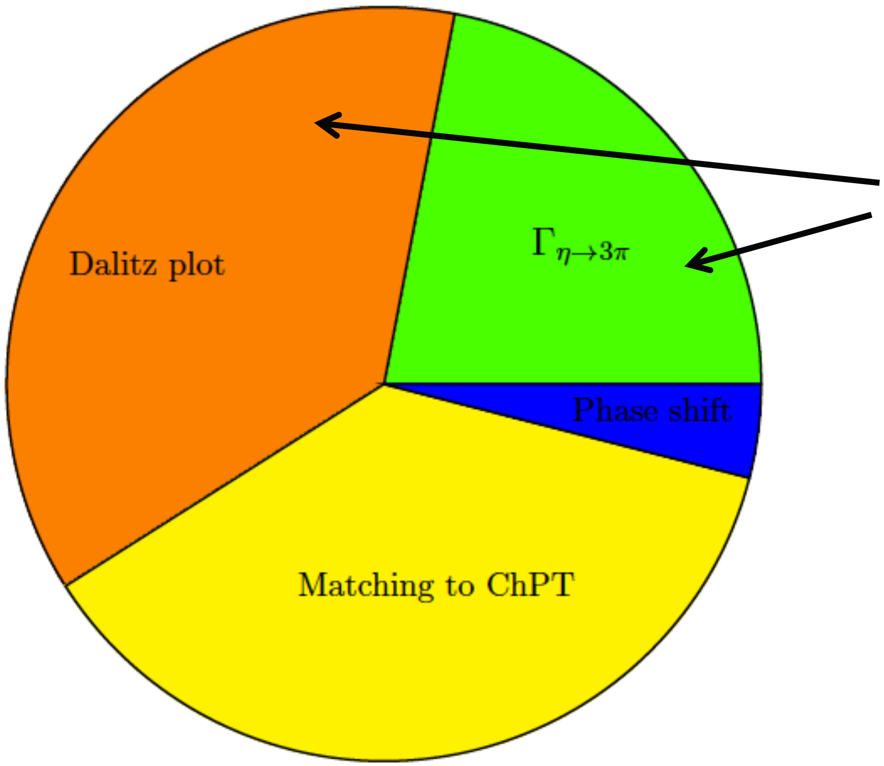


➡ Careful estimate of the uncertainties in progress

6.1 Introduction

- Attempt to quantify roughly the uncertainties

Preliminary



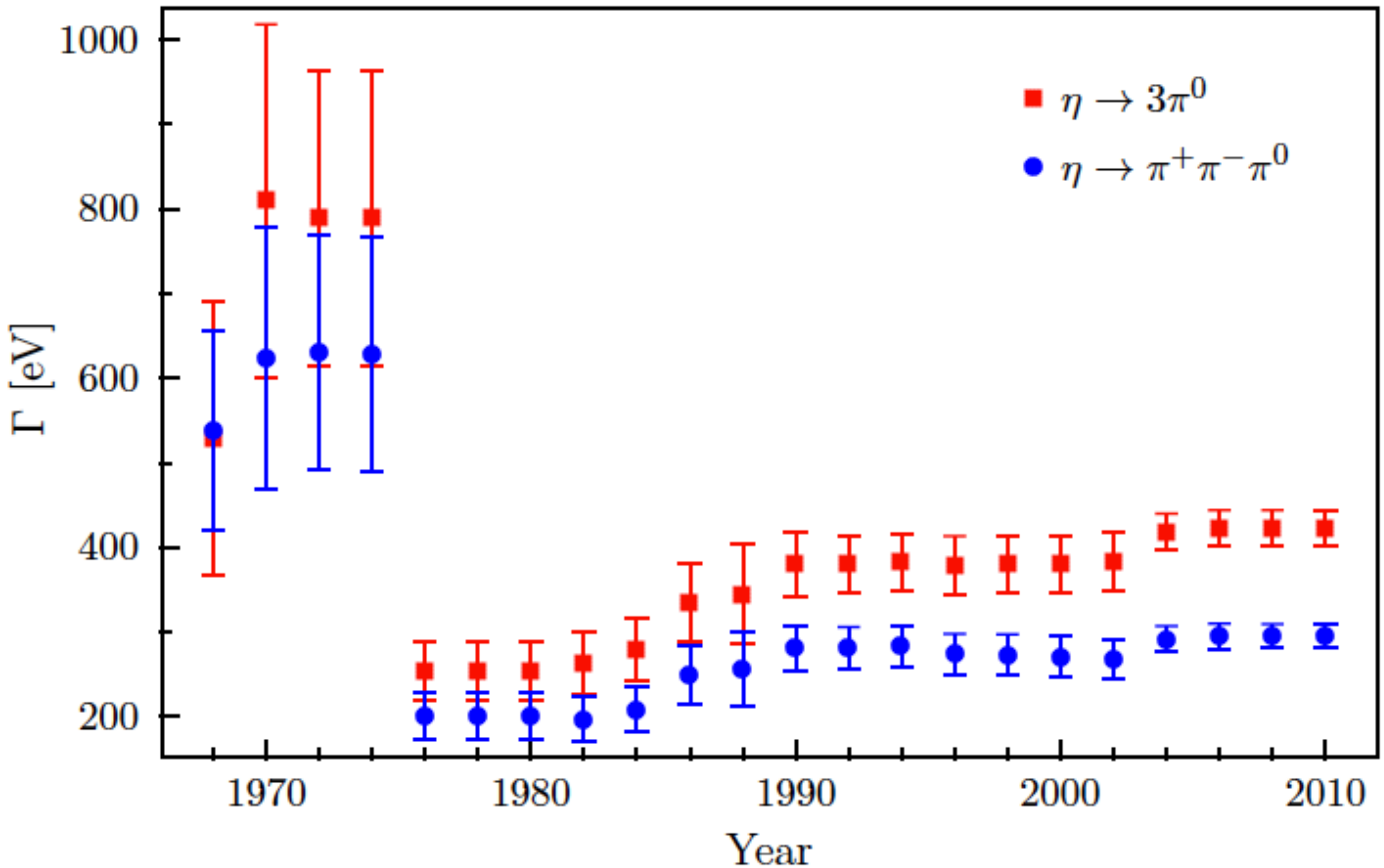
JLab experiments can help to reduce 2 sources of uncertainties

⇨ Careful estimate of the uncertainties in progress

6.2 $\eta \rightarrow 2\gamma$ via Primakoff experiment

- $\eta \rightarrow 2\gamma$ enters $\Gamma_{\eta\rightarrow 3\pi}$ determination :

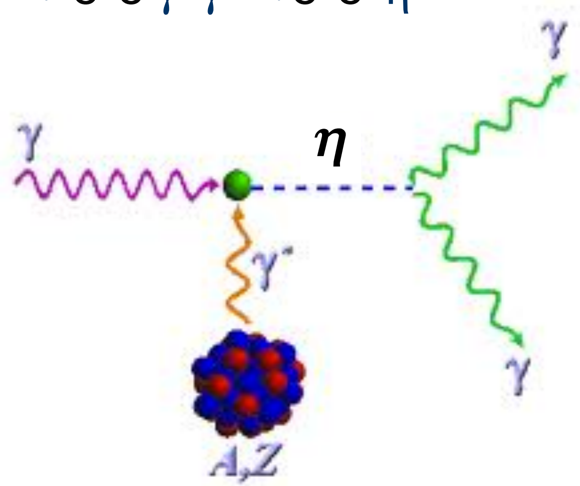
S. Lanz, PhD Thesis'11



- Large fluctuations mainly due to the total decay width fixed via the process $\eta \rightarrow 2\gamma$

6.2 $\eta \rightarrow 2\gamma$ via Primakoff experiment

- 2 different measurements:
  - 2 photons production: $e^+e^- \rightarrow e^+e^-\gamma^*\gamma^* \rightarrow e^+e^-\eta$
  - Primakoff production :



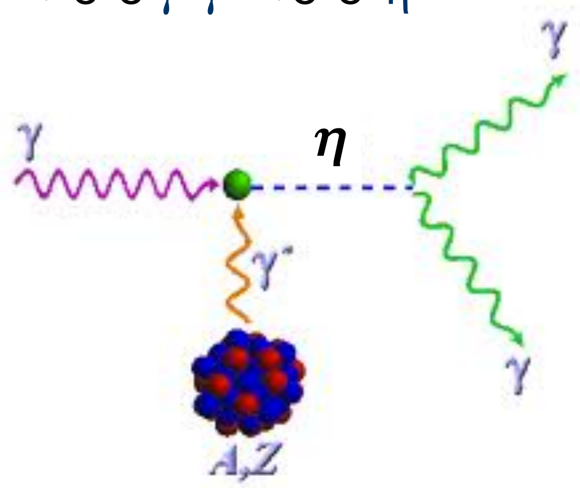
PrimEx

- 2 sets of measurements do not agree PDG'94:
  - 2 photons production, average : $\boxed{\Gamma(\eta \rightarrow 2\gamma) = 0.510 \pm 0.026 \text{ keV}}$
  - Primakoff measurement : $\boxed{\Gamma(\eta \rightarrow 2\gamma) = 0.324 \pm 0.046 \text{ keV}}$ *Browman'74*

- Primakoff measurement excluded from PDG average in 2004, need to be reamesured ⇨ *PrimEx* at Jlab!

6.2 $\eta \to 2\gamma$ via Primakoff experiment

- 2 different measurements:
  - 2 photons production: $e^+e^- \to e^+e^-\gamma^*\gamma^* \to e^+e^-\eta$
  - Primakoff production :



PrimEx

- Uncertainty on Q generated by the decay width input:

$$\Gamma_{\eta\to 3\pi} = 295 \pm 20 \text{ eV} \Rightarrow \boxed{Q \sim 22 \pm 0.31}$$

Overall expected uncertainty approximately ± 1.00

Possible improvement with new measurement?

6.3 Measurement of $\eta \to 3\pi$ at Jlab eta factory

- Only one recent published result for the Dalitz plot parameters in the charged channel by KLOE

$$\left|A_c(s,t,u)\right|^2 = N\left(1 + aY + bY^2 + cX + dX^2 + eXY + fY^3 + gX^3 + hX^2Y + lXY^2\right)$$

- ➢ Charge conjugation: ⇨ symmetry X ↔ -X
- ➢ h consistent with zero

Exp	a	b	d
KLOE	-1.090(-20)(+9)	0.124 (12)	0.057 (+9)(-17)
Crystal Barrel	-1.10 (4)	-	-
Layter	-1.08 (14)	-	-
Gormley	-1.15 (2)	0.16 (3)	-

a	-1.090 (5) (+ 8) (-19)
b	0.124 (6) (10)
c	0.002 (3) (1)
d	0.057 (6) (+7) (-16)
e	-0.006 (7) (5) (-3)
f	0.14 (1) (2)
$P(\chi^2)$	0,73

Talk by Ambrosino, Hadron'11

- One new analysis by WASA underway, CLAS?

6.3 Measurement of $\eta \rightarrow 3\pi$ at Jlab eta factory

- More information in the charged compared to the neutral channel
  ⇨ neutral channel sum over isospin:

$$\boxed{\overline{A}(s,t,u) = A(s,t,u) + A(t,u,s) + A(u,s,t)}$$

Only one Dalitz plot parameter determined α ⇨ $\boxed{\left|A_n(s,t,u)\right|^2 = N(1+2\alpha Z)}$

- Some possible inconsistencies between charged and neutral channel pointed out:

$$\alpha \leq \frac{1}{4}\left(b + d - \frac{1}{4}a^2\right) \Rightarrow \alpha = \frac{1}{4}\left(b + d - \frac{1}{4}a^2\right) + \Delta$$

Bijnens & Ghorbani'07

Schneider, Kubis, Ditche'11

  - Δ can be calculated using NREFT including $\pi\pi$ rescattering effects

    From KLOE Dalitz plot parameters ⇨ $\boxed{\alpha = -0.059(7)}$

    in disagreement with KLOE direct measurement and PDG average!

  - Disagreement due to predicted b two times larger than the experimental result : $\boxed{b_{\mathrm{NREFT}} = 0.308 > b_{KLOE} = 0.124}$

6.3 Measurement of $\eta \to 3\pi$ at Jlab eta factory

- Matching wih CHPT and experiment: main source of uncertainty on Q !
  Only statistical uncertainties ⇨ $\boxed{Q \sim 22 \pm 0.50}$

⇨ Improvement on the measurement of the charged channel would help to reduce the uncertainties on Q!

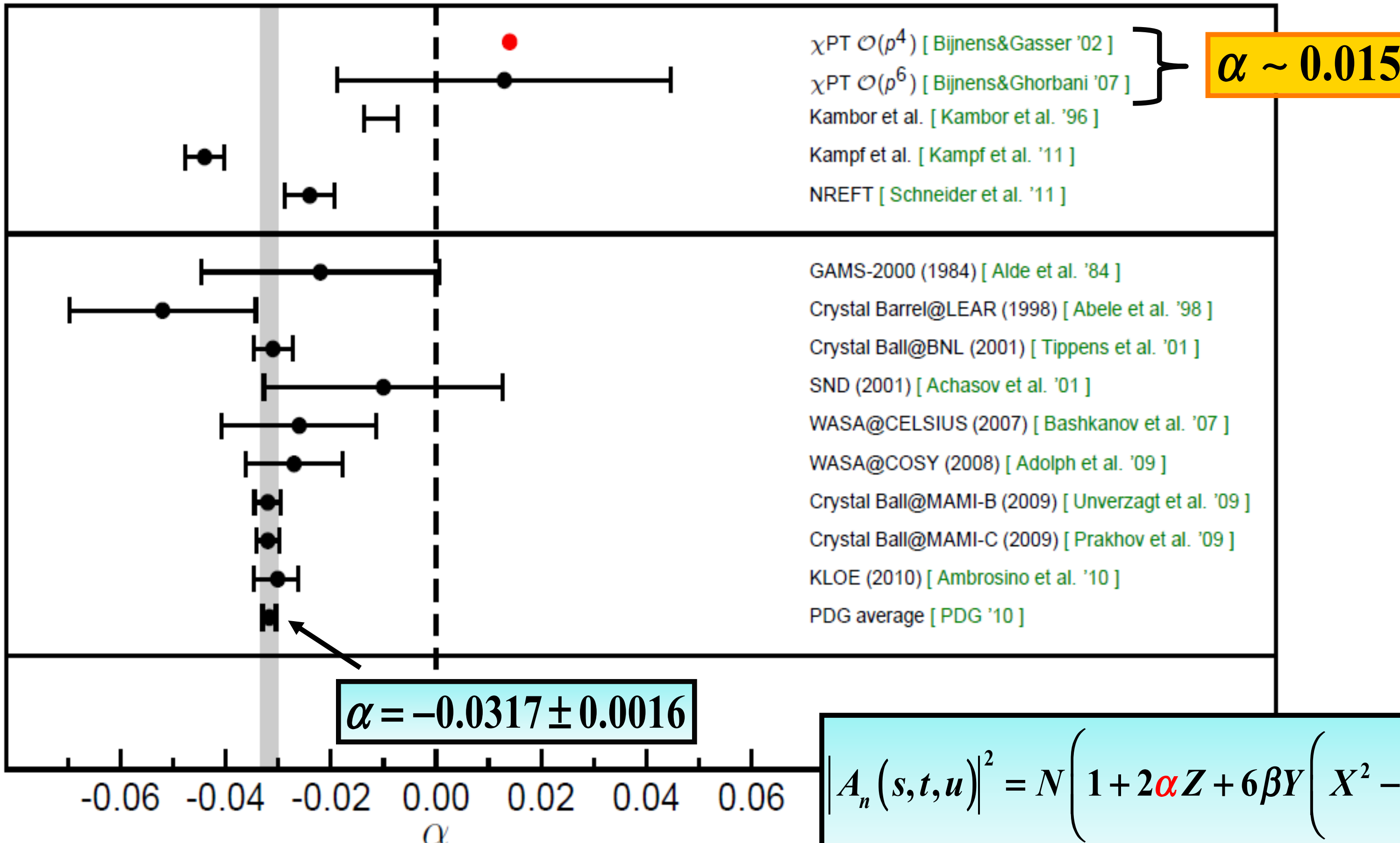
Can one do better at JLab?

- A dedicated experimental analysis using the dispersive approach to extract Q will allow for the *best determination,* systematics could be taken into account

  ⇨ use *basis functions*

6.3 Measurement of $\eta \to 3\pi$ at Jlab eta factory

- On the neutral channel: several experimental measurements:



$$\left|A_n(s,t,u)\right|^2 = N\left(1 + 2\alpha Z + 6\beta Y\left(X^2 - \frac{Y^2}{3}\right) + 2\gamma Z^2\right)$$

- *Any sensitivity to higher order coefficients?*

Comparison with original analysis

	$Q(\pi^+\pi^-\pi^0)$	$Q(3\pi^0)$	r
Results from Walker	22.8	22.9	1.43
My reproduction	22.74	22.87	1.425
$\delta_I(s)$	+0.14	+0.13	-0.004
L_3	+0.07	+0.11	+0.008
m_K	+0.22	+0.21	+0.000
m_π, m_η, F_π, Δ_F	+0.02	+0.02	-0.001
Γ	-0.45	-0.62	—
My result	22.74	22.72	1.428

$\mathbf{M_I}$

$$M_0(s) = \Omega_0(s)\left\{\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_0(s')\hat{M}_0(s')}{|\Omega_0(s')|(s'-s-i\epsilon)}\right\}$$

$$M_1(s) = \Omega_1(s)\left\{\beta_1 s + \frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta_1(s')\hat{M}_1(s')}{|\Omega_1(s')|(s'-s-i\epsilon)}\right\}$$

$$M_2(s) = \Omega_2(s)\frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta_2(s')\hat{M}_2(s')}{|\Omega_2(s')|(s'-s-i\epsilon)}$$

Comparison for Q

	Q	
dispersive (Walker)	22.8 ± 0.8	[Walker '98]
dispersive (Kambor et al.)	22.4 ± 0.9	[Kambor et al. '96]
dispersive (Kampf et al.)	23.3 ± 0.8	[Kampf et al. '11]
χPT, $\mathcal{O}(p^4)$	20.1	[Bijnens&Ghorbani '07]
χPT, $\mathcal{O}(p^6)$	22.9	[Bijnens&Ghorbani '07]
no Dashen violation	24.3	[Weinberg '77]
with Dashen violation	20.7 ± 1.2	[Anant et al. '04, Kastner&Neufeld '08]
lattice (FLAG average)	23.1 ± 1.5	[Colangelo et al. '10]
dispersive, matching	$22.74^{+0.68}_{-0.67}$	

Comparison for α

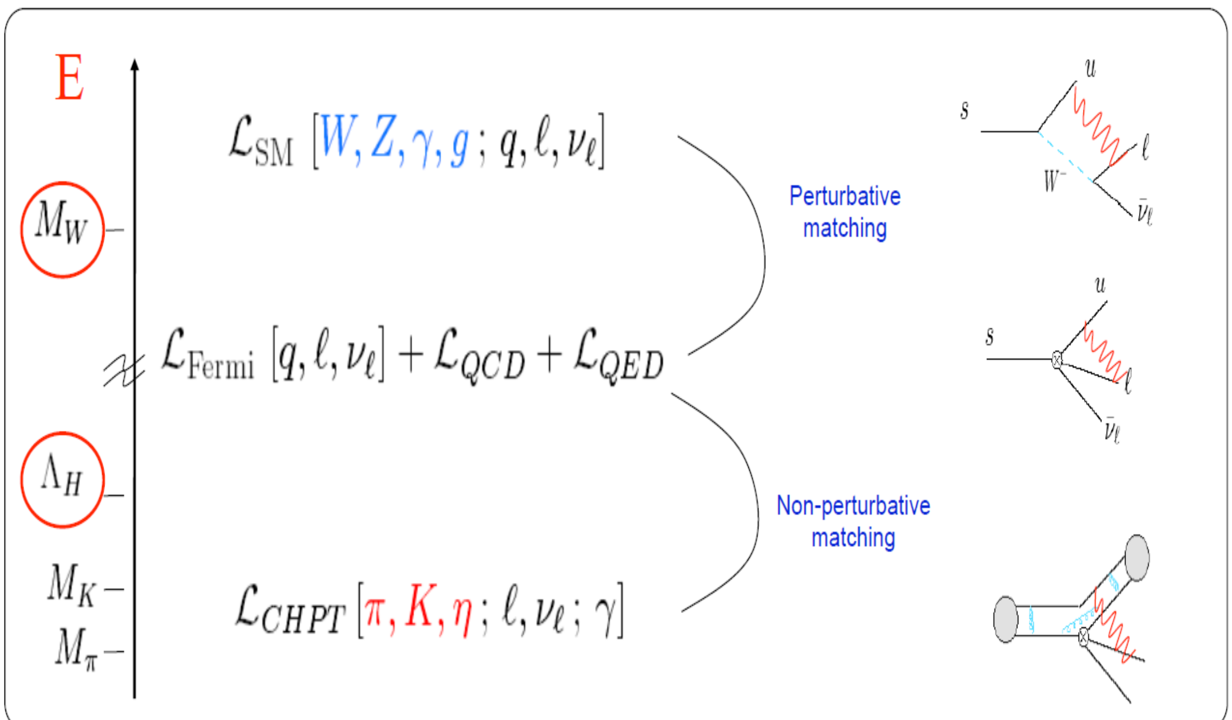
	α	
χPT $\mathcal{O}(p^4)$	0.014	[10]
χPT $\mathcal{O}(p^6)$	0.013 ± 0.032	[23]
Kambor et al.	$-0.014 \ldots -0.007$	[12]
Kampf et al.	-0.044 ± 0.004	[26]
NREFT	-0.024 ± 0.005	[28]
GAMS-2000 (1984)	-0.022 ± 0.023	[13]
Crystal Barrel@LEAR (1998)	-0.052 ± 0.018	[14]
Crystal Ball@BNL (2001)	-0.031 ± 0.004	[15]
SND (2001)	-0.010 ± 0.023	[16]
WASA@CELSIUS (2007)	-0.026 ± 0.015	[17]
WASA@COSY (2008)	-0.027 ± 0.0095	[18]
Crystal Ball@MAMI-B (2009)	-0.032 ± 0.0028	[19]
Crystal Ball@MAMI-C (2009)	-0.032 ± 0.0025	[20]
KLOE (2010)	$-0.0301\ ^{+0.0042}_{-0.0049}$	[21]
PDG average	-0.0317 ± 0.0016	[22]

1.6 Construction of an effective theory: ChPT

- Effective Field Theory approach: At a given *energy scale*
  - Degrees of freedom
  - Symmetries

  ⇒ *Decoupling* : Ex : To play pool you don't need to know the movement of earth around the sun

- *Ch*iral *P*erturbation *T*heory (*ChPT*)



Method: Representation of the amplitude

- Consider the s channel ⇨ Partial wave expansion of M(s,t,u):

$$\boxed{M(s,t,u) = f_0(s) + f_1(s)\cos\theta + \ldots}$$

- Elastic unitarity *Watson's theorem*

⇨ $$disc\left[f_1(s)\right] \propto t_1^*(s)\, f_1(s)$$

with $t_1(s)$ partial wave of elastic ππ scattering

- M(s,t,u) right-hand branch cut in the complex s-plane starting at the ππ threshold

- Left-hand cut present due to crossing

- Same situation in the t- and u-channel

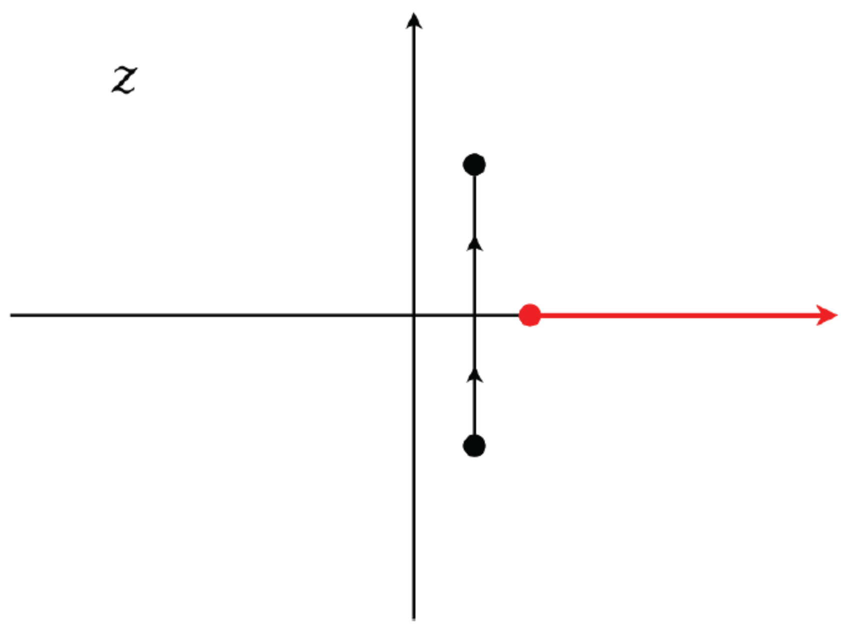
Discontinuities of the $M_I(s)$

- Ex: $\hat{M}_0(s) = \frac{2}{3}\langle M_0 \rangle + 2(s - s_0)\langle M_1 \rangle + \frac{20}{9}\langle M_2 \rangle + \frac{2}{3}\kappa(s)\langle zM_1 \rangle$

  where $\langle z^n M_I \rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I(t(s,z))$, $z = \cos\theta$ scattering angle

  Non trivial angular averages ⇨ need to deform the integration path to avoid crossing cuts

  Anisovich & Anselm'66



Discontinuities of the $M_I(s)$

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  where $\langle z^n M_I \rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I(t(s,z))$, $z = \cos\theta$ scattering angle

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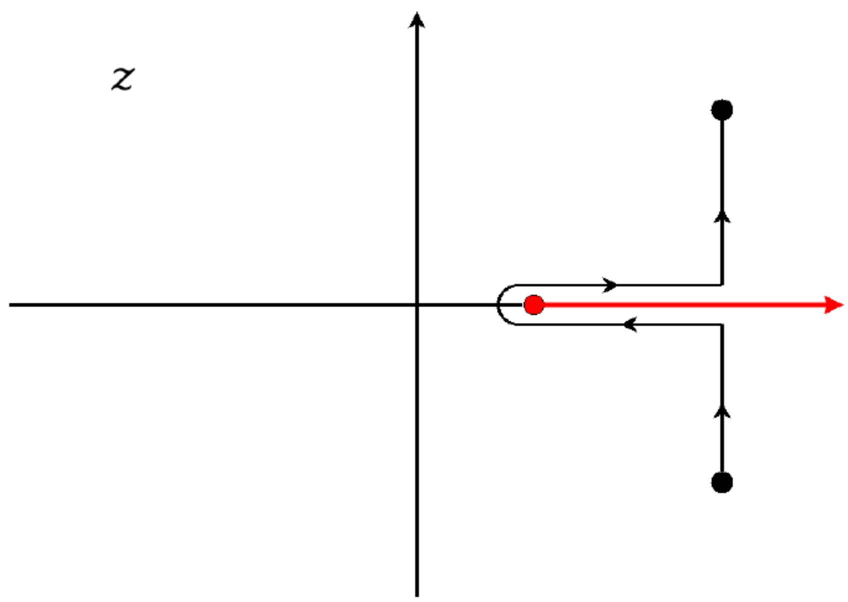
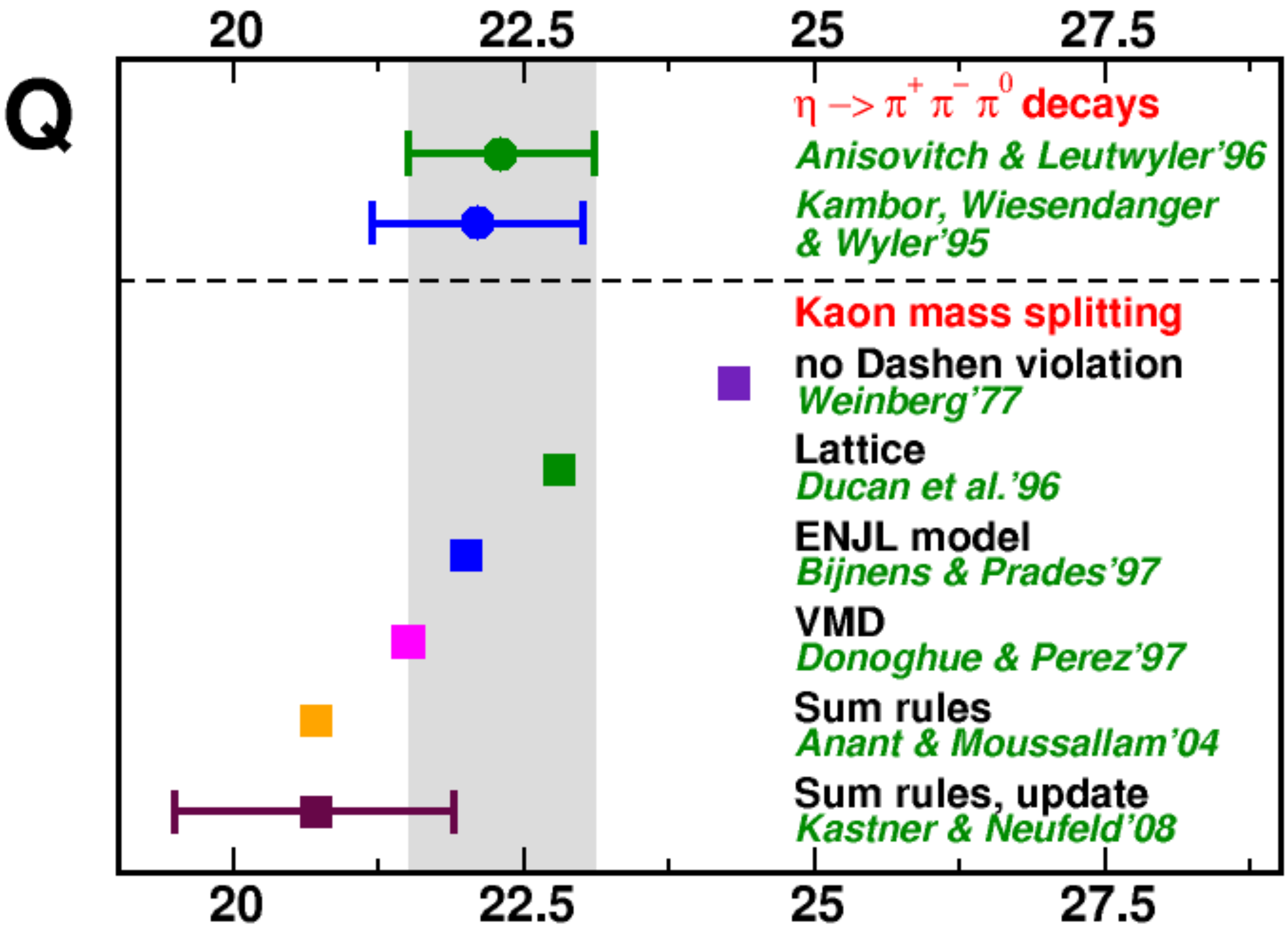


3.7 Comparison of values of Q



$Q = 22.3 \pm 0.9$

$Q = 22.1 \pm 0.9$

$Q = 24.2$

$Q = 22.8$

$Q = 22$

$Q = 21.5$

$Q = 20.7$

$Q = 20.7 \pm 1.2$

⇨ Fair agreement with the determination from meson masses

Comparison with Q from meson mass splitting

- $Q^2 = \dfrac{M_K^2}{M_\pi^2}\dfrac{M_K^2 - M_\pi^2}{M_{K^0}^2 - M_{K^+}^2}\left[1 + O(m_q^2)\right]$ is only valid for e=0

- Including the electromagnetic corrections, one has

$$\boxed{Q_D^2 \equiv \frac{(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 + M_{\pi^0}^2)(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2 - M_{\pi^0}^2)}{4M_{\pi^0}^2(M_{K^0}^2 - M_{K^+}^2 + M_{\pi^+}^2 - M_{\pi^0}^2)}}$$

$\Rightarrow$ $Q_D = 24.2$

- Corrections to the Dashen's theorem

$\Rightarrow$ The corrections can be large due to e^2m_s corrections:

$$\boxed{\left(M_{K^+}^2 - M_{K^0}^2\right)_{\mathrm{em}} - \left(M_{\pi^+}^2 - M_{\pi^0}^2\right)_{\mathrm{em}} = e^2 M_K^2\left(A_1 + A_2 + A_3\right) + O\left(e^2 M_\pi^2\right)}$$

Urech'98,
Ananthanarayan & Moussallam'04

3.6 Corrections to Dashen's theorem

- Dashen's Theorem

$$\left(M^2_{K^+} - M^2_{K^0}\right)_{\text{em}} = \left(M^2_{\pi^+} - M^2_{\pi^0}\right)_{\text{em}} \Rightarrow \boxed{\left(M_{K^+} - M_{K^0}\right)_{\text{em}} = 1.3 \text{ MeV}}$$

- With higher order corrections
  - Lattice : $\left(M_{K^+} - M_{K^0}\right)_{\text{em}} = 1.9 \text{ MeV}, Q = 22.8$ *Ducan et al.'96*
  - ENJL model: $\left(M_{K^+} - M_{K^0}\right)_{\text{em}} = 2.3 \text{ MeV}, Q = 22$ *Bijnens & Prades'97*
  - VMD: $\left(M_{K^+} - M_{K^0}\right)_{\text{em}} = 2.6 \text{ MeV}, Q = 21.5$ *Donoghue & Perez'97*
  - Sum Rules: $\left(M_{K^+} - M_{K^0}\right)_{\text{em}} = 3.2 \text{ MeV}, Q = 20.7$ *Anant & Moussallam'04*

  Update $\Rightarrow$ $Q = 20.7 \pm 1.2$ *Kastner & Neufeld'07*

4.2 Method: Representation of the amplitude

- Decomposition of the amplitude as a function of isospin states

$$\boxed{M(s,t,u) = M_0(s) + (s-u)M_1(t) + (s-t)M_1(u) + M_2(t) + M_2(u) - \frac{2}{3}M_2(s)}$$

Fuchs, Sazdjian & Stern'93

Anisovich & Leutwyler'96

  - M_I isospin *I* rescattering in two particles
  - Amplitude in terms of S and P waves ⇨ exact up to NNLO ($\mathcal{O}(p^6)$)
  - Main two body rescattering corrections inside M_I

- Functions of only one variable with only right-hand cut of the partial wave ⇨ $disc\left[M_I(s)\right] \equiv disc\left[f_1^I(s)\right]$

- Elastic unitarity *Watson's theorem*

$$\boxed{disc\left[f_1^I(s)\right] \propto t_1^*(s) f_1^I(s)}$$

with $t_1(s)$ partial wave of elastic $\pi\pi$ scattering

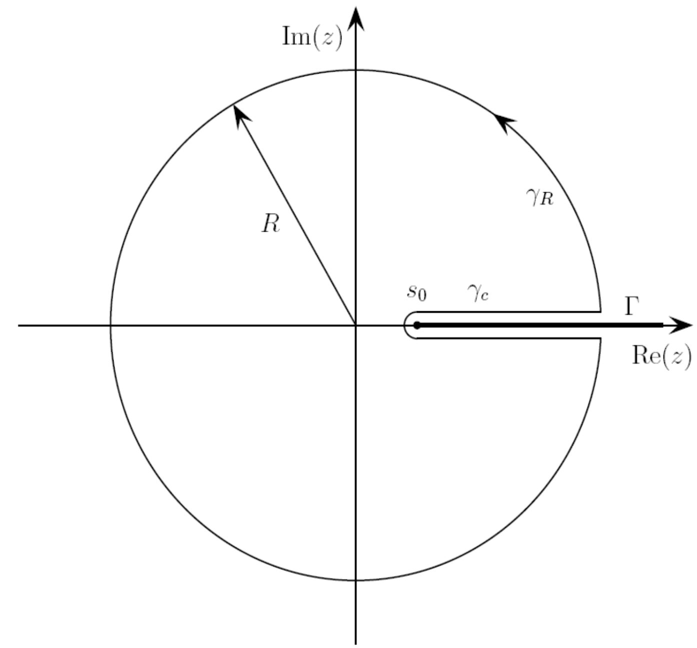
4.2 Method: Representation of the amplitude

- Knowing the discontinuity of M_I ⇨ write a dispersion relation for it

- Cauchy Theorem and Schwarz reflection principle

⇨ $$M_I(s) = \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} \frac{disc\left[M_I(s')\right]}{s'-s-i\varepsilon} ds'$$

M_I can be reconstructed everywhere from the knowledge of $disc\left[M_I(s)\right]$



- If M_I doesn't converge fast enought for $|s| \to \infty$ ⇨ subtract the dispersion relation

$$M_I(s) = P_{n-1}(s) + \frac{s^n}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'^n} \frac{disc\left[M_I(s')\right]}{(s'-s-i\varepsilon)}$$

$P_{n-1}(s)$ polynomial

4.3 Hat functions

- Discontinuity of M_I: by definition $disc\left[M_I(s)\right] \equiv disc\left[f_1^I(s)\right]$

  $\Rightarrow$ $\boxed{f_1^I(s) = M_I(s) + \hat{M}_I(s)}$

  with $\hat{M}_I(s)$ real on the right-hand cut

- The left-hand cut is contained in $\hat{M}_I(s)$

- Determination of $\hat{M}_I(s)$:
  subtract M_I from the partial wave projection of $M(s,t,u)$

  $$M(s,t,u) = M_0(s) + \left(s-u\right) M_1(t) + \ldots$$

- $\hat{M}_I(s)$ singularities in the t and u channels, depend on the other M_I
  Angular averages of the other functions $\Rightarrow$ Coupled equations

4.3 Hat functions

- Ex: $$\boxed{\hat{M}_0(s) = \frac{2}{3}\langle M_0\rangle + 2(s-s_0)\langle M_1\rangle + \frac{20}{9}\langle M_2\rangle + \frac{2}{3}\kappa(s)\langle zM_1\rangle}$$

where $$\langle z^n M_I\rangle(s) = \frac{1}{2}\int_{-1}^{1} dz\, z^n M_I(t(s,z)),$$

$$z = \cos\theta$$ scattering angle

Non trivial angular averages ⇨ need to deform the integration path to avoid crossing cuts

Anisovich & Anselm'66

4.4 Dispersion Relations for the $M_I(s)$

- Elastic Unitarity

$$[l = 1 \text{ for } I = 1,\ l = 0 \text{ otherwise}]$$

$$\boxed{disc\left[M_I\right] = disc\left[f_l^I(s)\right] = \theta\left(s - 4M_\pi^2\right)\left[M_I(s) + \hat{M}_I(s)\right]\sin\delta_l^I(s)e^{-i\delta_l^I(s)}}$$

$\pi\pi$ phase shift

δ_l^I phase of the partial wave $f_l^I(s)$

Watson theorem: elastic $\pi\pi$ scattering phase shifts

- Solution: Inhommogeneous Omnès problem

$$\boxed{M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0^0(s')\,\hat{M}_0(s')}{\left|\Omega_0(s')\right|\left(s' - s - i\varepsilon\right)}\right)}$$

Omnès function

$$\left[\Omega_I(s) = \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds'\frac{\delta_l^I(s')}{s'(s' - s - i\varepsilon)}\right)\right]$$

Similarly for M_1 and M_2

4.4 Dispersion Relations for the $M_I(s)$

- $$\boxed{M_0(s) = \Omega_0(s)\left(\alpha_0 + \beta_0 s + \gamma_0 s^2 + \frac{s^3}{\pi}\int_{4M_\pi^2}^{\infty}\frac{ds'}{s'^3}\frac{\sin\delta_0^0(s')\,\hat{M}_0(s')}{|\Omega_0(s')|\left(s'-s-i\varepsilon\right)}\right)}$$

  Omnès function

  Similarly for M_1 and M_2

- Four subtraction constants to be determined: α_0, β_0, γ_0 and one more in M_1 (β_1)

- Inputs needed for these and for the ππ phase shifts δ_l^I
  - M_0: ππ scattering, ℓ=0, I=0
  - M_1: ππ scattering, ℓ=1, I=1
  - M_2: ππ scattering, ℓ=0, I=2

- Solve dispersion relations numerically by an iterative procedure

5.4 Comparison with KKNZ

- Adler zero not reproduced!