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Radiation from a charge rotating inside a cylindrical grating

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Spectral-angular distribution is investigated for the radiation emitted by a point charge moving along a helical trajectory inside a cylindrical grating with conducting strips

Smith-Purcell effect usually is considered for sources with rectilinear motion and the radiation is not superposed with other types of the radiation

In the problem under consideration two types of the radiation processes are realized: Synchrotron and Smith-Purcell radiations

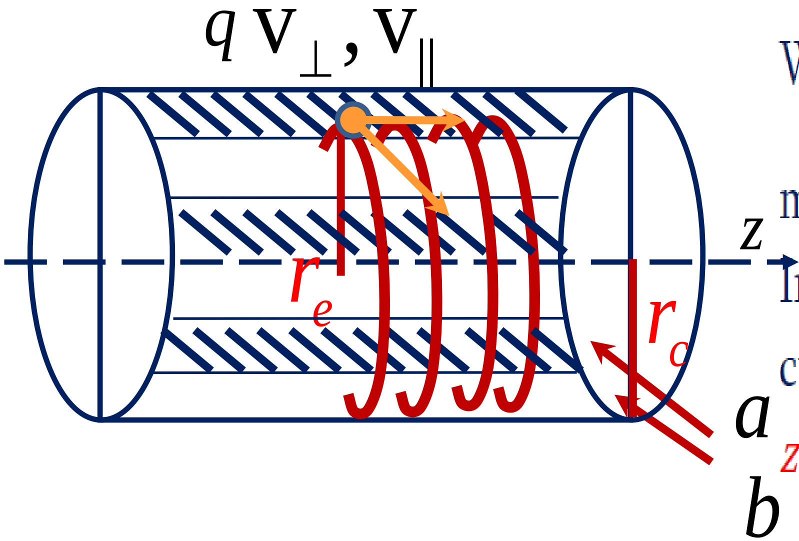
Primary source for both types of these emissions is the electromagnetic field of the charged particle, unlike to the synchrotron radiation the Smith-Purcell radiation is formed by the medium as a result of its polarization by the charge field

Helical motion of an electron beam is employed in helical undulators to produce circularly polarized radiation in narrow angular cone in the forward direction

Helical undulator radiation was used to generate a polarized positron beam

Synchrotron radiation from relativistic electrons spiraling in magnetic fields is the main mechanism to explain the emissions of many objects in radio astronomy

Geometry of the problem



We assume that the system is immersed in a homogeneous medium with dielectric permittivity ε.
In accordance with the symmetry of the problem, cylindrical coordinates $(r; \varphi; z)$ will be used with the axis z directed along the axis of the grating.

We consider the radiation of a relativistic charged particle q, moving along a circular helical trajectory of radius r_e inside a infinitely long coaxial cylindrical grating. The latter consists metallic strips with width a situated on the rings with radius $r_c < r_e$. The distance between the strips will be denoted by b.

Geometry of the problem

The radius of the circular orbit and the angular frequency of motion are expressed in terms of the external magnetic field and of the particle velocity by the formulas

$$r_c = \frac{m_e c v_\perp}{qH_{ext}}\gamma, \quad \omega_0 = \frac{v_\perp}{r_e} = \frac{qH_{ext}}{m_e c}$$

The strips are located in the angular regions

$$\frac{2\pi m}{N} \le \varphi \le \varphi_0 + \frac{2\pi m}{N}$$

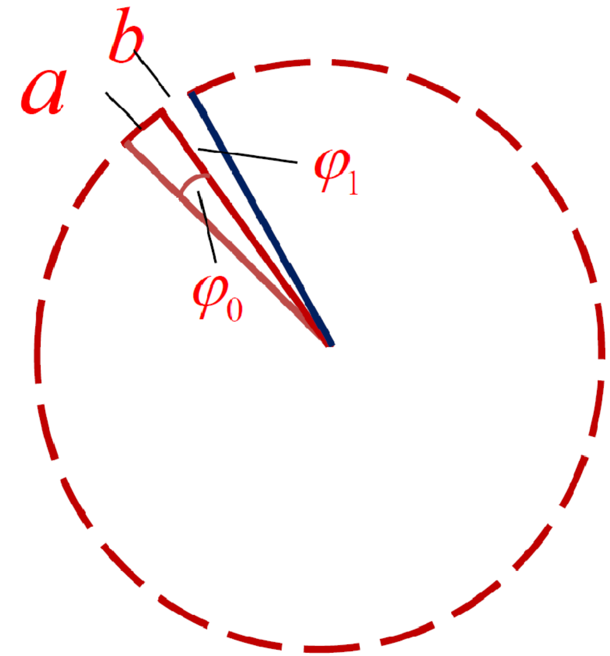
where $m = 0, 1, 2, \ldots, N$

Angular width of the strips $\varphi_0 = \dfrac{a}{r_e}$

Angular period of the grating $\varphi_1 = \dfrac{a+b}{r_e}$

Number of periods in the grating

$$N = \frac{2\pi r_c}{a+b}$$



Method for Calculation

- The problem is mathematically complicated and an exact analytical result is not available. This is already the case for simpler problems of the Smith-Purcell radiation from planar gratings. For the latter geometry various approximate analytic and numerical methods have been developed for the evaluation of the spectral-angular distribution of the radiation intensity (for the comparison of different models in the calculations of the Smith-Purcell radiation intensity see [1]).

- The approximation by the surface currents induced on the strips by the field of the moving charge for planar gratings has been discussed in [2, 3] (for further developments of the method see [4]).

[1] D.V. Karlovets and A.P. Potylitsyn, Phys. Rev. ST Accel. Beams 9, 080701 (2006).
[2] J. Walsh, K. Woods, and S. Yeager, Nucl. Instrum. Methods Phys. Res., Sect. A 341, 277 (1994).
[3] J.H. Brownell, J. Walsh, and G. Doucas, Phys. Rev. E 57, 1075 (1998).
[4] D.V. Karlovets and A.P. Potylitsyn, Phys. Lett. A 373, 1988 (2009);
D.V. Karlovets and A.P. Potylitsyn, arXiv:0908.2336; D.V. Karlovets, JETP 113, 27 (2011).

Electromagnetic fields

We are interested in the radiation intensity at large distances from the grating

We will assume that the charge trajectory is sufficiently close to the grating and will approximate the radiation field by the field from the source

Region between the strips

$$j_l = \begin{cases} \dfrac{q}{r} v_l \delta(r - r_e)\delta(\varphi - \omega_0 t)\delta(z - v_\| t), & \text{for} \quad m\varphi_0 + m\varphi_1 \le \varphi \le (m+1)\varphi_1 \\ 0, & \text{for} \quad m\varphi_1 \le \varphi \le m\varphi_1 + \varphi_0 \end{cases}$$

$m = 0, 1, 2, \ldots, N - 1,$

Locations of the strips

$$v_r = 0$$

$$v_\varphi = \omega_0 r_e$$

$$v_z = v_\|$$

This corresponds to rectangular oscillations of the charge

Approximation allows us to obtain closed expressions for the electromagnetic fields and radiation intensity in the region outside the grating

Electromagnetic fields

Having the current density, the vector potential is expressed as

$$A_i(\mathbf{r},t) = -\frac{1}{2\pi^2 c}\int d\mathbf{r}' dt' G_{il}(\mathbf{r},t,\mathbf{r}',t') j_l(\mathbf{r}',t')$$

Electromagnetic field Green function in a homogeneous medium with permittivity ε and summation over *l* is understood

Vector potential is presented as the Fourier expansion

$$A_l(x) = \sum_{n,m=-\infty}^{+\infty}\int_{-\infty}^{+\infty} dk_z\, e^{i(n+mN)\varphi - i\omega_n(k_z)t + ik_z z} A_{nml}(k_z,r),$$

Fourier coefficients

$$A_{nml}(k_z,r) = \frac{i^{1-l} q v_\perp}{4c} s_m \sum_{\alpha=\pm 1} \alpha^l J_{n+Nm+\alpha}(\lambda r_e) H_{n+Nm+\alpha}(\lambda r),$$

$$A_{nm3}(k_z,r) = \frac{q v_\parallel}{2ic} s_m J_{n+Nm}(\lambda r_e) H_{n+Nm}(\lambda r),$$

$$\boxed{s_m = \begin{cases} \frac{1}{\pi m} e^{-imN\varphi_0/2} \sin(mN\varphi_0/2), & m \neq 0, \\ N\varphi_0/2\pi - 1, & m = 0. \end{cases}}$$

$$\boxed{\lambda = \sqrt{\omega_n^2(k_z)\varepsilon/c^2 - k_z^2}} \quad \boxed{\omega_n(k_z) = n\omega_0 + k_z v_\parallel}$$

Radiation intensity

Average energy flux per unit time through the cylindrical surface of radius **r > r$_c$**, coaxial with the grating, is given by the expression

$$I = \frac{c}{4\pi T}\int_0^{2\pi} d\varphi \int_0^T dt \int_{-\infty}^{\infty} dz\, r\, [\mathbf{E}\times\mathbf{H}]_r$$

$T = 2\pi/\omega_0$ and $\mathbf{n}_r$ is the unit vector along the radial direction r

Angular density of the radiation intensity on a given harmonic

$$\frac{dI_n}{d\Omega} = \frac{q^2 n^2 \omega_0^2}{2\pi c\sqrt{\varepsilon}\left|1-\beta_{\|}\cos\theta\right|^3}\sum_{m=-\infty}^{+\infty}|s_m|^2\left\{\beta_\perp^2 J'^2_{n+Nm}(nu(\theta))\right.$$

$$\left. + \left[\cos\theta\left(1+\left(1-\beta_{\|}\cos\theta\right)Nm/n\right)-\beta_{\|}\right]^2 \frac{J^2_{n+Nm}\left(nu(\theta)\right)}{\sin^2\theta}\right\}$$

$$d\Omega = \sin\theta d\theta d\phi \qquad\qquad u(\theta) = \frac{\beta_\perp \sin\theta}{1-\beta_{\|}\cos\theta}.$$

The spectrum of radiation

For the radiation at $n \neq 0$ harmonics, the allowed values for k_z are determined from the inequality $\lambda^2 > 0$. The latter is rewritten as

$$k_z^2(1 - \beta_{\|}^{-2}) + 2k_z n\omega_0/v_{\|} + (n\omega_0/v_{\|})^2 > 0$$

Introducing a new angular variable θ, $0 \leqslant \theta \leqslant \pi$, the solution of this inequality is presented as

$$k_z = \frac{n\omega_0}{c} \frac{\sqrt{\varepsilon}\cos\theta}{1 - \beta_{\|}\cos\theta}$$

$$\omega_n(\theta) = |\omega_n(k_z)| = \frac{n\omega_0}{|1 - \beta_{\|}\cos\theta|}, \quad n = 1, 2, \ldots,$$

This corresponds to wave with frequency propagating at the angle ϑ with respect to the z-axis. Formula describes the normal Doppler effect in the cases $\beta_{\|} < 1$ and $\beta_{\|} > 1$, $\vartheta > \vartheta_C$, $\vartheta_C = \arccos(1/\beta_{\|})$, and anomalous Doppler effect inside the Cherenkov cone, $\vartheta < \vartheta_C$ in the case $\beta_{\|} > 1$.

Limiting case

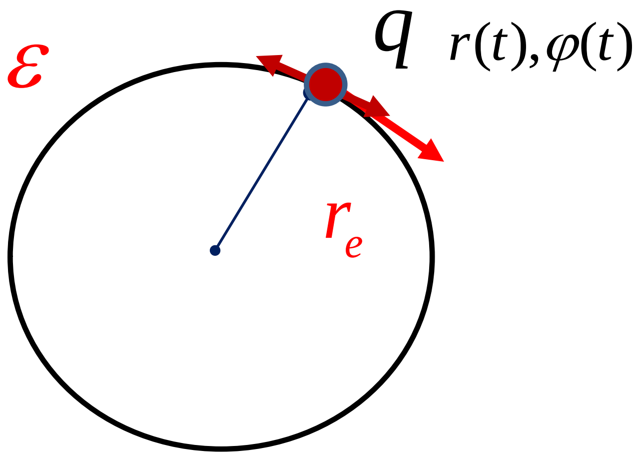
- For $a = 0$ the only nonzero contribution comes from the term $m = 0$, with $|s_0|^2 = 1$, and we obtain the radiation intensity for a charge moving in a homogeneous medium:

$$\frac{dI_n^{(0)}}{d\Omega} = \frac{q^2 n^2 \omega_0^2}{2\pi c\sqrt{\varepsilon}} \left[\beta^2 J_n'^2(n\beta \sin\theta) + \cot^2\theta J_n^2(n\beta \sin\theta)\right]$$

- Another limiting case $b \to 0$ corresponds to a charge rotating inside a cylindrical waveguide. In this case the field of the charge is screened by the waveguide and the radiation intensity in the exterior region vanishes

The radiation of rotating longitudinal oscillator

Consider the case when the electron moving by spiral trajectory in a homogeneous dielectric medium and at the same time is oscillating along the direction of the motion



$$v_\varphi(t) = v_0\left[1 + a\cdot cos\left(n_0\omega_0 t\right)\right],$$
$$v_r = 0,$$
$$v_z = v_{||}t:$$

The angular velocity of the rotation of the particle and the angular coordinate will be determined as follows:

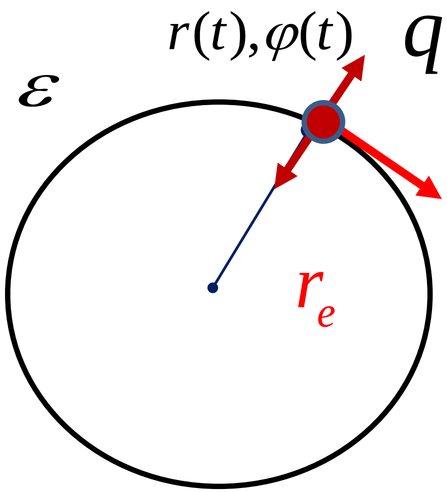
$$\Omega_\varphi(t) = \omega_0\left[1 + a\cdot cos\left(y\omega_0 t\right)\right]$$

$$\varphi_e(t) = \omega_0\left[t + \frac{a}{\omega_0 y} sin\left(y\omega_0 t\right)\right]$$

here n_0 is an integer

The radiation of rotating transversal oscillator

Consider the case when the electron moving by spiral trajectory in a homogeneous dielectric medium and at the same time is oscillating perpendicular the direction of the motion



$$r(t) = r_e\left[1 + a\cdot\cos\left(n_0\omega_0 t\right)\right]$$

$$\varphi(t) \approx \omega_0 t - \frac{a}{n_0}\cdot\sin\left(n_0\omega_0 t\right)$$

$$z(t) = \mathrm{v}_{\|} t$$

The angular velocity of the rotation of the particle and the angular coordinate will be determined as follows:

$$\mathrm{v}_r(t) = -n_0\mathrm{v}_0 a\cdot\sin\left(n_0\omega_0 t\right)$$

$$\mathrm{v}_\varphi = \mathrm{v}$$

$$\mathrm{v}_z \approx \mathrm{v}_{\|}$$

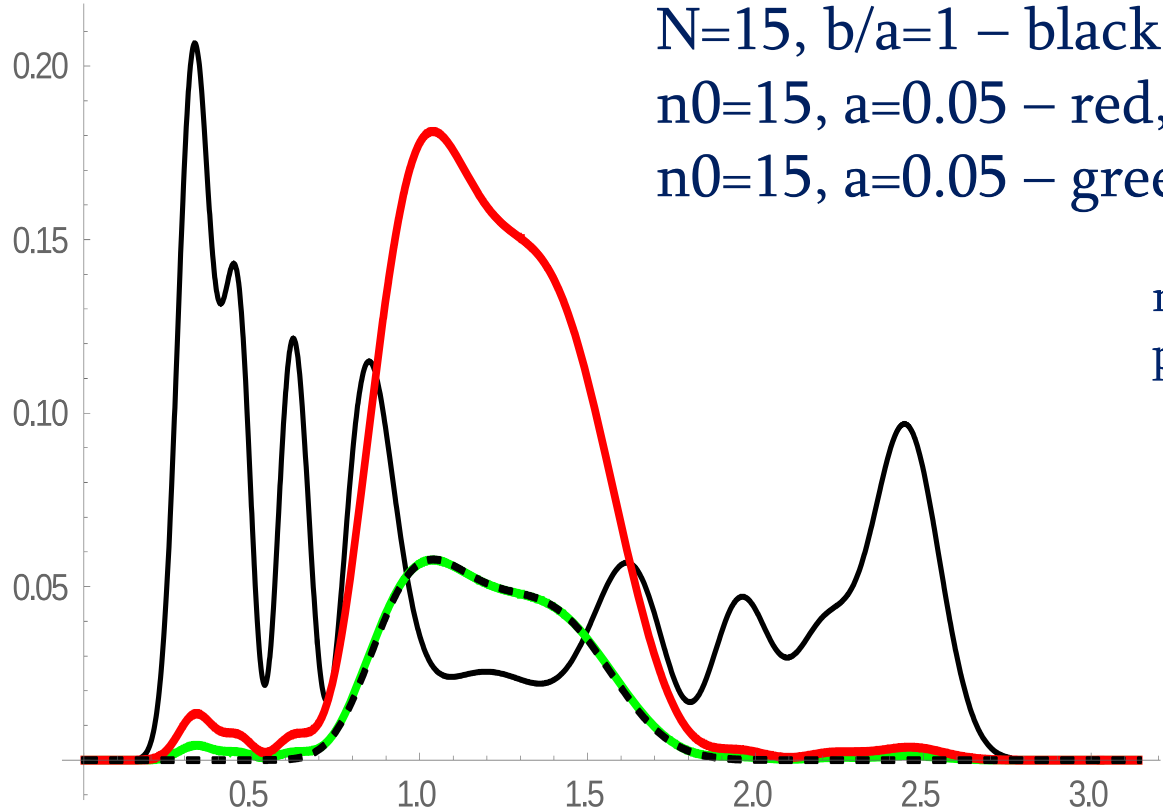
$$\boxed{a << 1}$$

Estimates

Number of the radiated quanta on the harmonic *n=25* versus θ

for the electron with the energy 2 MeV, ($\beta_{\parallel} = 0.35, \quad \beta_{\perp} = 0.9$)

For the values of the other parameters we have taken



N=15, b/a=1 – black, with grating

n0=15, a=0.05 – red, longitudinal oscillator

n0=15, a=0.05 – green, transversal oscillator

n0 - Number of oscillations per period of rotation

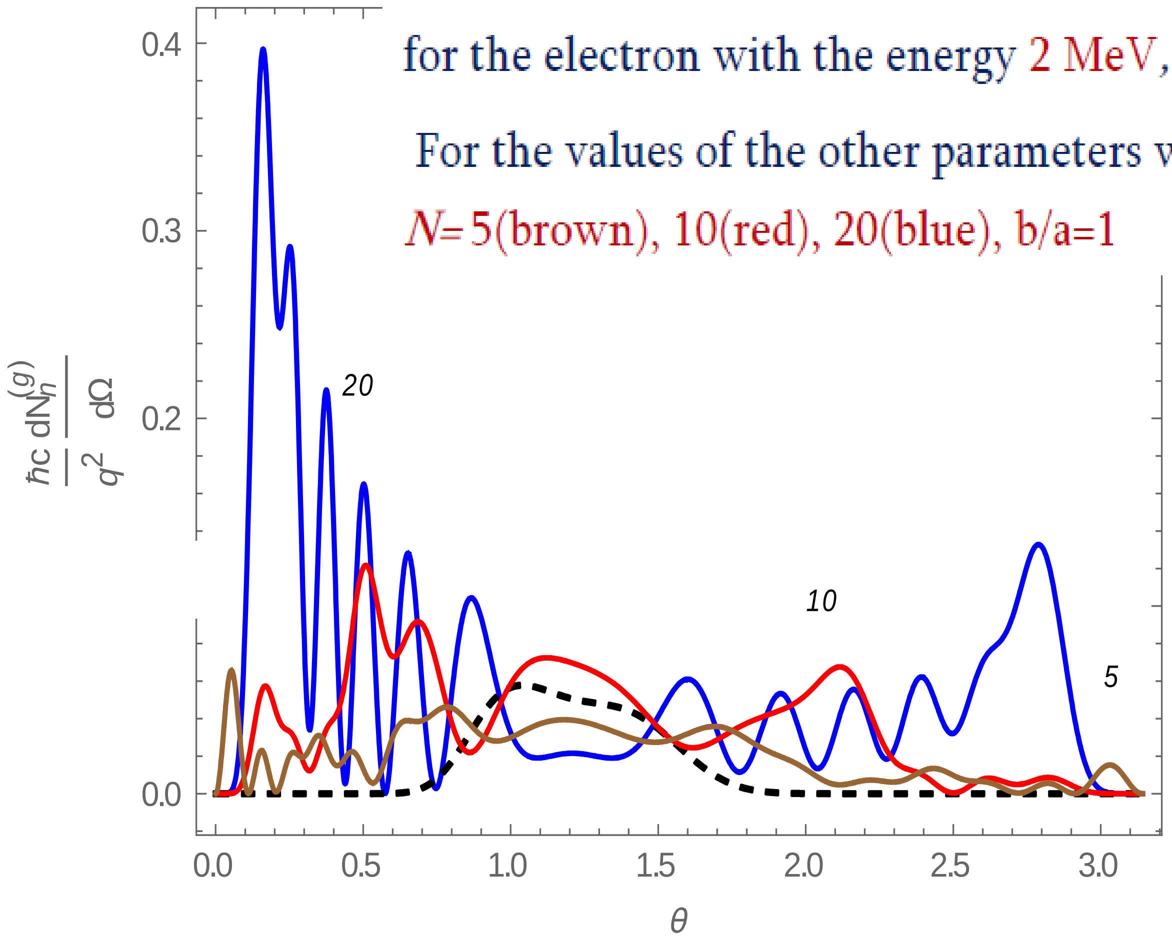
Numerical analysis

Number of the radiated quanta on the harmonic *n=25* versus θ

for the electron with the energy 2 MeV, ($\beta_{\parallel} = 0.35, \quad \beta_{\perp} = 0.9$)

For the values of the other parameters we have taken

N=5(brown), 10(red), 20(blue), b/a=1

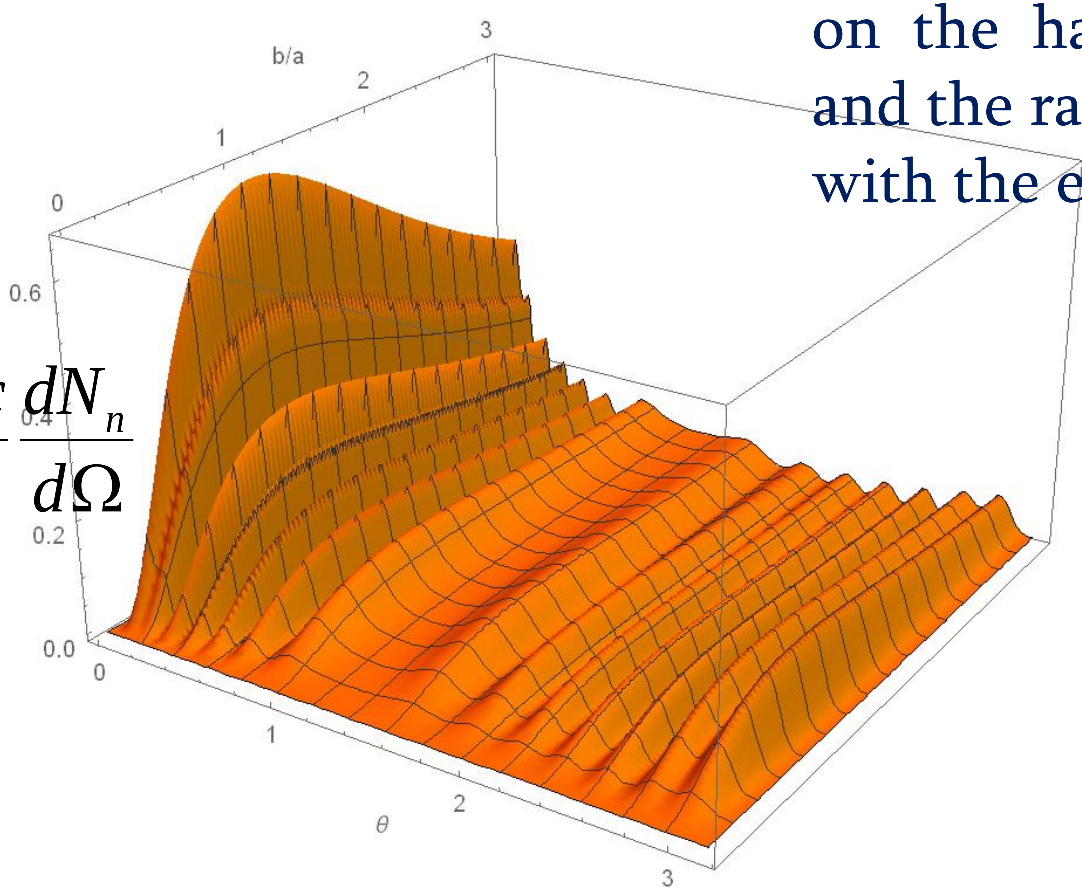


For the angles not close to the rotation plane, the radiation intensity is dominated by the Smith-Purcell part

Numerical analysis

Number of the radiated quanta on the harmonic *n= 25* versus θ and the ratio *b/a* for the electron with the energy *2 MeV*, $N = 28$

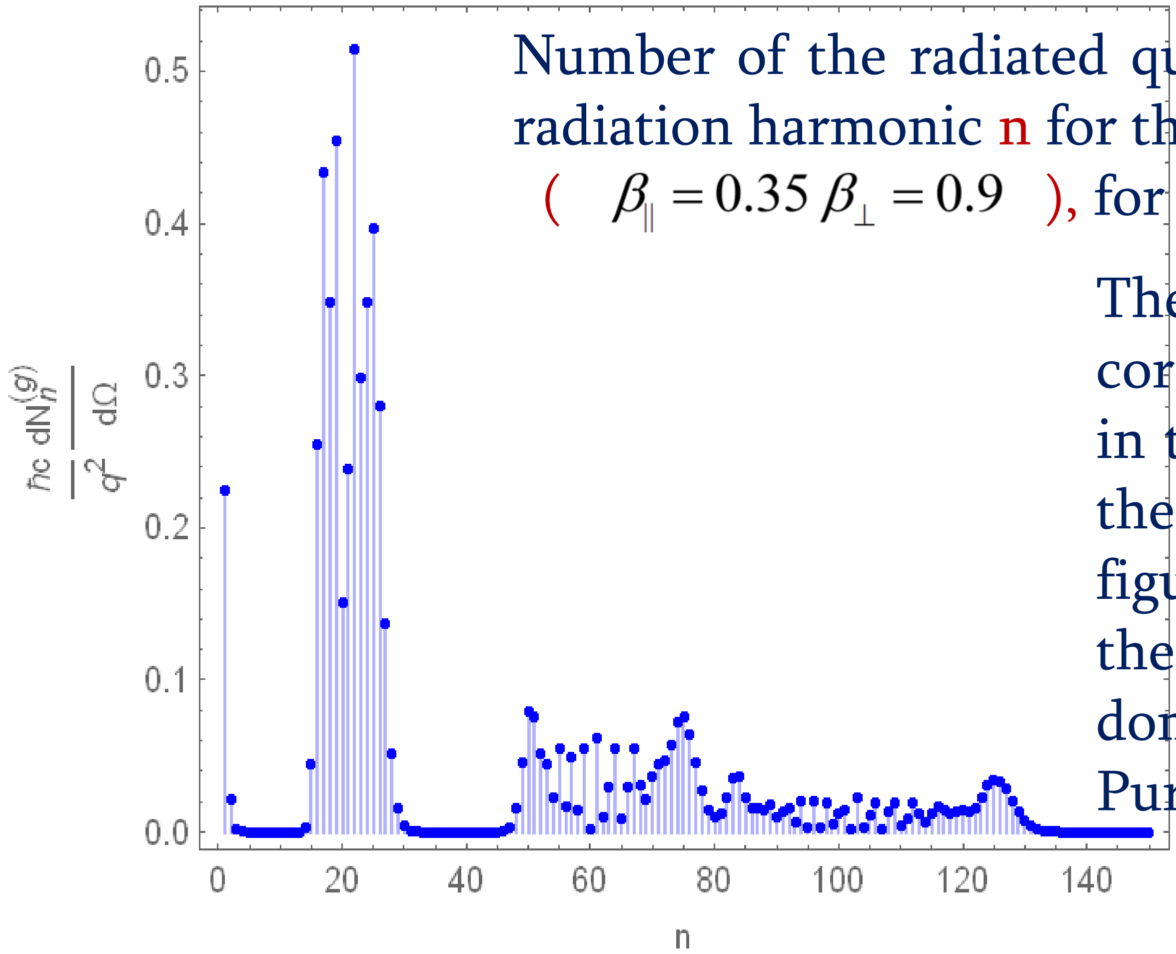
$$\frac{\hbar c}{q^2}\frac{dN_n}{d\Omega}$$



Locations of the angular peaks are not sensitive to the ratio *b/a.*

As a function of *b/a*, the radiation intensity takes its maximum value for *b/a* close to *1.*

Numerical analysis



Number of the radiated quanta as a function of the radiation harmonic n for the electron energy 2 MeV, ($\beta_{\parallel} = 0.35$ $\beta_{\perp} = 0.9$), for N=20, b/a=1. $\theta = 0.16$

These value of the angle corresponds to the local peak in the angular distribution for the examples presented in figure above. For this angle the radiation is mainly dominated by the Smith-Purcell part.

Synchrotron part of the radiation is mainly located near the angle $\theta = \pi/2$

Estimates

Maximal magnetic fields available in laboratory H≈45T which corresponds to $H \approx 4.5 \times 10^5$G. For the radius of the circular orbit one has (1 MeV=1.6×10^{-6} erg)

$$r = \frac{\beta E_e}{eH} = \frac{1.6\times10^{-6}}{4.8\times10^{-10}}\frac{\beta E_e(MeV)}{H(G)}cm = 3.3\times10^{3}\frac{\beta E_e(MeV)}{H(G)}cm$$

For the electron energy 2MeV for the magnetic field $H \approx 5 \times 10^3$G the radius of the orbit is ≈1 cm. For ω_0 one has $\omega_0 = v_\perp / r_0$.

For $v_\perp \approx c$, $\omega_0 \approx 3 \times 10^{10}$Hz

For the radiation frequency one has $\omega_n(\theta) = \dfrac{n\omega_0}{1-\beta_{\|}\cos\theta}, \quad n = 1, 2, \ldots,$

Two cases should be considered separately

Estimates

✦If the angle is not close to θ=0, then for the radiation frequency

$$\nu_n = \frac{\omega_n}{2\pi} \approx n\frac{\omega_0}{2\pi}$$

For the radiation harmonic n of the order 100 the radiation frequency is in the terahertz range, $\nu \sim 10^{12}$Hz

✦In the second case for the relativistic longitudinal motion and for the radiation along small angles θ (this case is realized in helical undulators) one has

$$1-\beta_{\parallel}\cos\theta \approx 1-\beta_{\parallel}+\beta_{\parallel}\theta^2/2 \approx \left(\gamma_{\parallel}^{2}+\theta^2\right)/2 \qquad \omega_n = \frac{2n\omega_0\gamma_{\parallel}^{2}}{1+\gamma_{\parallel}^{2}\theta^2}$$

In this case the radiation is mainly along the angles $\theta \sim 1/\gamma_{\parallel}$ and the radiation frequency is in the range

$$\nu_n = \frac{\omega_n}{2\pi} \approx \frac{n\omega_0}{2\pi}\gamma_{\parallel}^{2}$$

For the longitudinal energy of the order 10MeV one has $\gamma_{\parallel} \approx 20$ and the radiation is in the optical range

Radiation in helical undulators

For helical undulators it is introduced the parameter K_u

$$\beta_\perp = \frac{K_u}{\gamma}, \ \beta_\| = \sqrt{1 - \frac{1+K_u^2}{\gamma^2}}, \ \omega_0 = ck_u\beta_\|, \ r_0 = \frac{K_u}{\gamma k_u \beta_\|}$$

For helical undulators $K_u < 1$

For $0 < 1 - \beta_\| \ll 1$ and for small angles one has

$$\omega_n = \frac{2n\omega_0\gamma_\|^2}{1+\gamma_\|^2\theta^2} \quad u \approx \frac{2K_u\theta\gamma}{1+K_u^2+\gamma^2\theta^2}$$

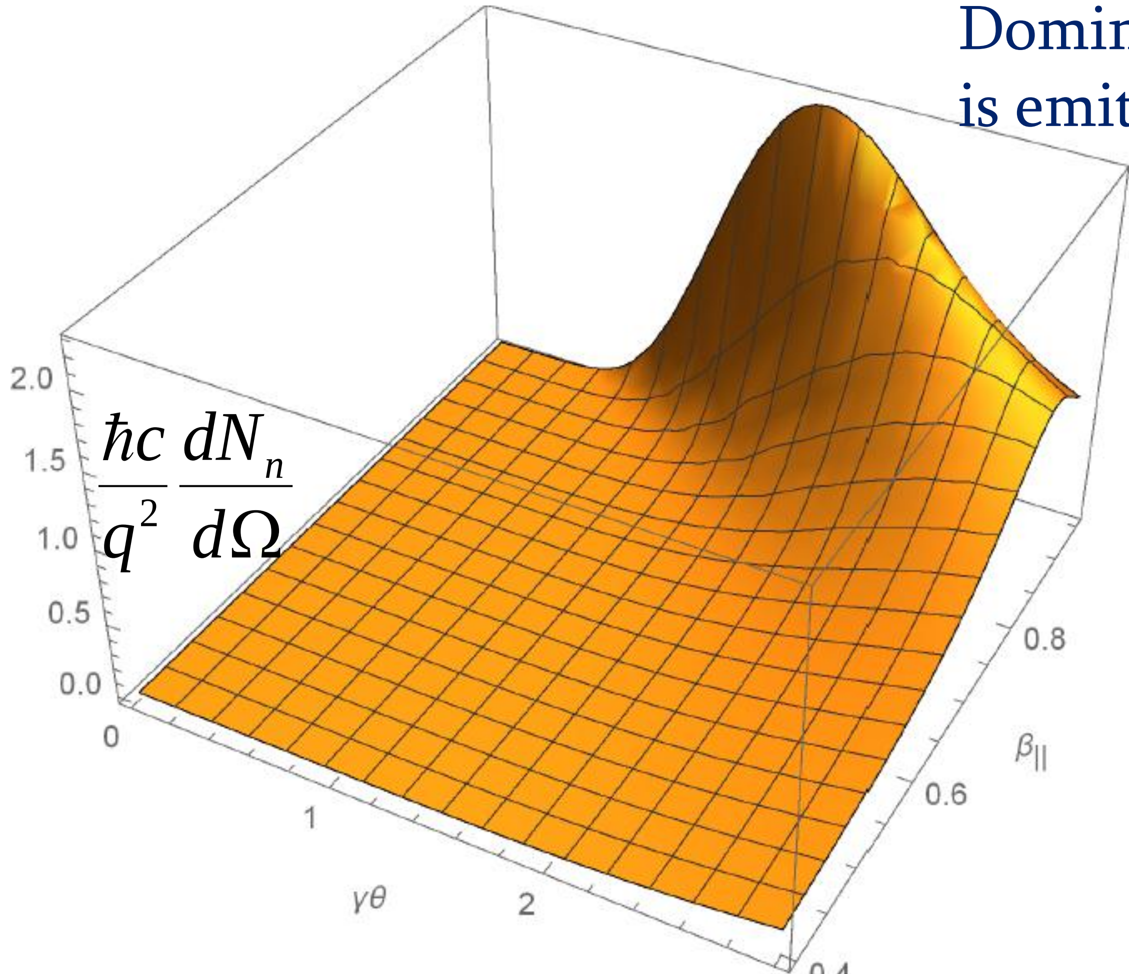
$$\frac{dI_n}{d\Omega} = \frac{4q^2n^2\omega_0^2\gamma^4}{\pi c\left(1+K_u^2+\gamma^2\theta^2\right)^3}\sum_{m=-\infty}^{+\infty}|s_m|^2\left[K_u^2 J'^2_{n+Nm}(nu)\right.$$

$$\left.+\frac{\gamma^2\theta^2}{4}\left(\left(1+K_u^2+\gamma^2\theta^2\right)\frac{Nm}{n}+1+K_u^2-\gamma^2\theta^2\right)^2 J^2_{n+Nm}(nu)\right]$$

Radiation is peaked along the $\theta \sim 1/\gamma$ angles

Numerical example: Pure undulator radiation

Number of the radiated quanta on the harmonic *n=5* versus θ and $\beta_{||}$ for the $E_e = 5\ \mathrm{MeV}, K_u = 0.7$



Dominant part of the radiation is emitted along small angles

$$\theta \sim 1/\gamma$$

Undulator radiation in the presence of grating

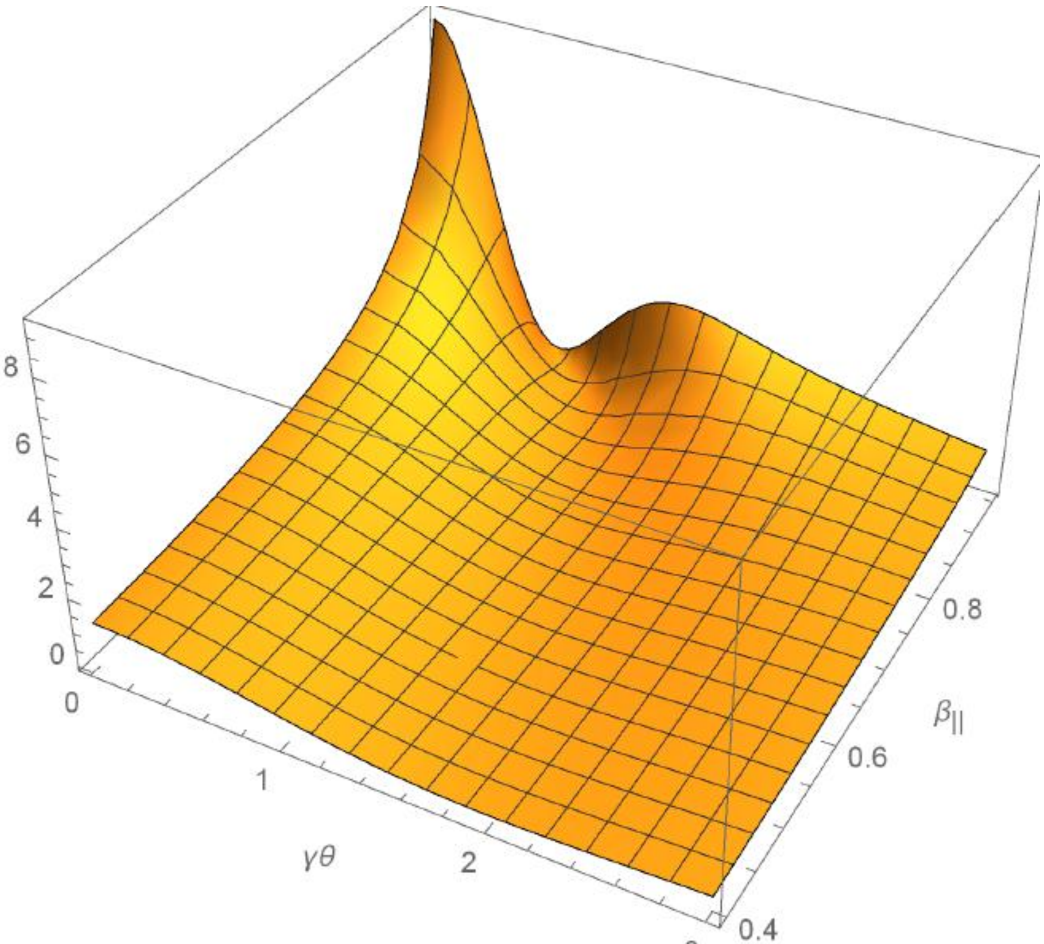
Number of the radiated quanta on the harmonic *n=5* versus θ and $\beta_{\|}$ for the $E_e = 5\ \text{MeV}$

$$N = 4,\ b = a$$

$$K_u = 0.7$$

Presence of grating gives rise an additional peak at small angles

$$\frac{\hbar c}{q^2}\frac{dN_n}{d\Omega}$$



Conclusion

- We have investigated the radiation from a charged particle moving along a helical trajectory inside a cylindrical metallic grating

- In the problem under consideration two mechanisms of the radiation, synchrotron and Smith-Purcell ones, work simultaneously

- Both regimes of the undulator and wiggler are considered

- For given characteristics of the charge, by the choice of the parameters of the diffraction grating, one can have highly directional radiation on a given harmonic directed near the normal to the plane of the charge rotation