The conformal bootstrap approach to critical phenomena

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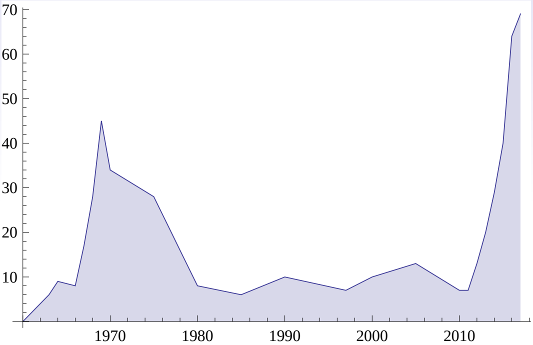
50 years of Veneziano Model
from dual models to strings, M-theory and beyond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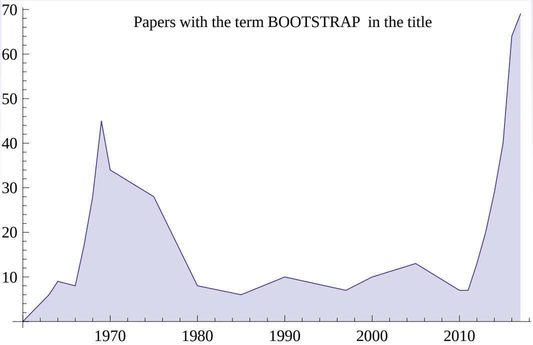
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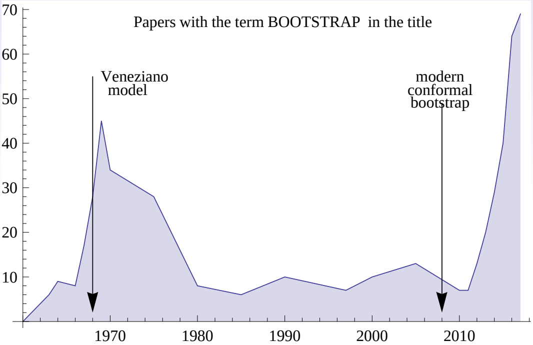
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modern conformal bootstrap
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The Conformal Bootstrap

A reincarnation of the bootstrap program for the strong interactions of Chew and Frautchi (1960)

- Ferrara, Gatto, Grillo, Parisi (1972), Polyakov (1973)...
- Significant analytic results in 2d CFTs: Belavin, Polyakov, Zamolodchikov (1984)
- It was not expected to succeed in $d > 2$ due to lack of Virasoro algebra
- Unexpectedly, conformal bootstrap started producing concrete numerical results first in $d = 4$ (Rattazzi, Rychkov, Tonni, Vichi (2008)) and then in $d = 3$ (El-Showk, Paulos, Poland, Rychkov, Simmon-Dufffin, Vichi (2012)).
- Since then many new numerical and analytic results in diverse space dimensions.
- Purpose of this approach:
  Assume conformal invariance of critical systems and explore consequences

- A CFT in d dimensions is defined by a set of local operators $\{\mathcal{O}_k(x)\}$ $x \in \mathcal{R}^d$ and their correlation functions

$$\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle$$

- Local operators can be multiplied. Operator Product Expansion:

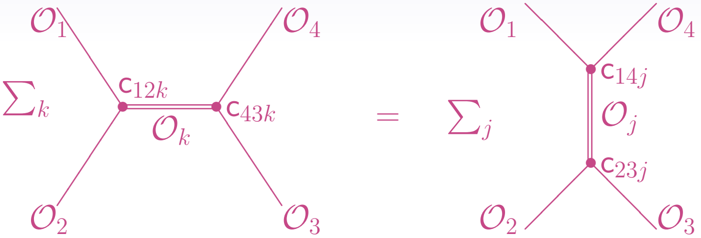
$$\mathcal{O}_i(x)\mathcal{O}_j(0) \sim \sum_k c_{ijk} f_k(x) \mathcal{O}_k(0)$$

- $\mathcal{O}_{\Delta,\ell,f}(x)$ are labelled by a scaling dimension Δ

$$\mathcal{O}_{\Delta,\ell,f}(\lambda x) = \lambda^{-\Delta} \mathcal{O}_{\Delta,\ell,f}(x)$$

  an $SO(d) \subset SO(d+1,1)$ representation ℓ (spin), and possibly a flavor index f
- Acting with the Lie algebra of the conformal group $\{J_{\mu\nu}, P_\mu, K_\mu, D\}$ on a local operator generates a whole representation of the conformal group.
- A local operator $\mathcal{O}$ with $[K_\mu, \mathcal{O}(0)] = 0$ is said a *primary*, the others are *descendants*

- The spectrum of the primaries $\{[\Delta_i, \ell_i]\}$ and the set of the OPE coefficients $\{c_{ijk}\}$ form the CFT *data* which completely characterise the CFT
- Consistent data must satisfy *crossing symmetry*
- Modern conformal bootstrap is the (analytic or/and numerical) study of



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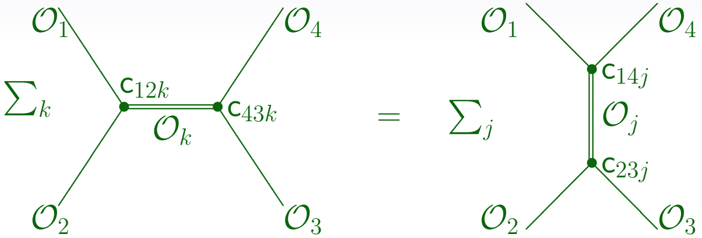

It knows almost everything about critical phenomena!

Crossing symmetry, simplest case

- The 4-pt function of a single scalar operator $\Phi(x)$ in a CFT can be parametrised as
$\langle\Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta_\Phi}|x_{34}|^{2\Delta_\Phi}}, \left(u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2},\ v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\right)$
- $g(u,v)$ can be expanded in terms of *conformal blocks* $G_k(\Delta_k, \ell_k; u, v)$ (eigenfunctions of the Casimir operator) :
$g(u,v) = 1 + \sum_k \mathrm{p}_k\, G_k. \quad \mathrm{p}_k = \mathrm{c}^2_{\Phi\Phi\mathcal{O}_k}$
- Crossing (or bootstrap) equations
  1. $g(u,v) = g(u/v, 1/v)$ projecting on states of even spin only
  2. $g(u,v)\, v^{\Delta_\Phi} = g(v,u)\, u^{\Delta_\Phi}$ encoding a huge amount of information
- sum rule $\boxed{\sum_k \mathrm{p}_k \frac{v^{\Delta_\Phi} G_k(u,v) - u^{\Delta_\Phi} G_k(v,u)}{u^{\Delta_\Phi} - v^{\Delta_\Phi}} = 1}$
- Put $u = z\bar{z}$, $v = (1-z)(1-\bar{z})$ and Taylor expand about the symmetric point $z = \bar{z} = \frac{1}{2}$

- the crossing symmetry constraint can then be rewritten as one inhomogeneous equation (normalization)
- $\sum_k \mathrm{p}_k \, \mathrm{f}^{(0,0)}_{\Delta_\Phi,\Delta_k} = 1$
  and an infinite set of homogeneous equations
- $\sum_k \mathrm{p}_k \, \mathrm{f}^{(m,n)}_{\Delta_\Phi,\Delta_k} = 0 \qquad (m, n \in \mathbb{N}, m+n \neq 0)$
  $\mathrm{f}^{(m,n)}_{\alpha,\beta} = \left(\partial^m_z \partial^n_{\bar{z}} \frac{v^\alpha G_\beta(u,v) - u^\alpha G_\beta(v,u)}{u^\alpha - v^\alpha}\right)_{z,\bar{z}=\frac{1}{2}}$ known functions of Δ_Φ and Δ_k
- Assuming unitarity ($\Rightarrow \mathrm{p}_k \geq 0$) allows to turn bootstrap equations into a powerful numerical algorithm *(linear programming)* producing data with rigorous error bars (see later)
- However many interesting critical systems do not correspond to unitary (or reflection positive) theories

The method of determinants (does not assume unitarity)

FG (2013); FG and A.Rago (2014); FG, P.Liendo, M.Meineri and A.Rago (2015); Y.Nakayama (2016);I.Eststerlis,A.L.Fitzpatrick and D.M.Ramirez (2016); S.Hikami(2017)

- Truncate the sum rule to a finite number N of terms
  $\sum_k^N f^{(0,0)}_{\Delta_\Phi,\Delta_k} p_k \approx 1 \qquad \sum_k^N f^{(m,n)}_{\Delta_\Phi,\Delta_k} p_k \approx 0 \qquad (m+n \neq 0)$
- Two kinds of unknowns: The low-lying spectrum $\{\Delta_\Phi, \Delta_1, \dots \Delta_N\}$ and the OPE coefficients $\{p_1 \dots p_N\}$
- If we knew all the Δ_k's ($k \leq N$) we could solve the truncated linear system and compute the p_k's
  How to get the Δ's ?
- ➩ look for solutions of an *overdetermined* system, i.e. with $M > N$ linear homogeneous equations

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  How to get the Δ's ?

➩ look for solutions of an *overdetermined* system, i.e. with $M > N$ linear homogeneous equations

- write the homogeneous system as
$\sum_k^N \left(\mathsf{f}^{(m,n)}_{\Delta_\Phi,\Delta_k}\right) \mathsf{p}_k \equiv \mathsf{F}\,\vec{\mathsf{p}} = 0$ (F = rectangular $M \times N$ matrix)
- a system of $M \geq N$ linear homogeneous equations with N unknowns admits a non-identically vanishing solution if and only if all the minors of order N are vanishing:
$\det_{\mathrm{N}}\mathsf{F}_i = f_i(\Delta_\Phi, [\Delta_1, \ell_1], \dots [\Delta_N, \ell_N]) = 0$
i labels the possible $0 < i \leq \binom{M}{N}$ minors (i.e. determinants) of order N. Each of them gives a constraint on the Δ's
- If the number of independent minors is enough *and* the associated constraints are compatible with each other we get a solution of the truncated bootstrap equations

Example: scalar free-field theory in d dimensions

From the OPE of two free fields $\varphi(x)\,\varphi(y)$ we can extract the fusion rule

$$[\varphi]\times[\varphi]\sim[1]+\lambda_0[\Delta_{\varphi^2},0]+\lambda_2[d,2]+\dots\lambda_\ell\,[d-2+\ell,\ell]+\dots$$

we get the 4-point expansion in conformal blocks

$$g(u,v)=1+u^{\Delta_\varphi}+\left(\frac{u}{v}\right)^{\Delta_\varphi}=1+\lambda_0^2G_{\Delta_{\varphi^2},0}+\sum_\ell\lambda_\ell^2G_{d-2+\ell,\ell}$$

The free Lagrangian tells us

$$\Delta_\varphi=\frac{d}{2}-1,\;\Delta_{\varphi^2}=2\Delta_\varphi$$

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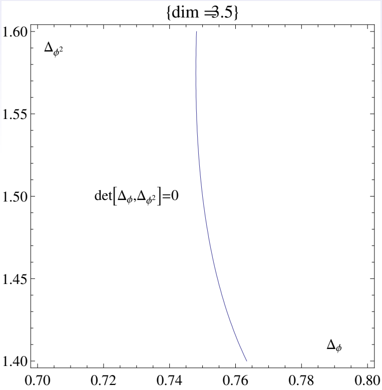
The free Lagrangian tells us

$$\Delta_\varphi = \frac{d}{2} - 1,\ \ \Delta_{\varphi^2} = 2\Delta_\varphi$$

How to see it in a pure CFT approach?

Assume Δ_{φ} and Δ_{φ^2} unknown and truncate the fusion rule

$$\boxed{[\varphi] \times [\varphi] \sim [1] + [\Delta_{\varphi^2}, 0] + [d, 2] + [d+2, 4]}$$



Put $d = 3.5$, for instance.
➩ 2 unknowns,
$N = 3$ conformal blocks,
$M = 5$ homogeneous equations,
10 different 3×3 minors.
Each curve represents the locus of vanishing of a minor in the $(\Delta_{\varphi}, \Delta_{\varphi^2})$ plane .

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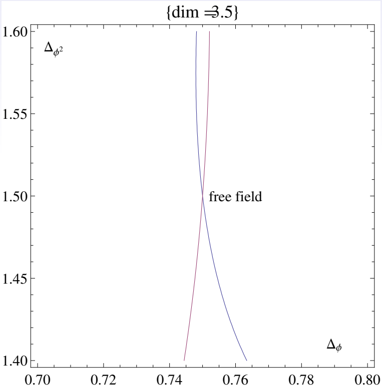

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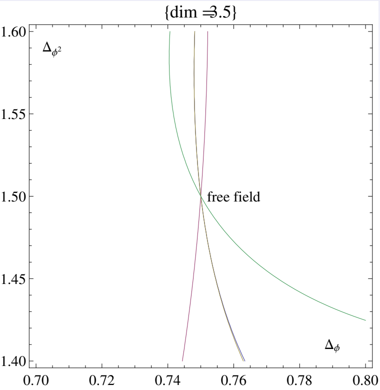

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Their mutual intersections coincide with the expected exact value.

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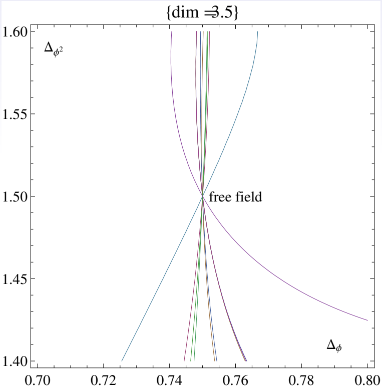

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are there other solutions?

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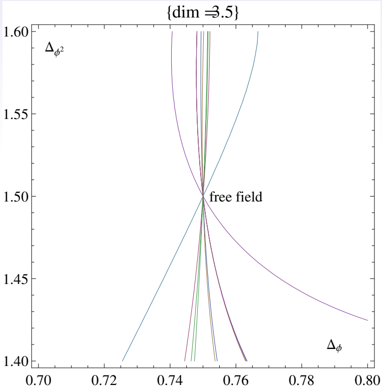

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look on a larger scale

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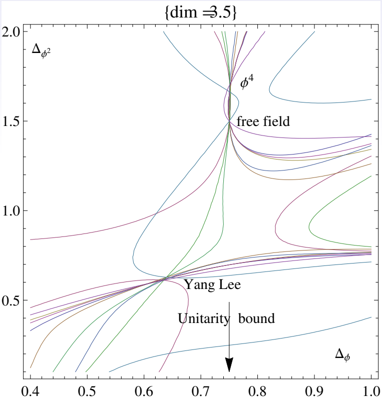

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look on a larger scale:

There are (approximate) solutions corresponding to other CFTs.

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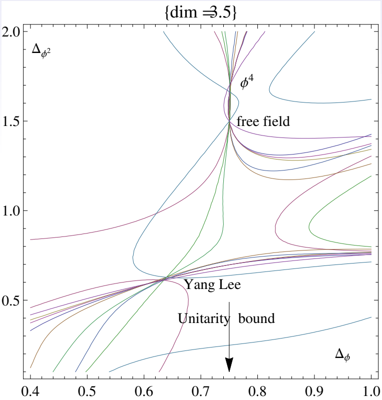

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There are (approximate) solutions corresponding to other CFTs.
Vary the space dimension d in order to see how these new solutions behave

Assume Δ_φ and Δ_{φ^2} unknown and truncate the fusion rule

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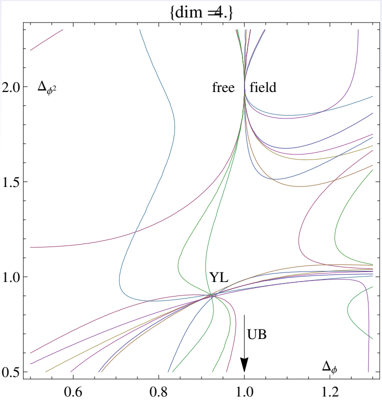

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At $d = 4$ the free scalar theory and the "ϕ^4" solution coalesce.

The merging of the two fixed points can be treated analytically

Assume Δ_φ and Δ_{φ^2} unknown and truncate the fusion rule

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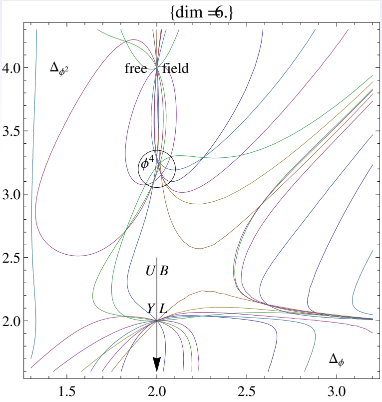

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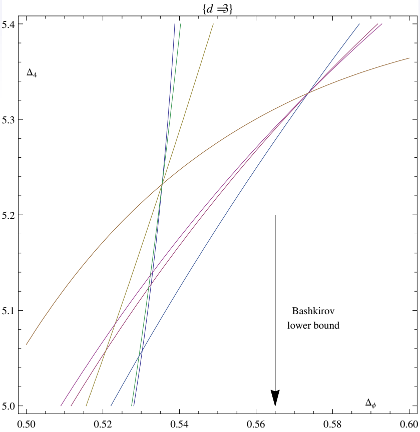
Each curve represents the locus of vanishing of a minor in the $(\Delta_\varphi, \Delta_{\varphi^2})$ plane .
At $d = 6$ the "Yang Lee" solution assumes the exact free field values

$\mathcal{N} = 1$ SuperCFT in three dimensions

$\mathcal{N} = 1$ supersymmetry implies $\Delta_{\phi^2} = \Delta_\phi + 1$

➯ truncated fusion rule

$$\boxed{[\phi] \times [\phi] \sim [1] + [\Delta_\phi + 1, 0] + [d = 3, 2] + [\Delta_4, 4]}$$



➩ Two unknowns: Δ_ϕ, Δ_4

Two isolated solutions

Bashkirov (arXiv:1310.8255), using supersymmetry and Ising bootstrap data, found the lower bound $\Delta_\phi \geq 0.565$

➯ $\Delta_\phi = 0.5826$, $\Delta_4 = 5.34$

Yang-Lee edge singularity

- switch on the interaction by adding to the action a φ^3 term with imaginary coupling: $S = \int d^D x \left[\frac{1}{2}(\partial\varphi)^2 + i(h - h_c)\varphi + ig\varphi^3\right]$.
- This non-unitary theory is known to describe in the infrared the universality class of the Yang-Lee edge singularity.
- Such a singularity occurs in any ferromagnetic D-dimensional Ising model above its critical temperature $T > T_c$.
- The zeros of the partition function in the complex plane of the magnetic field h are located on the imaginary ih axis above a critical value $ih_c(T)$.
- In the thermodynamic limit the density of these zeros behaves near h_c like $(h - h_c)^\sigma$ where $\sigma = \frac{\Delta_\varphi}{d-\Delta_\varphi}$ *(edge exponent)*
- σ is exactly known only in D=2 and D=6

Yang-Lee Universality Class

Besides the Ising model at $T > T_c$, the YL edge exponent is also related to other exponents of quite different systems:

* The pressure for $D-$ dimensional fluids with repulsive core has a singularity at negative values of activity with universal exponent $\phi(d) = \sigma(d) + 1$
* The number-per-site of large isotropic branched polymers in a good solvent (*undirected lattice animals*) obeys a power law associated with the exponent $\phi_l(d) = \sigma(d-2) + 2$

Monte Carlo simulations on these systems gives accurate results for σ

Recent calculations of high-temperature, low-field expansion (through 24^{th} order) improved the accuracy in the whole range $2 \leq d \leq 6$

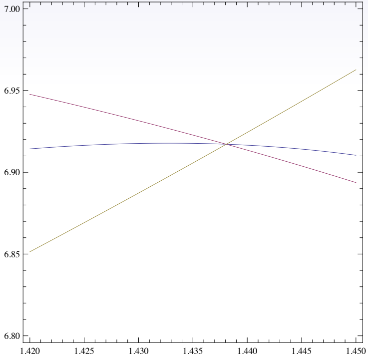
YL fusion algebra

- Improve the free-theory truncated fusion rule
  $\boxed{[\phi] \times [\phi] \sim 1 + [\phi^2] + [d, 2] + [d + 2, 4]}$
- Standard RG arguments tell us that the upper critical dimensionality of a model with φ^3 interaction is $D_u = 6$, above which the classical mean-field value $\sigma = \frac{1}{2}$ applies.
- ⇨ In $6 - \epsilon$ dimensions φ^2 is a redundant operator, as at the non-trivial φ^3 fixed point $\varphi^2 \propto \partial^2 \varphi$ by the equation of motion
- ⇨ φ^2 and its derivatives are descendant operators of the only relevant primary operator φ of this universality class
- ⇨ $\boxed{[\varphi] \times [\varphi] \sim 1 + [\varphi] + [d, 2] + [\Delta_4, 4] + \dots}$
- This fusion algebra characterizes the universality class of the Yang-Lee edge singularity in any space dimension.

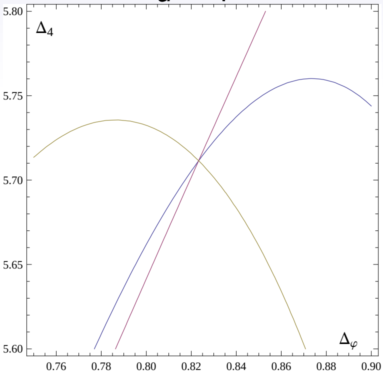
Yang-Lee in $2 \leq d \leq 6$

- For d near 6 the basic fusion rule $[\varphi] \times [\varphi] \sim 1 + [\varphi] + [d, 2] + [\Delta_4, 4]$ suffices in giving good results in accordance with the best numerical evaluations:

$d = 5$



$d = 4$



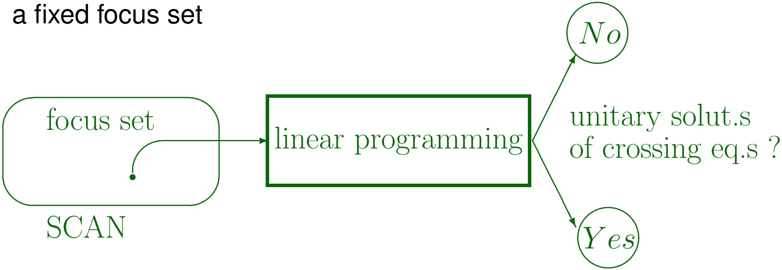
* To obtain accurate results also for $d < 4$ it is convenient to enlarge the fusion algebra to $[\varphi] \times [\varphi] \sim 1 + [\varphi] + [d, 2] + [\Delta_4, 4] + [\Delta, 0]$ where $[\Delta]$ can be associated with the scalar φ^3

		σ			
d	bootstrap	Ising in H	Fluids	Animals	$\epsilon-$expansion
2	-0.1664(5)	-0.1645(20)	-0.161(8)	-0.165(6)	(exact -1/6)
3	0.085(1)	0.077(2)	0.0877(25)	0.080(7)	0.079-0.091
4	0.2685(1)	0.258(5)	0.2648(15)	0.261(12)	0.262-0.266
5	0.4105(5)	0.401(9)	0.402(5)	0.40(2)	0.399-0.400
6	1/2	0.460(50)	0.465(35)	—	1/2

d	$\lambda^2_{\varphi\varphi\varphi}$	Δ_4	Δ_{ϕ^3}
3	-3.88(1)	4.75(1)	5.0(1)
4	-2.72(1)	5.848(1)	6.8(1)
5	-0.95(2)	6.961(1)	6.4(1)

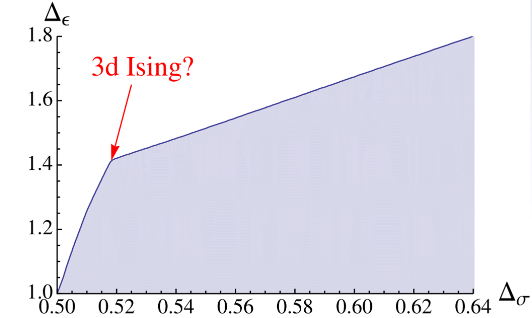
Unitary models

- Assuming unitarity ($\Rightarrow$ $p_k \geq 0$) allows to turn the bootstrap equations into a minimization problem (known as Linear Programming) which is much more powerful (but much more computationally challenging) than the method of determinants
- Schematically, split the unknowns Δ_k and p_k into two sets
  1. *FOCUS SET:* Δ's and/or p's of few particularly interesting operators, e.g.φ, φ^2
  2. *COMPLEMENT:* the rest of unknowns. One is interested whether a complement exists which makes the crossing equations satisfied for a fixed focus set



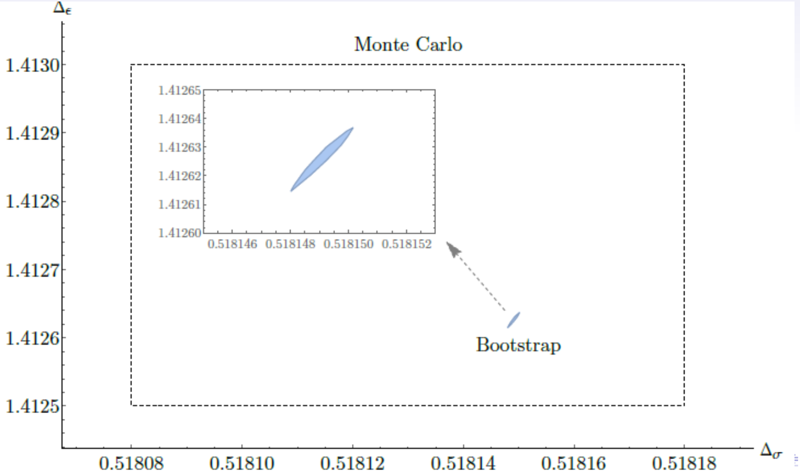
Boundary of the unitary solutions of crossing equations

S.El-Showk, M.F.Paulos,D.Poland,S.Rychkov,D.Simmons-Duffin and A.Vichi, (2012):

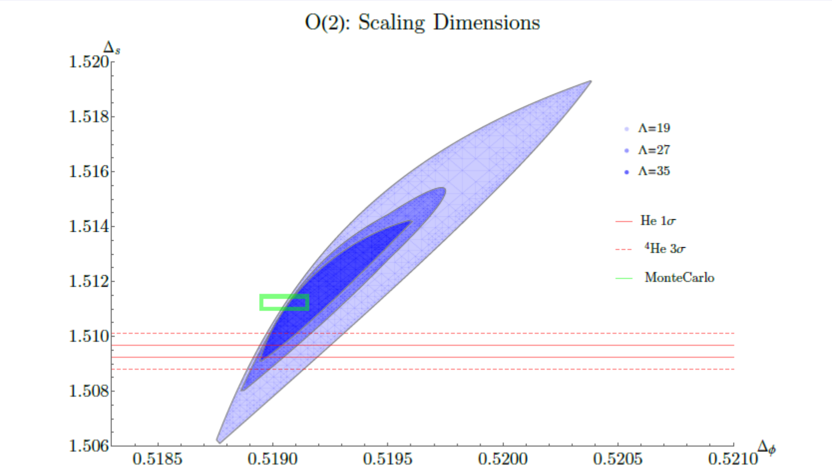


- Conjecture: the kink in the $\Delta_\sigma \equiv \Delta_\varphi$, $\Delta_\epsilon \equiv \Delta_{\varphi^2}$ plane is the CFT describing 3d critical Ising model
- According to QFT, it is described by a bosonic Lagrangian perturbed by a ϕ^4 potential at a specific value of the coupling constant. However CFT's have no much to do with Lagrangians and coupling constants
- ⇨ How to define the critical Ising model using only CFT notions?

* *The 3d critical Ising model corresponds to a $\mathbb{Z}_2$-symmetric CFT having only one relevant $\mathbb{Z}_2$-odd scalar (σ) and only one relevant $\mathbb{Z}_2$-even scalar (ϵ)*
* Adding this info, choosing as focus set σ, ϵ & $c_{\sigma\sigma\epsilon}/c_{\epsilon\epsilon\epsilon}$ and using also the mixed correlators $\langle\sigma\sigma\epsilon\epsilon\rangle$ &$\langle\epsilon\epsilon\epsilon\epsilon\rangle$ F.Kos, D.Poland, D.Simmons-Duffin and A.Vichi (2016) obtained:



* Similarly, the same authors obtained for the $O(2)$ symmetric 3d critical system (XY model):



Conclusions

1. Conformal bootstrap equations seem to know everything about critical phenomena
2. In the case of Yang-Lee edge singularity and other non-unitary CFTs the method of determinants gives accurate results in a wide range of space dimensions
3. The results are particularly impressive when the data are constrained by unitarity, in particular in 3d Ising model:
   $\Delta_\sigma = 0.5181489(10)$; $\Delta_\epsilon = 1.412625(10)$
   $c_{\sigma\sigma\epsilon} = 1.0518537(41)$; $c_{\epsilon\epsilon\epsilon} = 1.532435(19)$
   $\ldots\ldots$
4. How to get these results analytically?