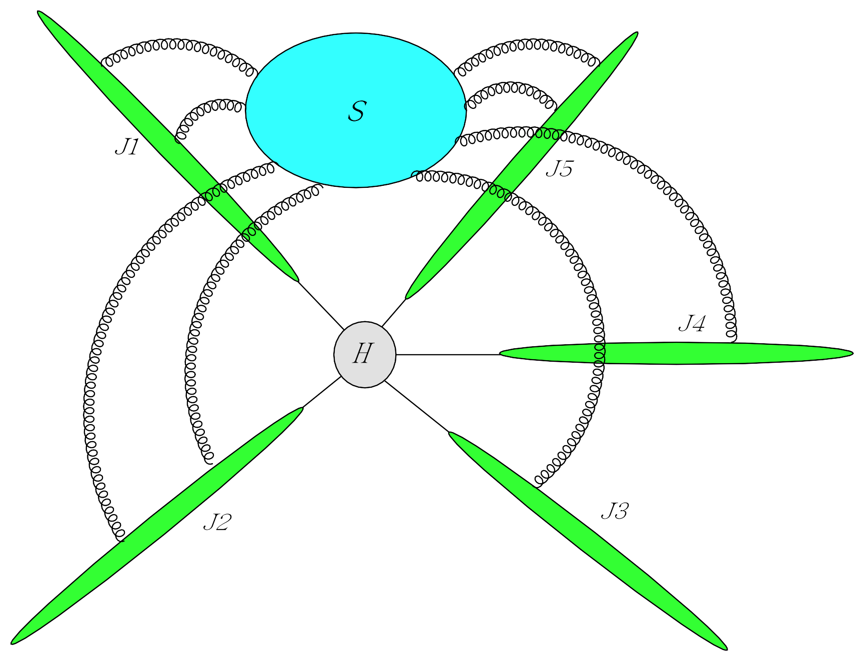
Infrared singularities in QCD amplitudes

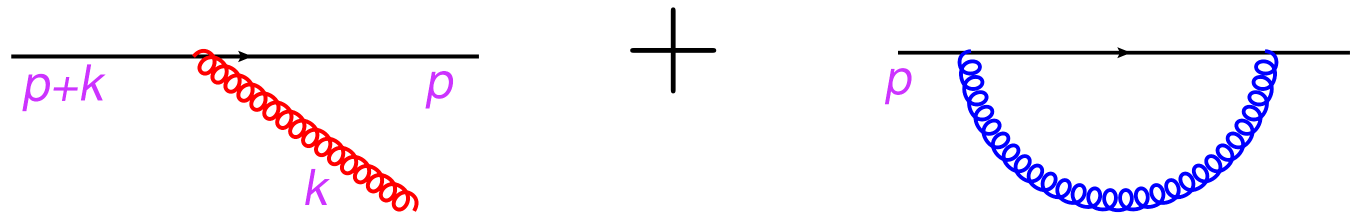
Einan Gardi (Edinburgh)



Based on a recent paper with Lorenzo Magnea: *Factorization constraints for soft anomalous dimensions in QCD scattering amplitudes*, [arXiv:0901.1091] to appear in JHEP.

Why study infrared singularities?

Cross sections are finite only upon summing real and virtual contributions

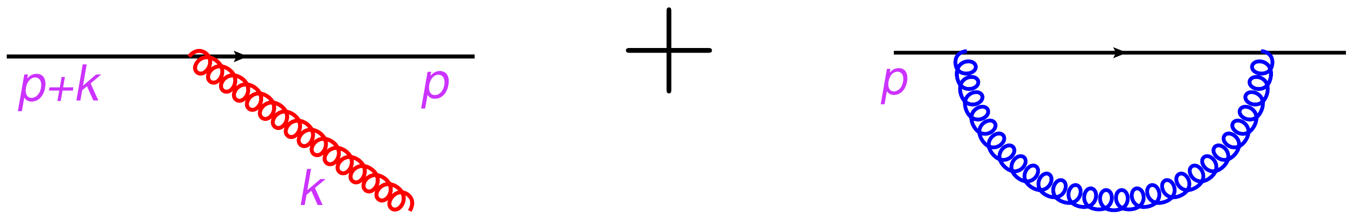


$$\text{Dim. reg.:} \quad \frac{1}{\epsilon}\left[\underbrace{\left(Q^2/m_j^2\right)^{\epsilon}}_{\text{real}} \underbrace{-\ 1}_{\text{virtual}}\right] \quad \Longrightarrow \quad \ln(Q^2/m_{\text{jet}}^2)$$

- cross sections calculations:
  - For general kinematics (and cuts!) phase–space integration must be done numerically. $\Longrightarrow$ need to know the singularities before we start the calculation.
  - It is possible: the singular terms are universal.
  - At one loop we have general algorithms, allowing to determine and subtract the singularities for general kinematics. Example: dipole subtraction [Catani Seymour (1996)].
  - Such are needed in multi-loop calculations.

Why study infrared singularities? (II) resummation

Cross sections are finite only upon summing real and virtual contributions

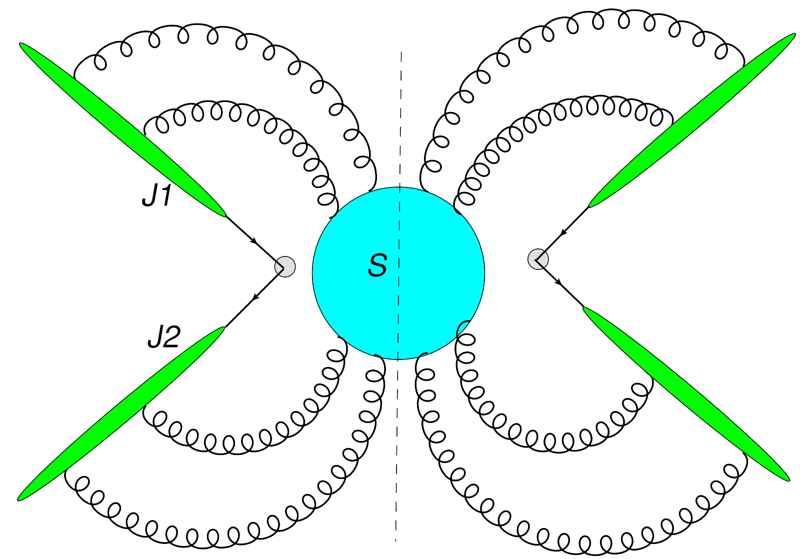


Dim. reg.: $$\frac{1}{\epsilon}\left[\underbrace{\left(Q^2/m_j^2\right)^\epsilon}_{\text{real}}\ \underbrace{-\ 1}_{\text{virtual}}\right] \qquad \Longrightarrow \qquad \ln(Q^2/m_{\text{jet}}^2)$$

- cross sections calculations.
- Resummation: The logarithms are often large, and they spoil the convergence of the expansion in α_s.
  But, knowing the singularity structure, they can be resummed to all orders: they exponentiate:
  - parton showers
  - dedicated precision calculations

Factorization in a two-jet process

- Sudakov resummation in inclusive cross sections is well understood
- Soft and Jet sub-processes are incoherent $\Longrightarrow$ factorization
- Each sub-process is associated with
  - a single scale
  - a unique anomalous dimension — a function of the running coupling only
  - the overlap: the cusp anomalous dimension γ_K



$$\mathrm{Sud}(m^2, N) = \exp\left\{ C_R \int_0^1 \frac{dr}{r} \Big[\underbrace{(1-r)^{N-1}}_{\text{real}} \underbrace{-1}_{\text{virtual}} \Big] R(m^2, r) \right\},$$

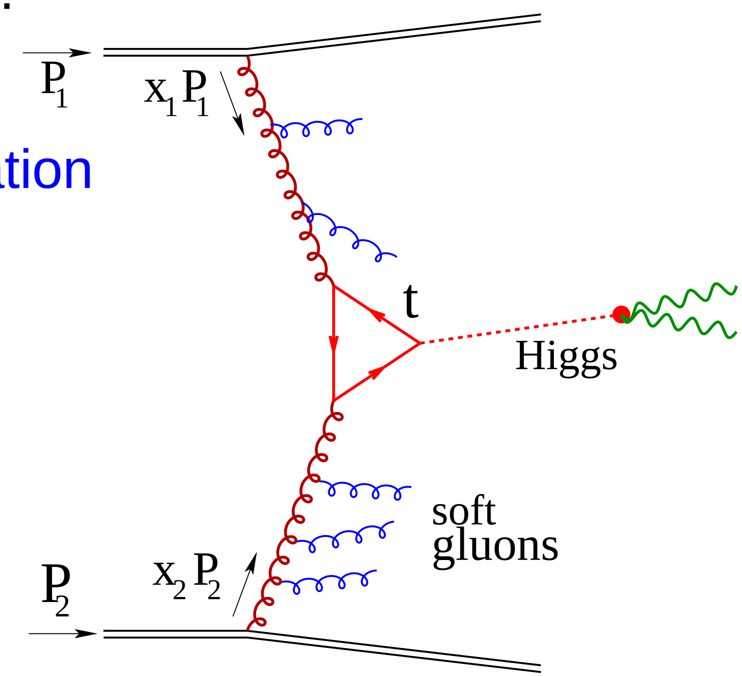
$$C_R \frac{R(m^2, r)}{r} = -\frac{1}{r} \left[\int_{r^2 m^2}^{r m^2} \frac{dk^2}{k^2} \gamma_K \left(\alpha_s(k^2)\right) + 2\mathcal{B}\left(\alpha_s(r m^2)\right) - 2\mathcal{D}\left(\alpha_s(r^2 m^2)\right) \right]$$

Resummation: Example 1 — Higgs production at the LHC

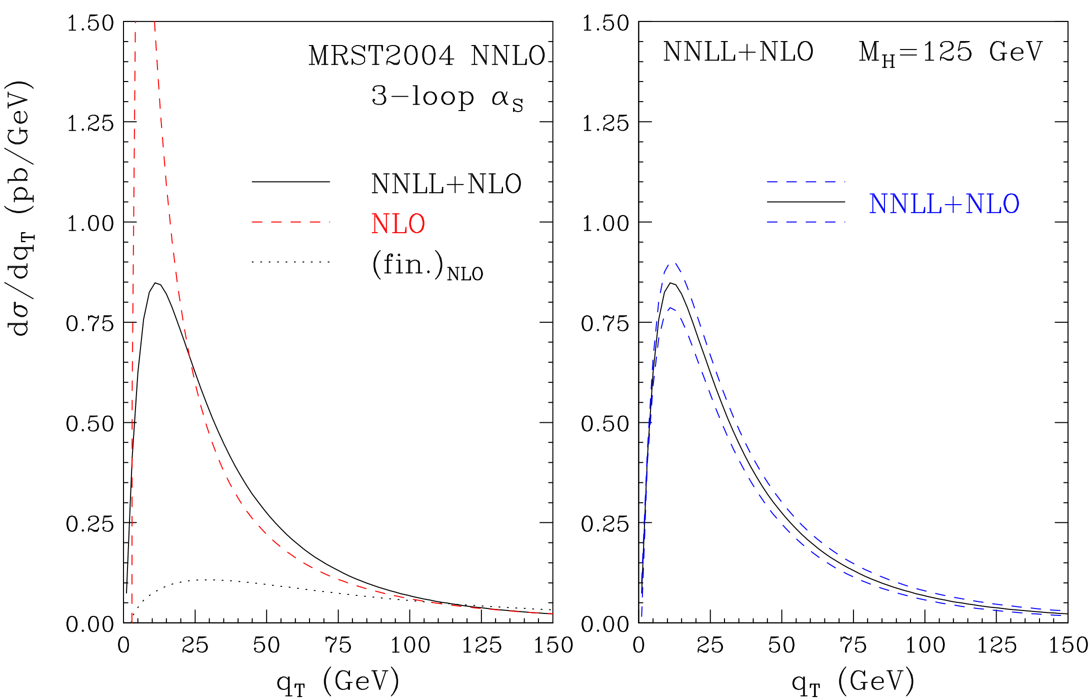
The main Higgs production channel: $gg \longrightarrow H+X$

gluon density $\Longrightarrow$ Higgs production occurs near partonic threshold:

- the total energy of gluons in the final state:
  $E_X = (\hat{s} - m_H^2)/2m_H \to 0$
- multiple soft gluon emission $\Longrightarrow$ resummation



Higgs P_T distribution [Bozzi et al. '05]

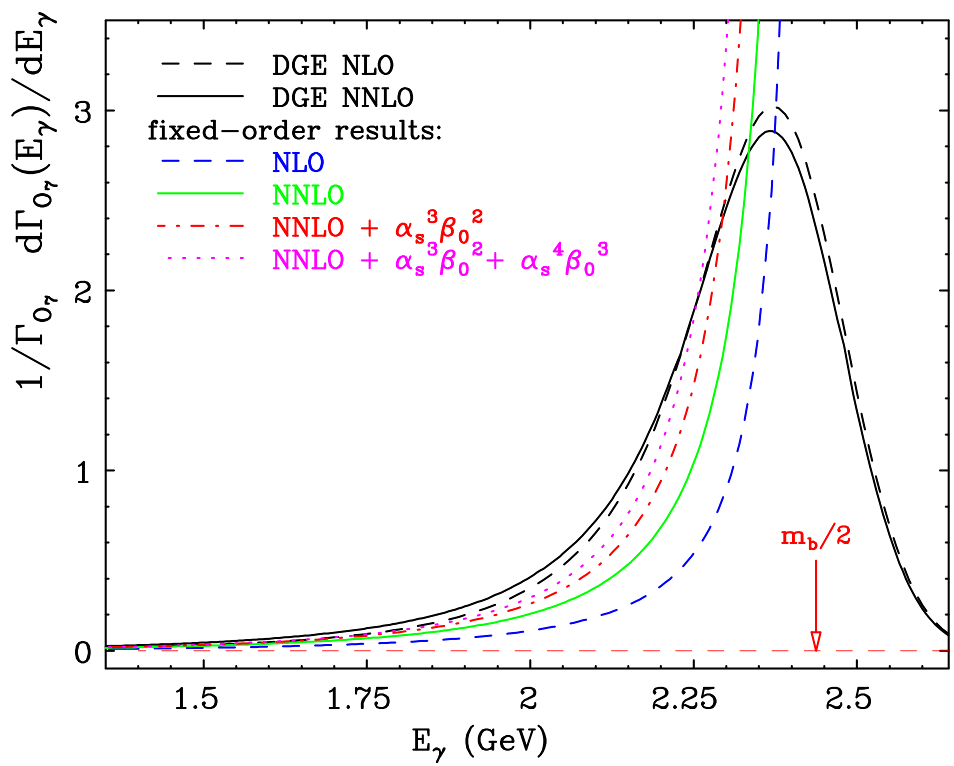


Resummation: Example 2 — precision flavour physics

Inclusive decay spectra in the B factories: $\bar{B} \to X_s \gamma$, $\bar{B} \to X_u l \nu$, ...
— Gambino's talk tomorrow.

Resummation: Korchemsky Sterman; Bauer Fleming Pirjol Stewart (SCET); Lange Neubert Paz; Andersen Gardi, Aglietti et al., ...

$\bar{B} \to X_s \gamma$ spectrum [Andersen & Gardi '06]

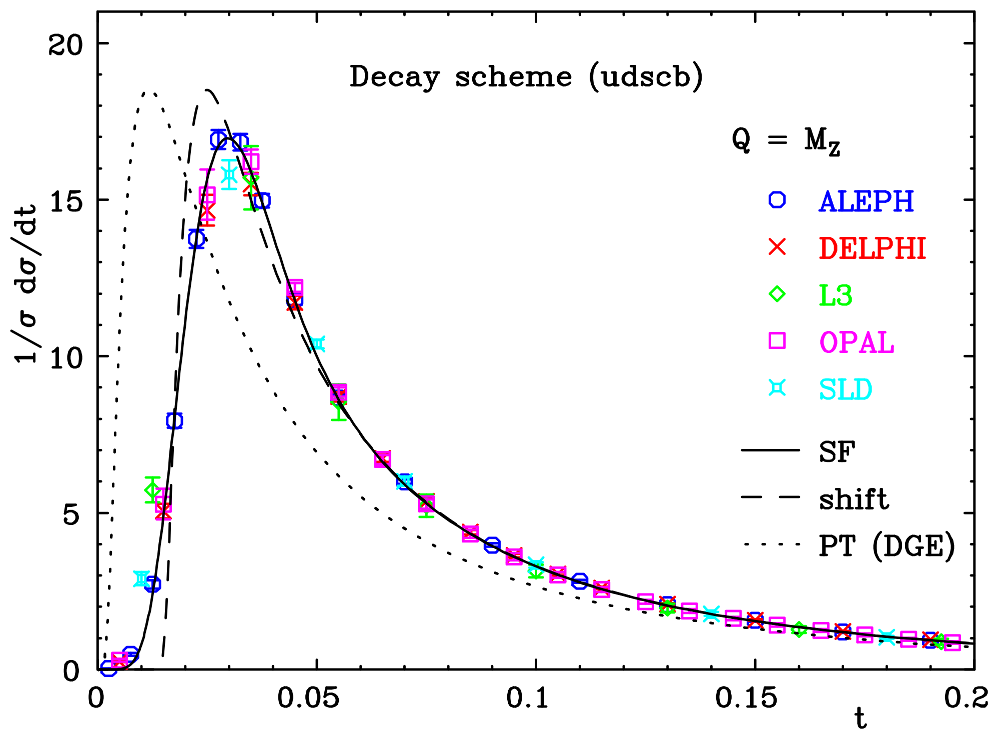


- $\bar{B} \to X_s \gamma$ branching fraction — bounds on new physics
- precise determination of $|V_{ub}|$!

Resummation: Example 3 — jet cross sections

Jets in $e^+e^- \to \mathrm{hadrons}$ — extensively studied at LEP [Catani Trentadue Turnock Webber (92); Korchsmsky Sterman (95); Dokshitzer Webber (95), Dokshitzer Marchesini Webber (96), ...]

thrust distribution [Gardi & Rathsman '02]

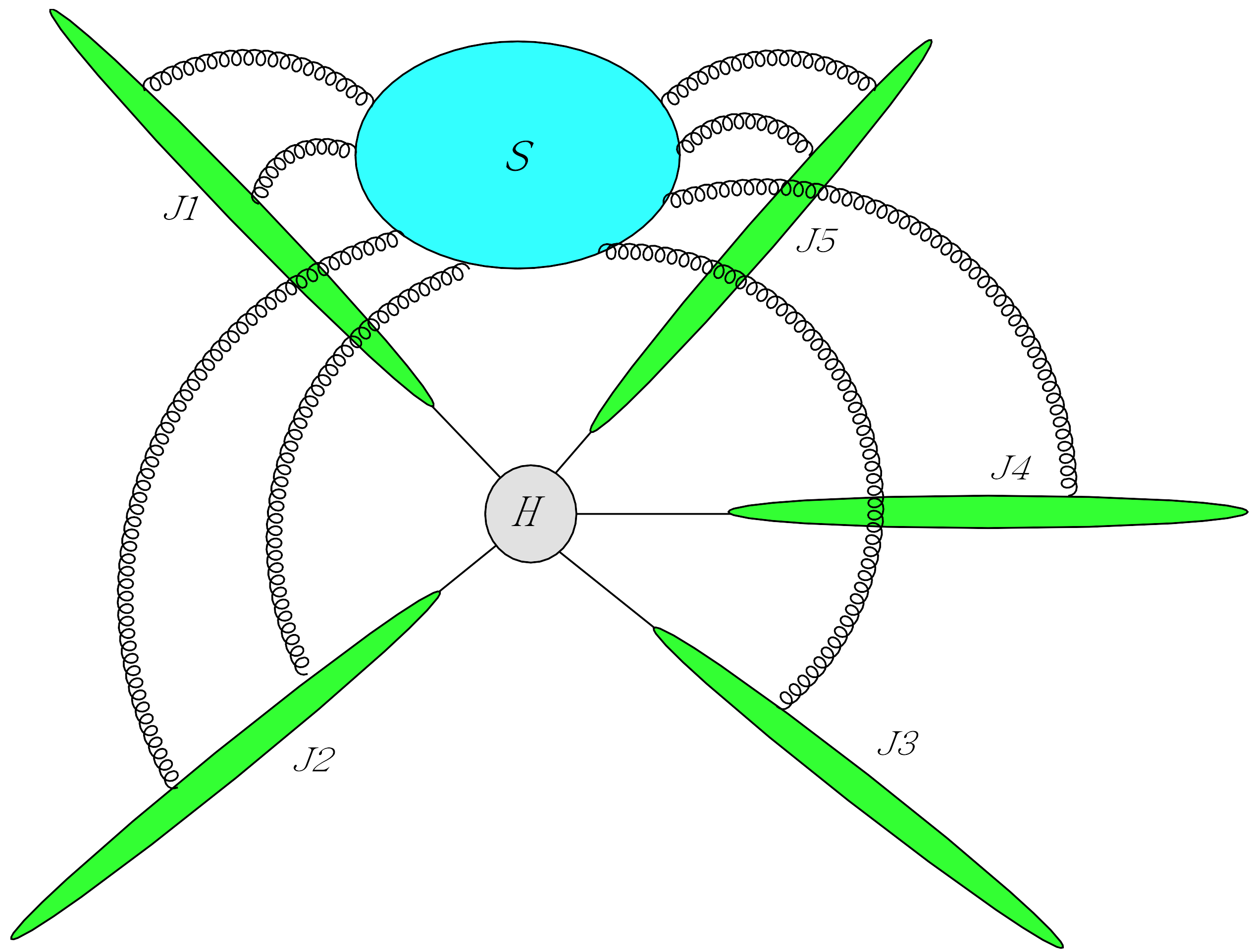


- Determination of the strong coupling
- Quantitative understanding hadronization corrections

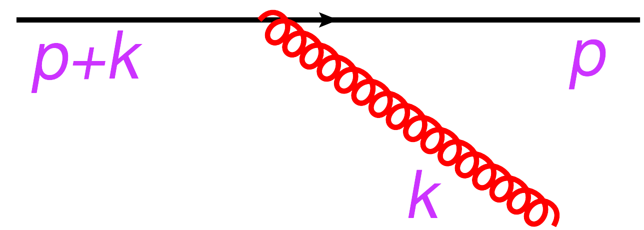
Factorization of a multi-leg amplitude

Fixed–angle scattering amplitude in a massless gauge theory ($p_i^2 = 0$)

Mueller (81)
Sen (83)
Botts Sterman (89)
Kidonakis Oderda Sterman (98)
Catani (98)
Tejeda-Yeomans Sterman (02)
Kosower (03)
Aybat Dixon Sterman (06)
Becher Neubert (09)
Gardi Magnea (09)



Eikonal approximation



Eikonal Feynman rules

gluon emission in the limit $k \to 0$:

$$\bar{u}(p)\left(-\mathrm{i}g_s T^{(a)}\gamma^\mu\right)\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+\mathrm{i}\varepsilon} \longrightarrow \bar{u}(p)g_s T^{(a)}\frac{p^\mu}{p\cdot k+\mathrm{i}\varepsilon}$$

- Valid when all momentum components of k are small (not valid when k is collinear to p but hard)
- Only the direction and the colour charge of the emitter are important. Rescaling invariance: $\beta \propto p$

$$g_s T^{(a)}\frac{p^\mu}{p\cdot k+i\varepsilon} = g_s T^{(a)}\frac{\beta^\mu}{\beta\cdot k+i\varepsilon}$$

- Equivalent to radiation off a Wilson line along the quark trajectory:

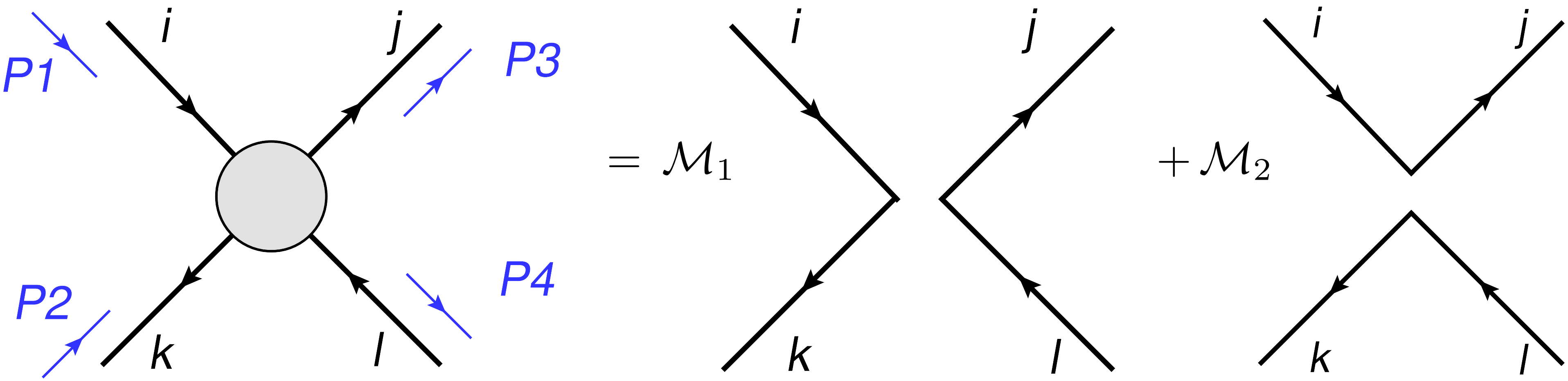
$$P\exp\left\{\mathrm{i}g_s\int_0^\infty d\lambda\beta\cdot A(\lambda\beta)\right\}$$

Colour flow

Decompose the amplitude in a colour basis (independent colour tensors with the index structure of the external partons):

Example:

$q(p_1)\bar{q}(p_2) \to q(p_3)\bar{q}(p_4)$ $\qquad c_1 = \delta_{ik}\delta_{jl}$ $\qquad c_2 = \delta_{ij}\delta_{kl}$



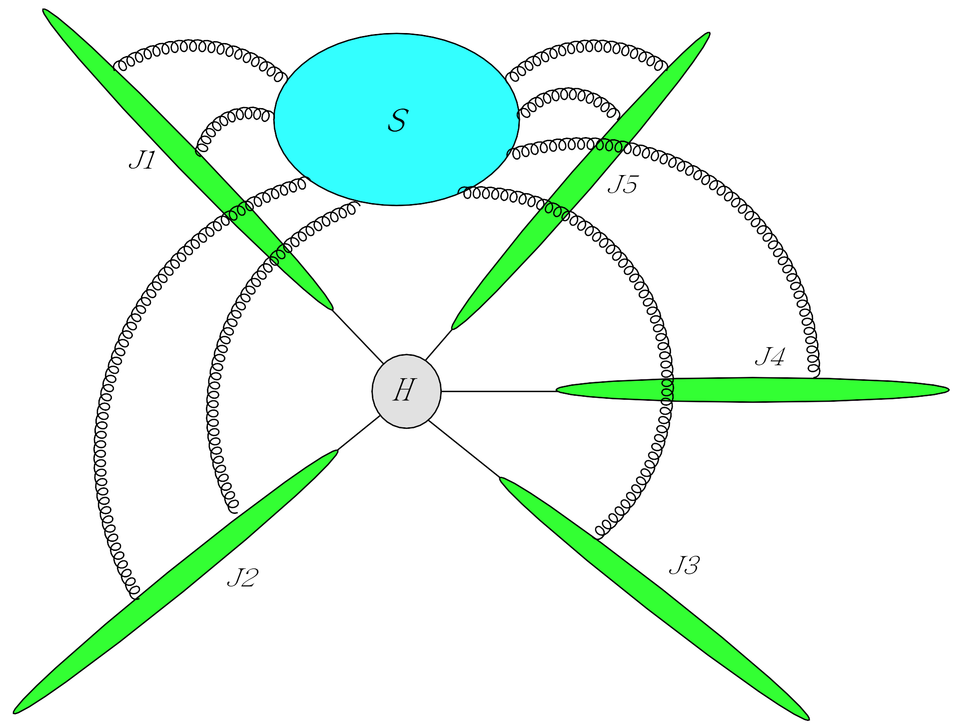
In general:

$$\mathcal{M}_{\{\alpha_i\}}\left(p_i/\mu, \epsilon\right) = \sum_{L=1}^{n_{\rm rep}} \mathcal{M}_L\left(p_i/\mu, \epsilon\right)\,\left(c_L\right)_{\{\alpha_i\}}$$

$n_{\rm rep}$ is the number of elements in the basis (number of irreducible representations that can be constructed with the given external particles).

Factorization of a multi-leg amplitude

- All singularities are in $\mathcal{S}$, $J_i/\mathcal{J}_i$.
- colour:
  $\mathcal{S}$ is a matrix acting on H
- kinematics:
  $\mathcal{S}$ depends on all velocities;
  $J_i/\mathcal{J}_i$ depends on a single p_i



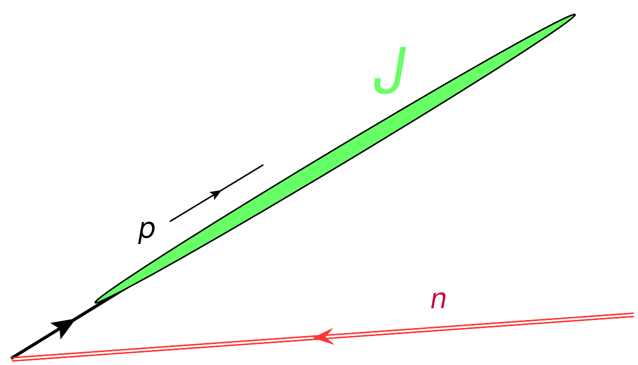
$$\mathcal{M}_N\left(p_i/\mu, \epsilon\right) = \sum_L \mathcal{S}_{NL}\left(\beta_i \cdot \beta_j, \epsilon\right)\; H_L\left(\frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}\right)$$

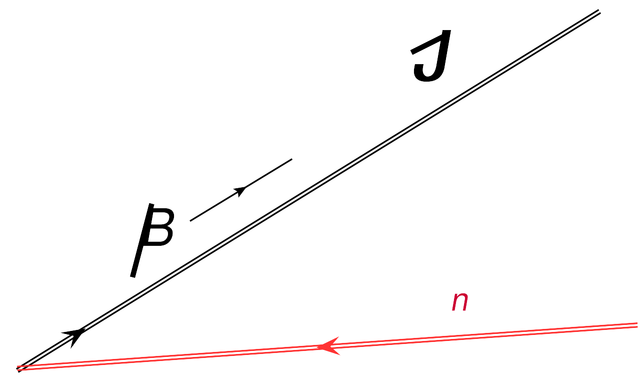
$$\times \prod_{i=1}^{n} J_i\left(\frac{(2p_i \cdot n_i)^2}{n_i^2\mu^2}, \epsilon\right) \Big/ \mathcal{J}_i\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \epsilon\right)$$

To avoid double counting of the soft-collinear region: $\mathcal{J}_i$ removes from J_i its eikonal part, which is already taken into account in $\mathcal{S}$.

The jet function: definition

- Introduce auxiliary vectors n_i ($n_i^2 \neq 0$) to separate collinear regions. Intuitive picture: jet i contains gluons (k) such that: $k \cdot p_i < n_i \cdot p_i$
- Define a gauge–invariant jet using a Wilson line along a ray n_i.





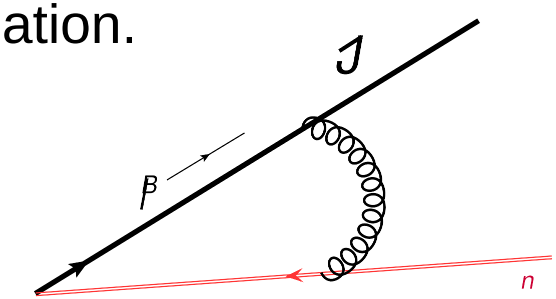
partonic jet: $$\overline{u}(p)\, J\left(\frac{(2p\cdot n)^2}{n^2\mu^2},\epsilon\right) = \langle p\,|\,\overline{\psi}(0)\,\Phi_n(0,-\infty)\,|0\rangle$$

where $$\Phi_n(\lambda_2,\lambda_1) = P\exp\left[\mathrm{i}g\int_{\lambda_1}^{\lambda_2} d\lambda\, n\cdot A(\lambda n)\right]$$

eikonal jet: $$\mathcal{J}\left(\frac{2(\beta\cdot n)^2}{n^2},\epsilon\right) = \langle 0|\,\Phi_\beta(\infty,0)\,\Phi_n(0,-\infty)\,|0\rangle$$

The eikonal jet and the cusp anomaly

- $\mathcal{J}$ doesn't depend on any kinematic scale; radiative corrections — only due to renormalization.

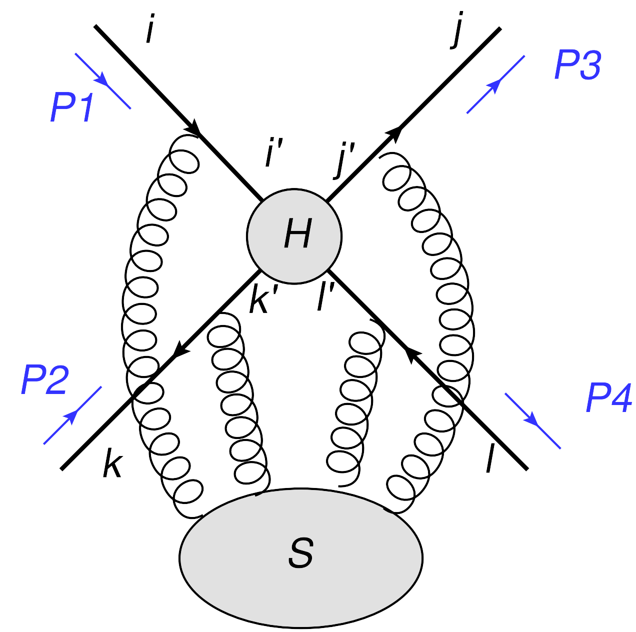


- multiplicatively renormalizable $\Longrightarrow$ evolution $\Longrightarrow$ exponentiation.
- Overlapping soft and collinear singularities $\Longrightarrow$
  - double poles
  - single poles that carry $(\beta \cdot n)^2/n^2$ dependence, violating classical rescaling symmetry wrt β. This is the cusp anomaly!

$$\mathcal{J}_i\left(\frac{2(\beta\cdot n)^2}{n^2},\epsilon\right) = \exp\left\{\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left[\frac{1}{4}\delta_{\mathcal{J}_i}\Big(\alpha_s(\lambda^2,\epsilon)\Big) - \frac{1}{8}\gamma_K^{(i)}\Big(\alpha_s(\lambda^2,\epsilon)\Big)\,\ln\left(\frac{2(\beta\cdot n)^2\mu^2}{n^2\lambda^2}\right)\right]\right\}$$

The double poles as well as the entire kinematic dependence of the simple poles are governed by $\gamma_K^{(i)}$! [EG & Magnea (09)]

The soft function $\mathcal{S}$



Definition:

$$(c_N)_{ijkl}\, \mathcal{S}_{NL}\left(\beta_a \cdot \beta_b, \alpha_s(\mu^2), \epsilon\right) =$$

$$\sum_{i'j'k'l'} \langle 0|\, \Phi^{k,k'}_{-\beta_2}(0,\infty)\, \Phi^{i,i'}_{\beta_1}(\infty,0)\Phi^{j,j'}_{\beta_3}(0,\infty)\Phi^{l,l'}_{-\beta_4}(\infty,0)\ |0\rangle\ (c_L)_{i'j'k'l'}$$

multiplicatively renormalizable $\Longrightarrow$ matrix evolution equation:

$$\mu\frac{d}{d\mu}\mathcal{S}_{JL}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) =$$

$$-\sum_N \left[\boldsymbol{\Gamma}_{\mathcal{S}}\right]_{JN}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)\ \mathcal{S}_{NL}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)$$

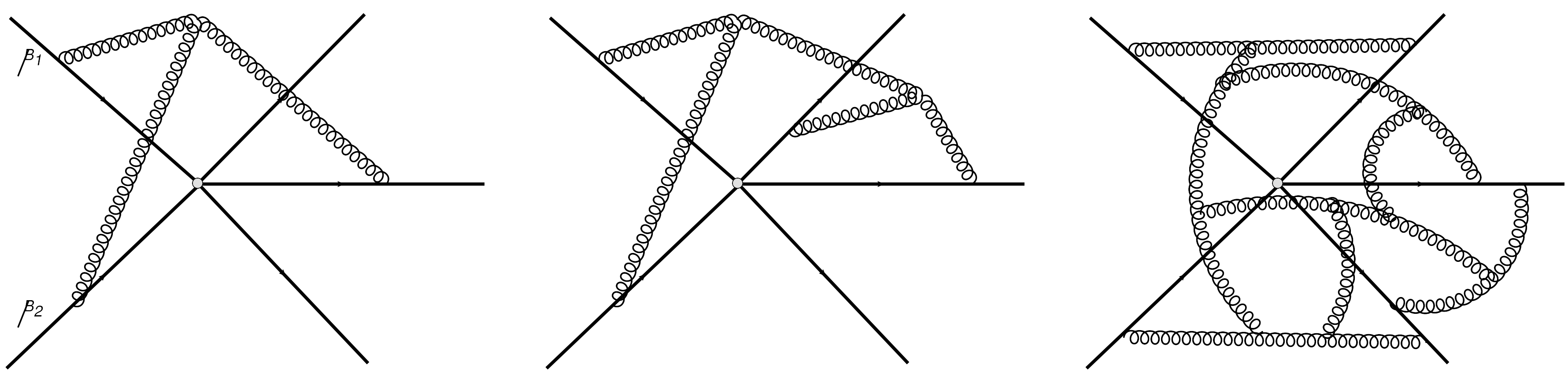
The soft function $\mathcal{S}$

Evolution $\Longrightarrow$ Exponentiation:

$$\mathcal{S}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = P \exp\left\{ -\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \mathbf{\Gamma}_{\mathcal{S}}\left(\beta_i \cdot \beta_j, \alpha_s(\lambda^2, \epsilon), \epsilon\right) \right\}$$

$\mathbf{\Gamma}_{\mathcal{S}}$ is a matrix of anomalous dimensions.

- A priori, $\mathbf{\Gamma}_{\mathcal{S}}$ can be very complicated: at each order in α_s it may contain new colour structures and kinematic dependence corresponding to sums of webs:



- In fact $\mathbf{\Gamma}^{\mathcal{S}}$ is (much?) simpler.

The soft anomalous dimension $\mathbf{\Gamma}_{\mathcal{S}}$ at two loops

Remarkable discovery: [Aybat Dixon Sterman (06)]

For any multi-leg amplitude:

$$\mathbf{\Gamma}_{\mathcal{S}}^{(2)} = \frac{K}{2}\mathbf{\Gamma}_{\mathcal{S}}^{(1)}$$

where $\mathbf{\Gamma}_{\mathcal{S}} = \sum_{n=1}^{\infty} \mathbf{\Gamma}_{\mathcal{S}}^{(n)} \left(\frac{\alpha_s(\mu)}{\pi}\right)^n$ and $K = \left(\frac{67}{18} - \zeta(2)\right) C_A - \frac{10}{9} T_F N_f$.

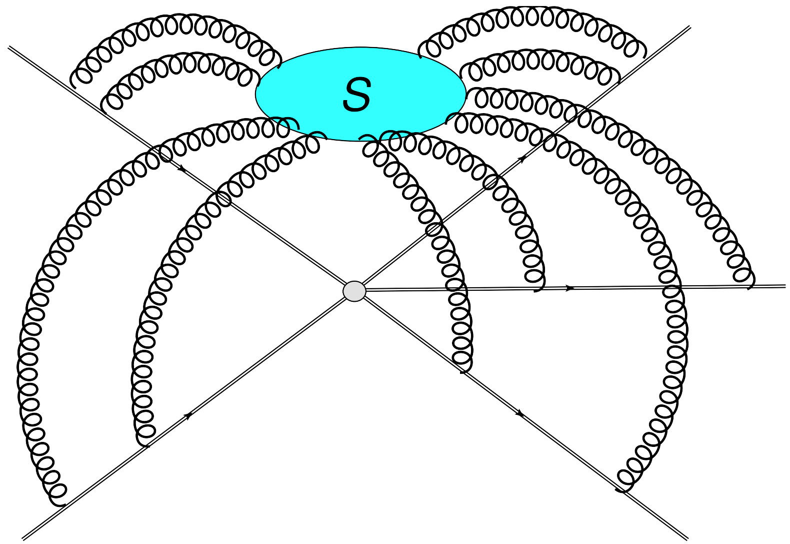
so at two loops: no new colour matrices, no new kinematic dependence...

- why?
- where is K coming from?
  This is the famous coefficient of the cusp anomalous dimension $\gamma_K^{(i)}$
  [Korchemsky Radyushkin (87), Kodaira Trentadue (82),...] :

$$\gamma_K^{(i)} = 2C_i \frac{\alpha_s}{\pi} + K C_i \left(\frac{\alpha_s}{\pi}\right)^2 + \cdots$$

very suggestive... does this extend to higher orders?

The soft function $\mathcal{S}$



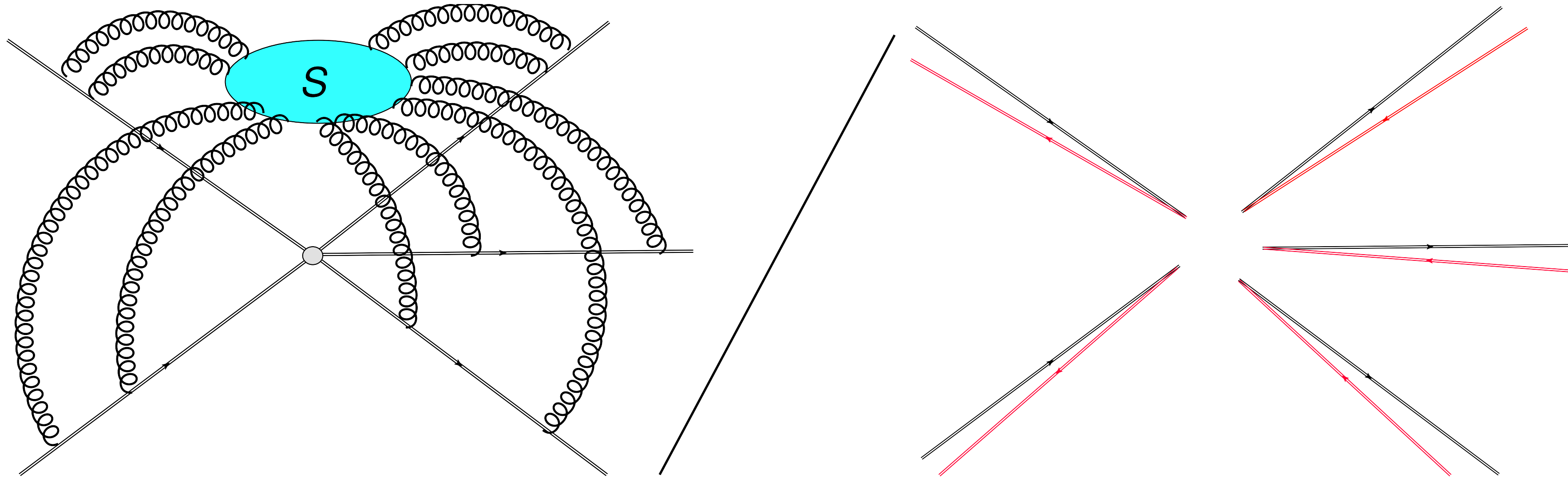
$$\mu \frac{d}{d\mu} \mathcal{S}_{JL} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) =$$
$$- \sum_N \left[\mathbf{\Gamma}_\mathcal{S} \right]_{JN} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) \mathcal{S}_{NL} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right)$$

$\mathbf{\Gamma}_\mathcal{S}$ has cusp singularities, and therefore, similarly to $\gamma_\mathcal{J}$

- it has poles in ϵ ($\mathcal{S}$ itself has double poles).
- it is not invariant with respect to $\beta_i \longrightarrow \kappa_i \beta_i$

Both these issues can be ‘fixed’ by dividing by appropriate eikonal jets...

The reduced soft function $\overline{\mathcal{S}}$

$\overline{\mathcal{S}} \;\equiv$ 

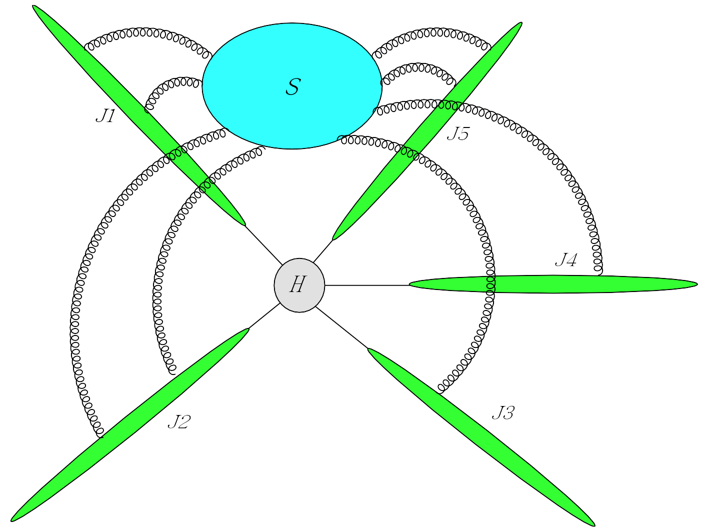
$$\overline{\mathcal{S}}_{JL}\left(\rho_{ij},\epsilon\right) = \frac{\mathcal{S}_{JL}\left(\beta_i\cdot\beta_j,\epsilon\right)}{\displaystyle\prod_{i=1}^{n}\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$

$\overline{\mathcal{S}}$ does not suffer from the cusp anomaly, and must therefore respect rescaling $\beta_i \longrightarrow \kappa_i\beta_i$:

$\Longrightarrow$ $\overline{\mathcal{S}}$ depends only on $$\rho_{ij} \equiv \frac{(\beta_i\cdot\beta_j)^2}{\left[2(\beta_i\cdot n_i)^2/n_i^2\right]\left[2(\beta_j\cdot n_j)^2/n_j^2\right]}$$

Factorization in terms of the reduced soft function $\overline{\mathcal{S}}$

$$\mathcal{M}_N\left(p_i/\mu,\epsilon\right) =$$



$$=$$

$$= \sum_L \mathcal{S}_{NL}\left(\beta_i\cdot\beta_j,\epsilon\right) H_L \prod_{i=1}^{n} \frac{J_i\left(\dfrac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)}{\mathcal{J}_i\left(\dfrac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$

$$= \sum_L \overline{\mathcal{S}}_{NL}\left(\rho_{ij},\epsilon\right) H_L \prod_{i=1}^{n} J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)$$

- $\overline{\mathcal{S}}$ has only single poles due to large-angle soft gluons.
- $\overline{\mathcal{S}}$ much like $\mathcal{M}$ cannot depend on the normalization of the velocities!

The equations for $\Gamma^{\overline{\mathcal{S}}}$

Factorization $+$ rescaling invariance imply:
$\Gamma^{\overline{\mathcal{S}}}$ for any multi-leg amplitude, in any colour basis, obeys:

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \mathbf{\Gamma}^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s) = \frac{1}{4} \gamma_K^{(i)}(\alpha_s) \ , \qquad \forall i$$

[Gardi Magnea (09)]

This is true to all orders, as well as at strong coupling.

- We have related the soft anomalous dimension of a general multi-leg amplitude to the cusp anomalous dimension.
- Intriguing relation between kinematics and colour.

Solving for $\Gamma^{\overline{\mathcal{S}}}$

$$\sum_{j\neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \mathbf{\Gamma}^{\overline{\mathcal{S}}}\left(\rho_{ij},\alpha_s\right) = \frac{1}{4}\,\gamma_K^{(i)}\left(\alpha_s\right)\ , \qquad\qquad \forall i$$

Does this set of differential equations have a unique solution?

- For two or three legs - yes! Then $\Gamma^{\overline{\mathcal{S}}}$ can be written in terms of γ_K, with explicitly determined kinematic dependence.
- For four or more legs - no: functions of conformal cross ratios

$$\rho_{ijkl} \equiv \frac{(\beta_i\cdot\beta_j)(\beta_k\cdot\beta_l)}{(\beta_i\cdot\beta_k)(\beta_j\cdot\beta_l)} = \left(\frac{\rho_{ij}\,\rho_{kl}}{\rho_{ik}\,\rho_{jl}}\right)^{1/2}$$

satisfy the homogeneous equation.

Yet, it has a simple all-order solution (minimal solution)

The sum-over-dipoles formula

$\gamma_K^{(i)}$ admits quadratic Casimir scaling ($C_i \equiv \mathrm{T}_i \cdot \mathrm{T}_i$) (at least to 3 loops):

$$\gamma_K^{(i)} = 2C_i \frac{\alpha_s}{\pi} + K C_i \left(\frac{\alpha_s}{\pi}\right)^2 + K^{(2)} C_i \left(\frac{\alpha_s}{\pi}\right)^3 + \cdots = C_i\, \widehat{\gamma}_K\left(\alpha_s\right) + \underbrace{\widetilde{\gamma}_K^{(i)}\left(\alpha_s\right)}_{\text{Higher Casimirs}}$$

The equations: $$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma_{\text{Q.C.}}^{\overline{\mathcal{S}}}\left(\rho_{ij}, \alpha_s\right) = \frac{1}{4} \mathrm{T}_i \cdot \mathrm{T}_i\, \widehat{\gamma}_K\left(\alpha_s\right)\,, \qquad \forall i$$

are solved by the sum-over-dipoles formula [Gardi Magnea (09)]:

$$\Gamma_{\text{Q.C.}}^{\overline{\mathcal{S}}}\left(\rho_{ij}, \alpha_s\right) = -\frac{1}{8} \widehat{\gamma}_K\left(\alpha_s\right) \sum_{i \neq j} \ln(\rho_{ij})\, \mathrm{T}_i \cdot \mathrm{T}_j + \frac{1}{2} \widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s) \sum_{i=1}^{n} \mathrm{T}_i \cdot \mathrm{T}_i\,,$$

- Generalises the two loop result to all orders (minimal solution!)
- Kinematics and colour are directly correlated.

The same formula was simultaneously proposed by Becher and Neubert.

Conclusions

- Detailed understanding of infrared singularities in QCD amplitudes is needed for cross section calculations and for resummation.
- Recent progress:
  - Remarkable simplicity at two loops — now better understood.
  - A completely general constraint was derived based on factorization and rescaling symmetry.
    It relates soft singularities in any amplitude, and any loop order, to the cusp anomalous dimension.
  - An all-loop sum-over-dipoles formula naturally emerges as a minimal solution.
- Several research avenues have opened up. The full beauty of gauge theory amplitudes is not yet revealed...

Beyond the minimal solution

Corrections to the sum-over-dipoles formula are of two kinds

- terms that are induced by higher Casimir contributions to γ_K — they may appear starting at four loops and must satisfy the equations

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}_{\text{H.C.}} (\rho_{ij}, \alpha_s) = \frac{1}{4} \widetilde{\gamma}_K^{(i)} (\alpha_s) \, , \qquad \forall i,$$

- solutions of the homogeneous equations

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}} (\rho_{ij}, \alpha_s) = 0 \qquad \forall i$$

  namely, functions of conformal cross ratios. These may appear starting at three loops, four legs.

Absence of $\hat{\mathbf{H}}^{(2)}_{[\text{f}]} = \sum_{j,k,l} \sum_{a,b,c} \mathrm{i} \, f_{abc} \, \mathrm{T}_j^a \mathrm{T}_k^b \mathrm{T}_l^c \, \ln (\rho_{ijkl}) \, \ln (\rho_{iklj}) \, \ln (\rho_{iljk})$

at the two-loops $\Gamma^{\overline{\mathcal{S}}}$ supports the minimal solution!