Accurate Simulation of the Finite Density Lattice Thirring Model

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Introduction

- Calculations at finite density of fermions often encounter sign problems
- In the 2D lattice Thirring model the sign problem can be solved with
  - Massless fermions
  - Open spatial boundary conditions
- Similar to QCD
  - Asymptotically free
  - Dynamically generated fermion mass
  - Massless boson
  - But no chiral symmetry breaking[1]
- Useful as a benchmark for testing other solutions to sign problems[2]

Model

We study the lattice Thirring model with the action

$$S = \sum_{x,y} \bar{\chi}_x \left(D^{KS}_{x,y}(\mu)\right) \chi_y + U \sum_{x,\nu} \bar{\chi}_x \chi_x \bar{\chi}_{x+\nu} \chi_{x+\nu} \quad (1)$$

$$D^{KS}_{x,y} = \frac{1}{2}\eta_{x,\nu} e^{\mu\delta_{\nu,0}} \delta_{y,x+\nu} - \frac{1}{2}\eta^{\dagger}_{x,\nu} e^{-\mu\delta_{\nu,0}} \delta y, x-\nu \quad (2)$$

$$\eta_{x,\nu} = (-1)^{\sum_{\mu>\nu} x_\mu} \quad (3)$$

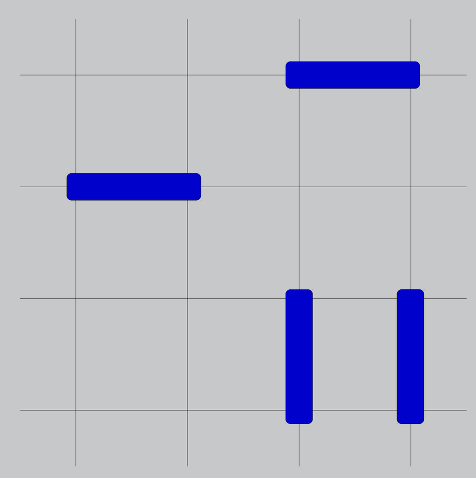
Fermion bag representation:[3]

- Expand partition function

$$Z = \int d\bar{\chi} d\chi e^{-S} = \int d\bar{\chi} d\chi e^{-\sum_{x,y} \bar{\chi}_x D^{KS}_{x,y}(\mu) \chi_y} \prod_{x,\nu} \left(1 + U \bar{\chi}_x \chi_x \bar{\chi}_{x+\nu} \chi_{x+\nu}\right) \quad (4)$$

$$= \sum_{[d]} U^{N_d} \int d\bar{\chi} d\chi \left(\bar{\chi}_x \chi_x \bar{\chi}_{x+\nu} \chi_{x+\nu}\right)^{d_{x,\nu}} e^{-\sum_{x,y \in [f]} \bar{\chi}_x D^{KS}_{x,y} \chi_y} \quad (5)$$

$$= \sum_{[d]} U^{N_d} \det\left(W([f], \mu)\right) \quad (6)$$



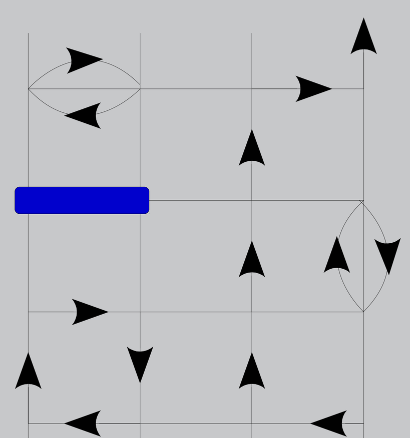
- New dimer variable $d_{x,\nu}$
- Determinant only over free sites
- Local updates, calculate change in determinant

Worldline Representation of the Determinant

$$\det\left(W([f], \mu)\right) \quad (7)$$

$$= \prod_{x \in [f]} \left(\int d\bar{\chi}_x d\chi_x\right) \exp\left(-\sum_{x,y \in [f]} \left[\frac{1}{2}\eta_{x,\nu} e^{\mu\delta_{\nu,0}} \bar{\chi}_x \chi_{x+\nu} - \frac{1}{2}\eta^{\dagger}_{x,\nu} e^{-\mu\delta_{\nu,0}} \bar{\chi}_{x+\nu} \chi_x\right]\right) \quad (8)$$

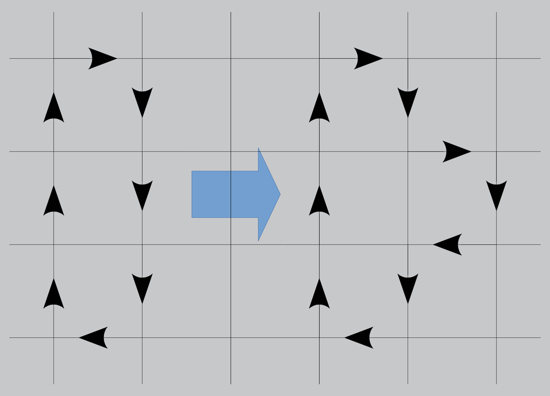
$$= \sum_{[l]} \prod_{loop \in l} \left(-\prod_{x,\alpha \in loop} e^{\pm\mu\delta_{\pm\alpha,0}} \frac{\eta_{x,\alpha}}{2}\right). \quad (9)$$



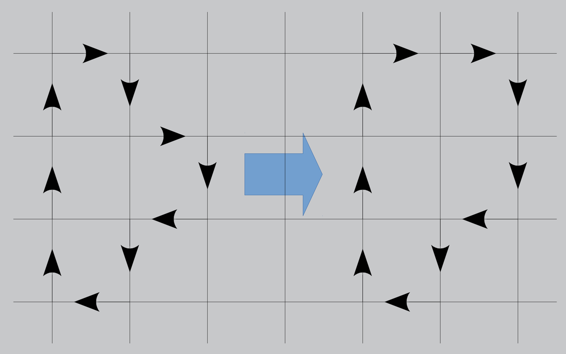
- $l_{x,\nu}$ are worldline variables of fermions
- Worldlines form closed loops because fermion number is conserved

Sign problems are absent with open boundary conditions

- Open boundary conditions: $\bar{\chi}_{x=0} = \chi_{x=0} = \bar{\chi}_{x=N_x} = \chi_{x=N_x} = 0$
- All loops can be generated from simple ones using two deformations:



(a) does not change the sign or the volume



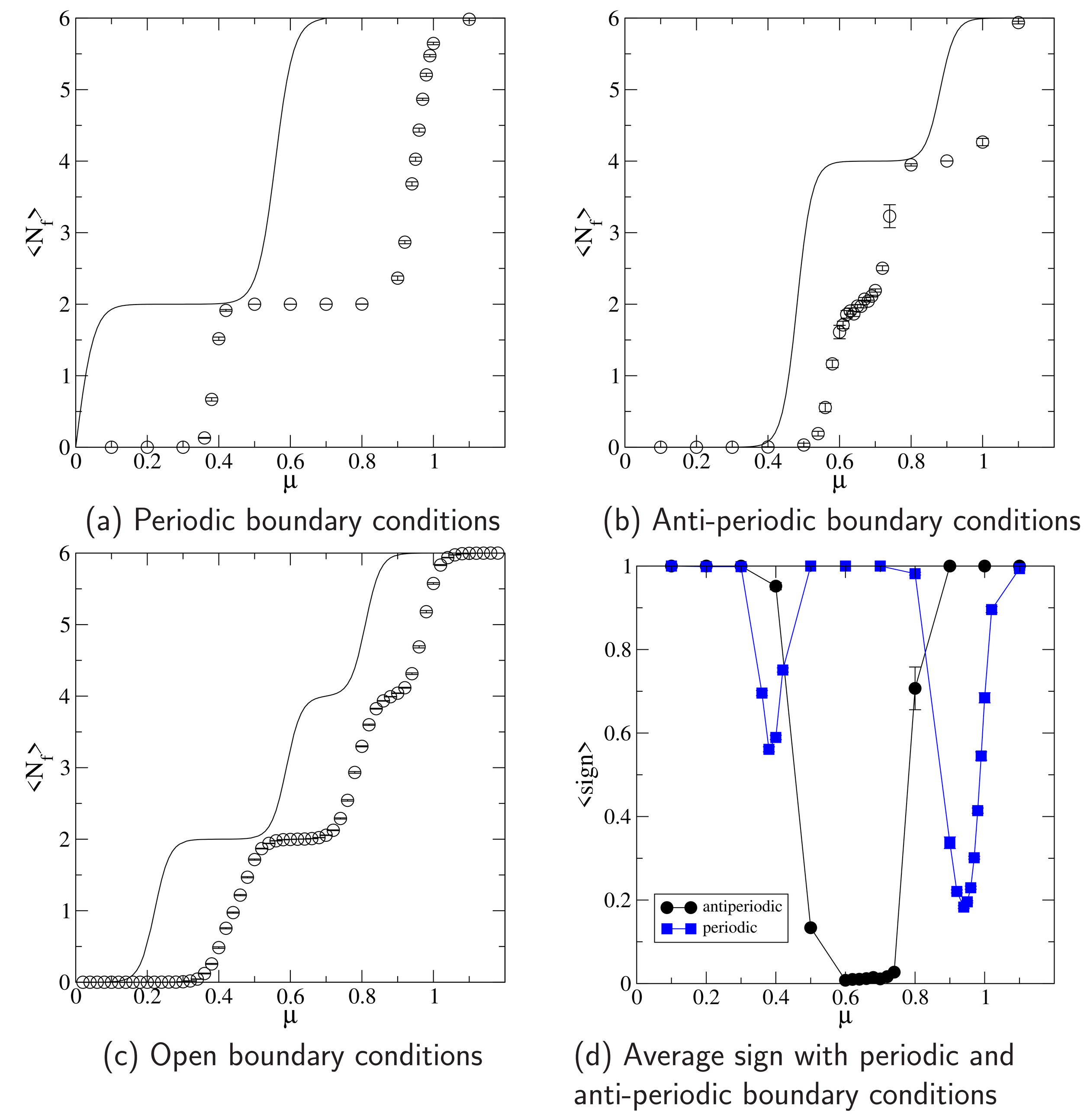
(b) changes the sign and the enclosed volume

Two classes of loops:

- Time wrapping loops
- Non-wrapping loops

Only even volumes can be enclosed → only positive signs

- Update with an efficient worm algorithm



(a) Periodic boundary conditions (b) Anti-periodic boundary conditions

(c) Open boundary conditions (d) Average sign with periodic and anti-periodic boundary conditions

Measurements of $< N_f >$ with $U = 0.3$, $L_X = 6$ and $L_T = 48$. The solid lines shows the value of $\langle N_f \rangle$ at $U = 0$.

Results

U	$2-\eta$	$\langle Q^2_\chi \rangle$	m_f	$m_b^{L_X=32}$	$m_f^{L_X=32}$
0	0	0	0	0.067393	0.045(5)
0.1	0.90(1)	0.499(7)	0.0098(4)	0.061(4)	0.0705(6)
0.2	1.201(4)	0.61(1)	0.081(1)	0.06(1)	0.1397(3)
0.3	1.303(4)	0.780(8)	0.183(1)	0.066(7)	0.247(2)
0.4	1.371(7)	0.895(4)	0.290(1)	0.060(3)	0.356(1)
0.5	1.393(3)	0.972(3)	0.395(3)	0.057(2)	0.465(1)
0.6	1.423(4)	1.024(3)	0.491(1)	0.049(5)	0.557(2)
1.0	1.467(4)	1.128(2)	0.793(1)	0.050(4)	0.842(2)
∞	1.5	1.208(8)	∞	0.0476(9)	∞

- Fermion number

$$\langle N_f \rangle = \left\langle \sum_{x \in S} \frac{\eta_{x,\alpha}}{2} \left[e^{\mu} \bar{\psi}_x \psi_{x+\alpha} - e^{-\mu} \bar{\psi}_{x+\alpha} \psi\right] \right\rangle = \left\langle \sum_{x \in S} l_{x,\hat{t}} - l_{x+\hat{t},-\hat{t}} \right\rangle \quad (10)$$

- Fermion mass m_f
  - First jump in $\langle N_f \rangle$ on an anisotropic lattice
  - On a large square lattice $\langle N_f \rangle = \max(a(\mu - m_f), 0)$
- Fermion number susceptibility

$$\chi = \frac{U}{V} \sum_{x,y} \langle \bar{\psi}_x \psi_x \bar{\psi}_y \psi_y \rangle \quad (11)$$

which scales as

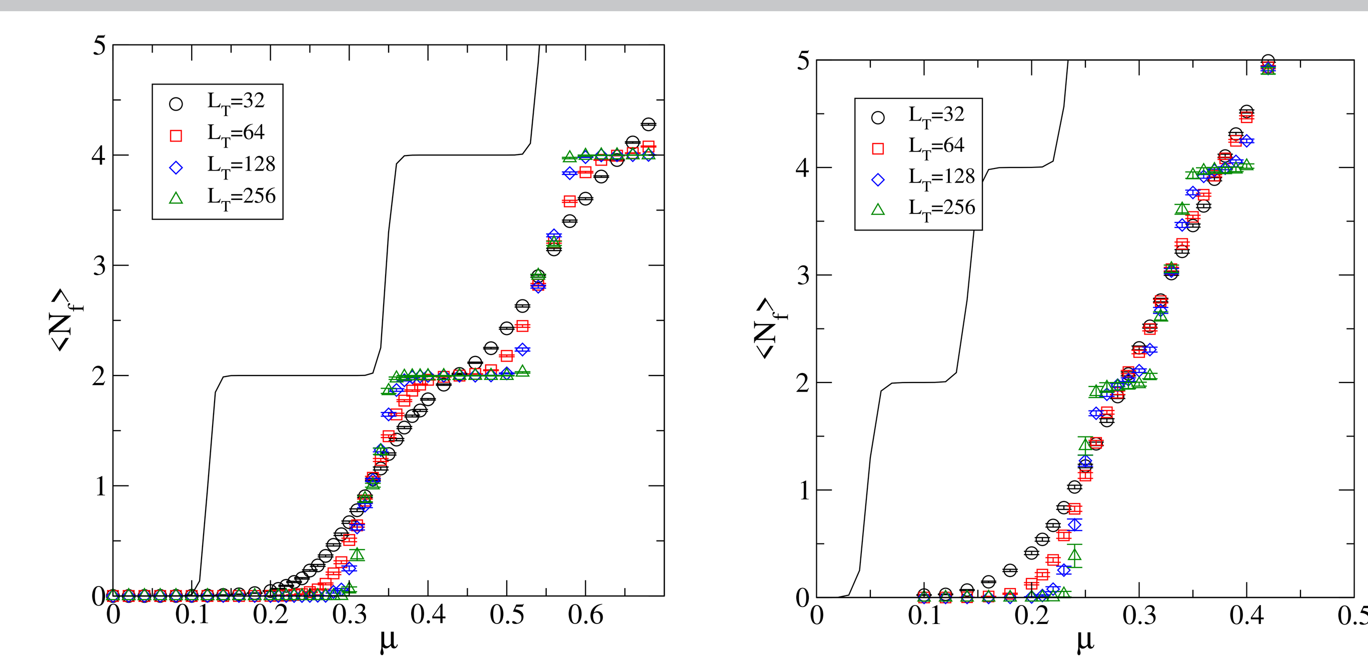
$$\chi = AL^{2-\eta}, \quad (12)$$

- Boson mass

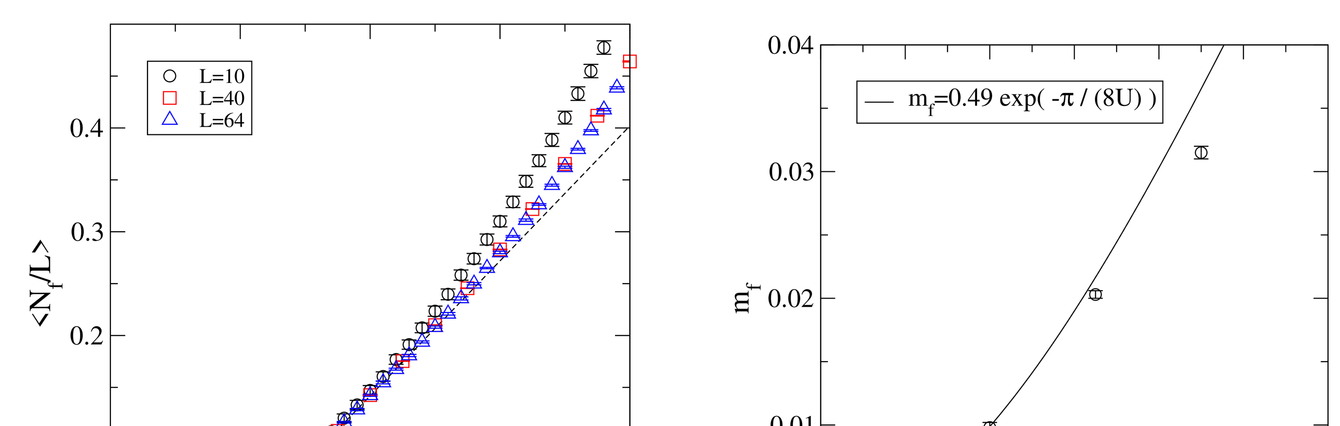
$$\chi(L_T) = \chi(\infty) - ce^{-m_b L_T} \quad (13)$$

- Chiral charge

$$\langle Q^2_\chi \rangle = \left\langle \left(\sum_x \epsilon_x \bar{\psi}_x \psi_{x+\hat{t}} - \epsilon_x \bar{\psi}_{x+\hat{t}} \psi_x\right)^2 \right\rangle \quad (14)$$



The fermion number at $U = 0.3$ and $L_X = 12$ (left) and 32 (right)



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