Gauge/Gravity Duality 2015

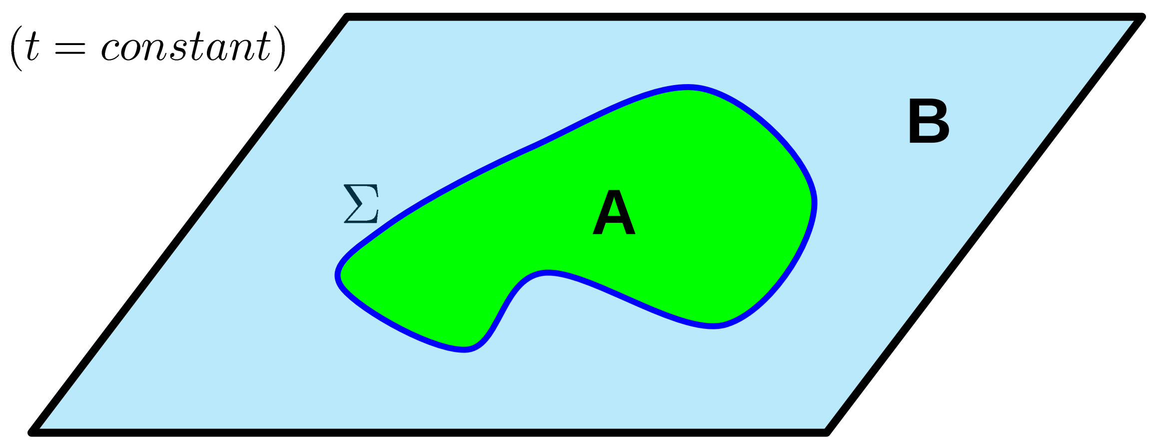
# Mutual Information & the F-Theorem

with Casini, Huerta & Yale

## Entanglement Entropy

- general tool; divide quantum system into two parts and use entropy as measure of correlations between subsystems
- in QFT, typically introduce a (smooth) boundary or entangling surface $\Sigma$ which divides the space into two separate regions
- integrate out degrees of freedom in "outside" region
- remaining dof are described by a density matrix $\rho_A$

$\longrightarrow$ calculate von Neumann entropy: $S_{EE} = -Tr\,[\rho_A\,\log\rho_A]$



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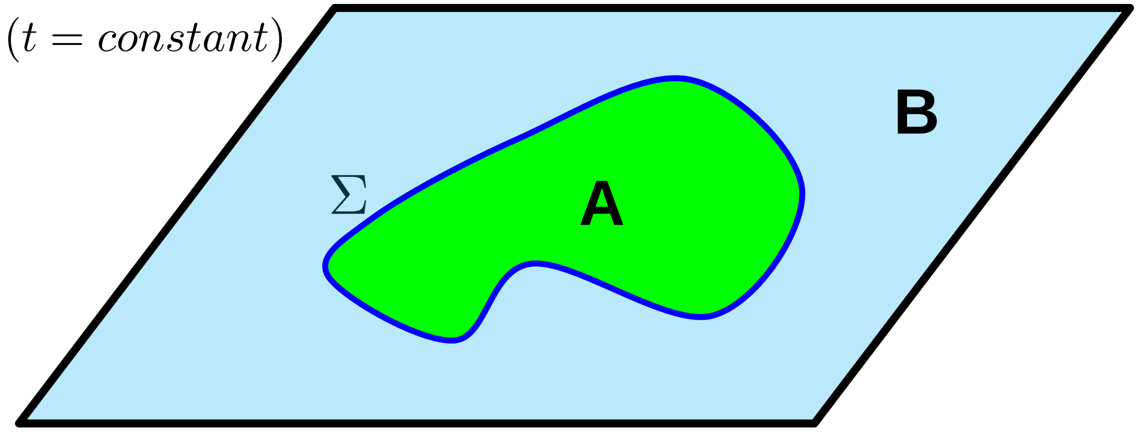


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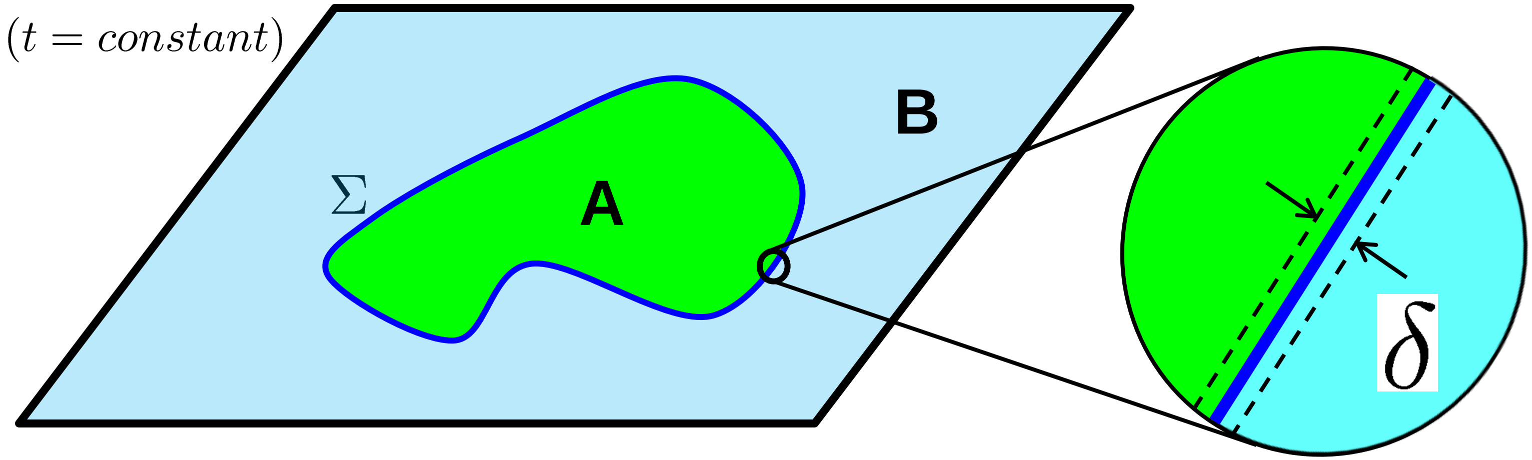


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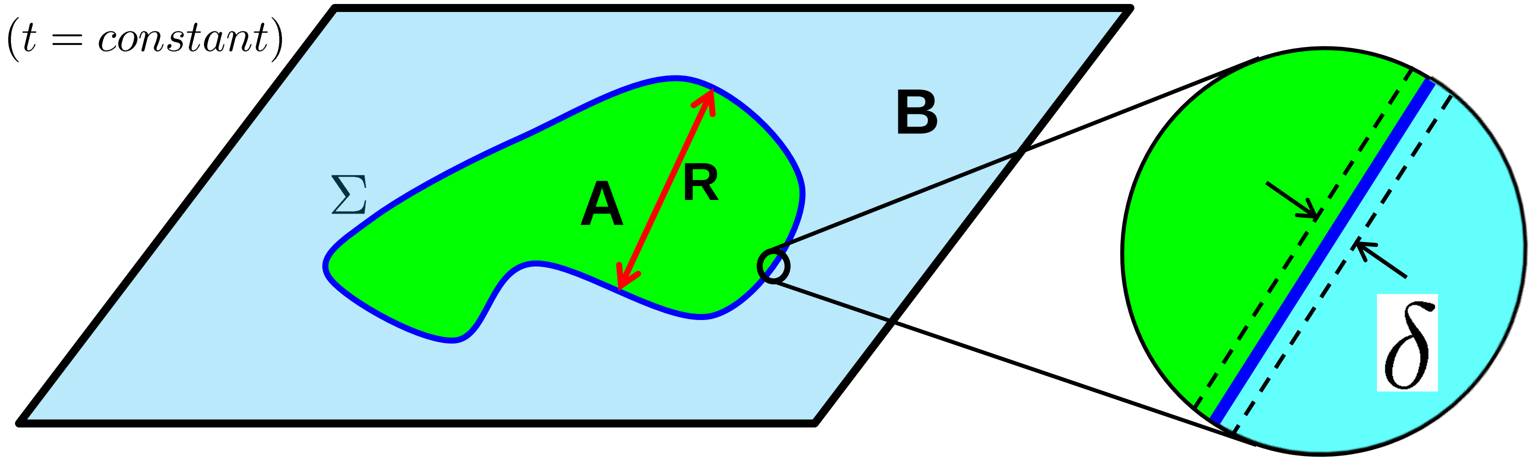


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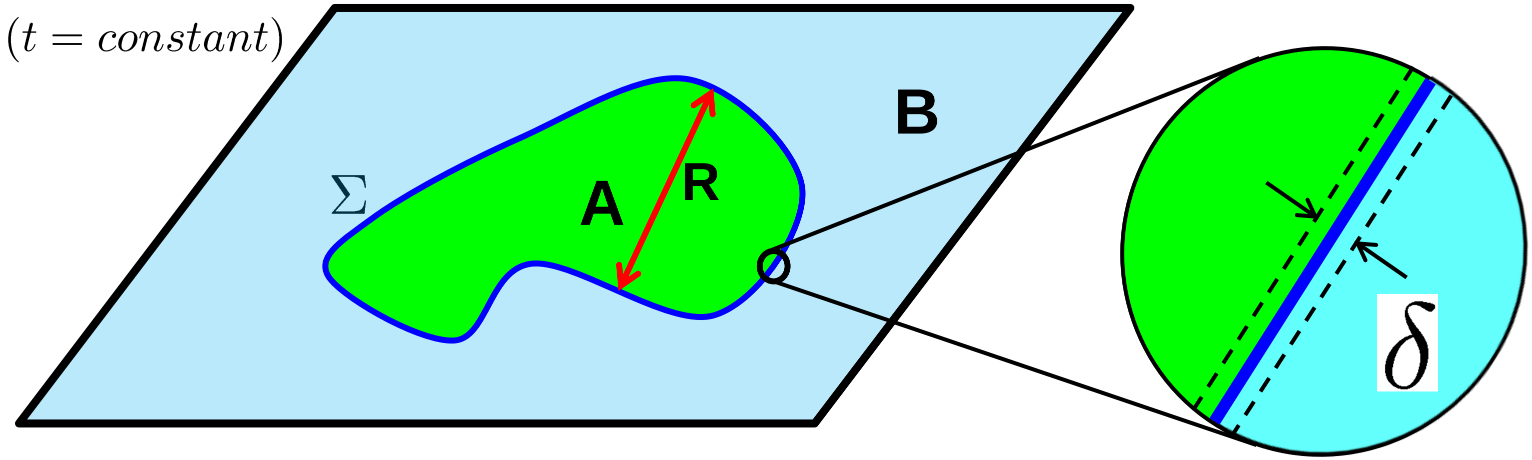


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- careful analysis reveals geometric structure, eg, $S = \tilde{c}_0 \frac{\mathcal{A}_\Sigma}{\delta^{d-2}} + \cdots$

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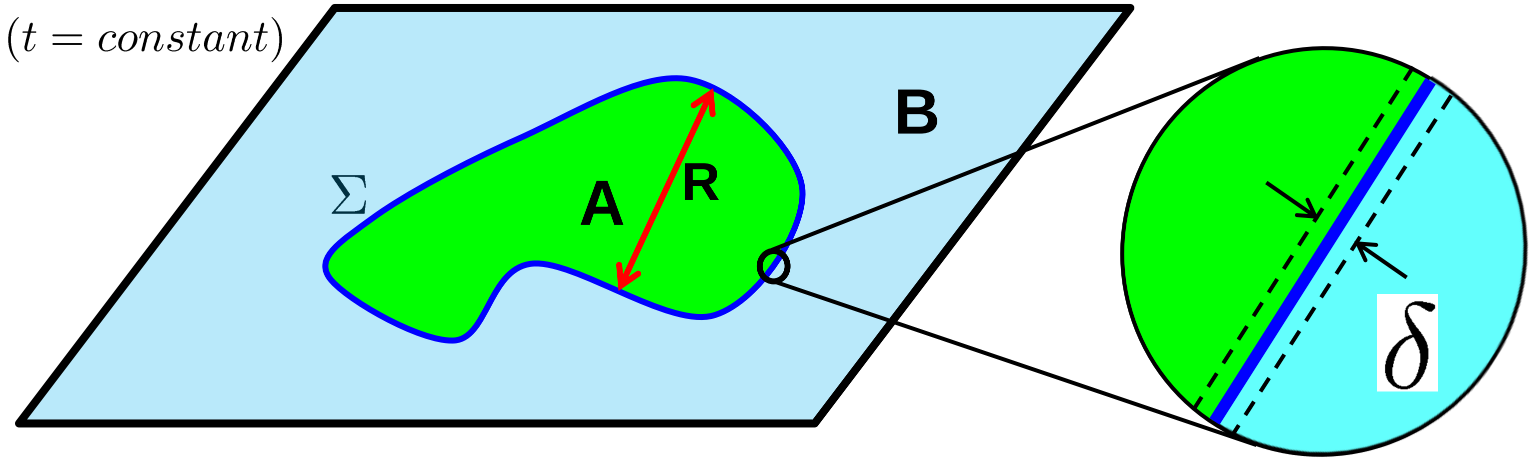


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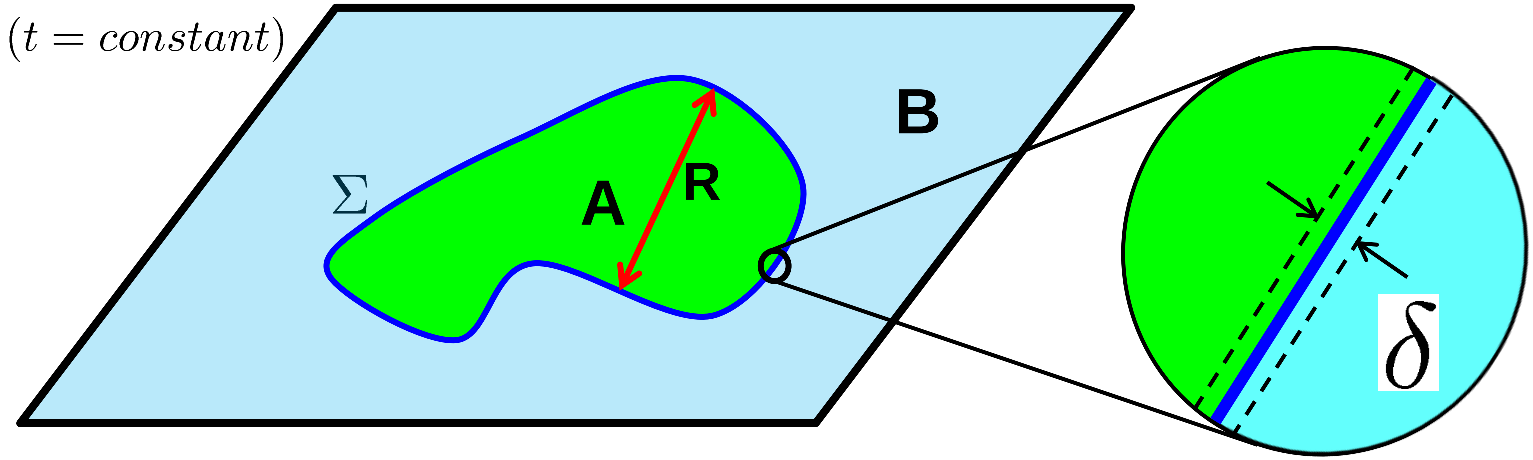


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- leading coefficients sensitive to details of regulator, eg, $\delta \to 2\delta$
- find universal information characterizing underlying QFT in subleading terms, eg, $S = \cdots + c_d \log\left(R/\delta\right) + \cdots$

## Entanglement C-theorem conjecture:

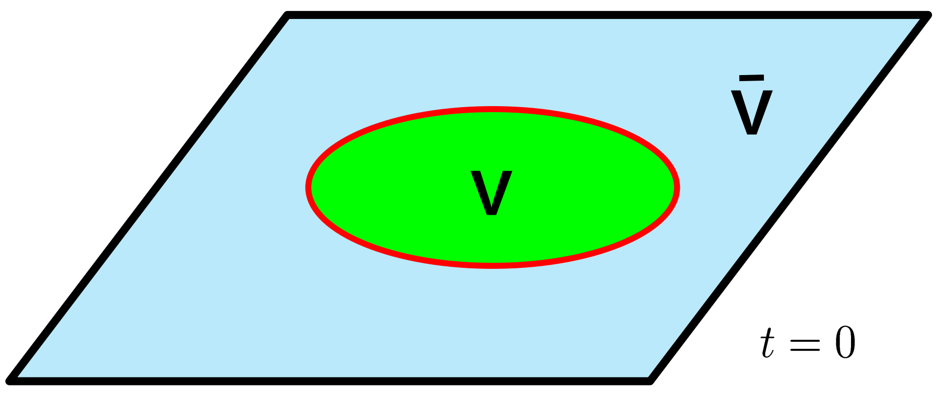
(RM & Sinha '10)

- identify central charge with universal contribution in entanglement entropy of ground state of CFT across sphere $S^{d-2}$ of radius R:

$$S_{univ} = \begin{cases} (-)^{\frac{d}{2}-1} 4\, a_d \log(2R/\delta) & \textbf{for even } \boldsymbol{d} \\ (-)^{\frac{d-1}{2}} 2\pi\, a_d & \textbf{for odd } \boldsymbol{d} \end{cases}$$

- for RG flows connecting two fixed points

$$(a_d)_{UV} \geq (a_d)_{IR}$$



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eg, d=2 ⟶ $a_2 = \dfrac{c}{12}$, Zamolodchikov's c-theorem

(entanglement proof: Casini & Huerta, '04)

d=4 ⟶ $a_4 = a$, a-theorem (Komargodski & Schwimmer, '11)

## F-theorem: d=3

(Jafferis, Klebanov, Pufu & Safdi '11)

- examine partition function for broad classes of 3-dimensional quantum field theories on three-sphere (SUSY gauge theories, perturbed CFT's & O(N) models)
- in all examples, F= – log $Z(S^3)>0$ and decreases along RG flows

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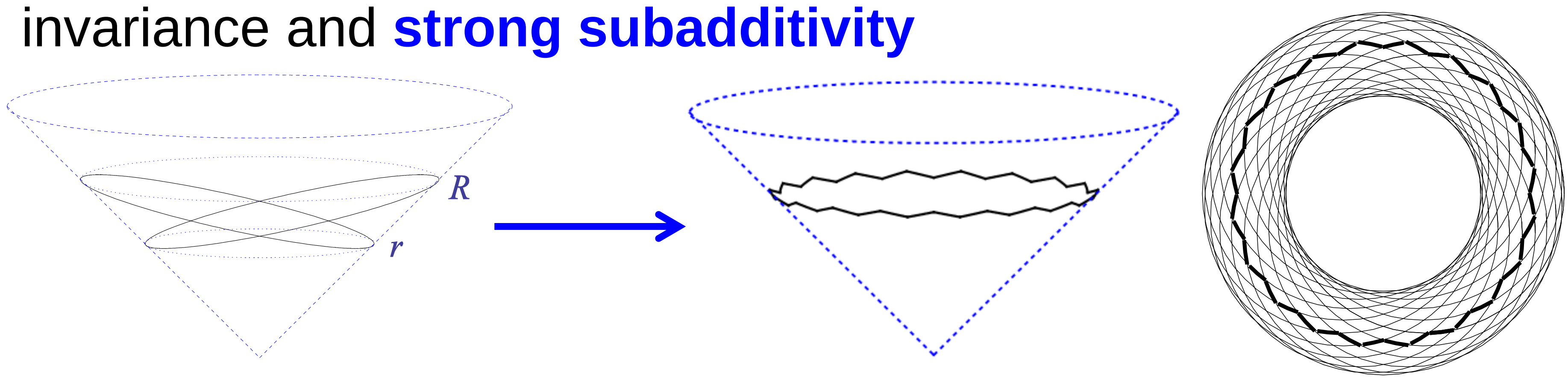
- **entanglement proof of F-theorem:** (Casini & Huerta '12)

  F-theorem for d=3 RG flows established using unitarity, Lorentz invariance and **strong subadditivity**



## Why is universal term in $S_{EE}$ universal?

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$\longrightarrow$ scales from RG flow can appear in final $S_{EE}$!!

(eg, Hertzberg & Wilczek; Banerjee)

**d=3:** $S(R) = 2\pi R\left(\frac{c_0}{\delta} + m\, c_1\right) - 2\pi a_3$

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- in regulators, tension between Lorentz inv. and unitarity

  $\longrightarrow$ latter emerge in $\delta \to 0$ limit, but regulator exposed in $S_{EE}$

## **“Renormalized” Entanglement Entropy:** (Liu & Mezei)

- divergences determined by local geometry of entangling surface with covariant regulator, eg,

$$S_{CFT} = c_0 \frac{R^{d-2}}{\delta^{d-2}} + c_2 \frac{R^{d-4}}{\delta^{d-4}} + \cdots + (-)^{\frac{d-1}{2}}\, 2\pi\, a_d + O(\delta/R)$$

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d=3: $\mathcal{S}_3(R) = R S'(R) - S(R)$

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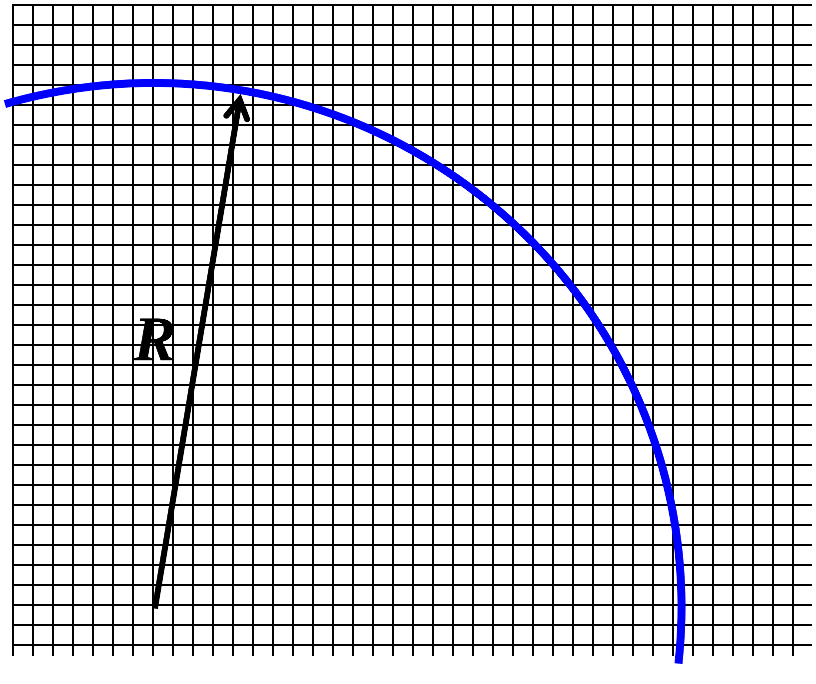
- approach demands special class of regulators: “covariant”

$\longrightarrow$ is result artifact of choosing “nice” regulator??

- if $a_d$ is physical, we should be able to use any regularization which defines the continuum QFT

• consider defining $a_3$ in presence of lattice regulator

$$d = 3: \; S(R) = \frac{2\pi R}{\delta}\, c_0 - 2\pi a_3$$



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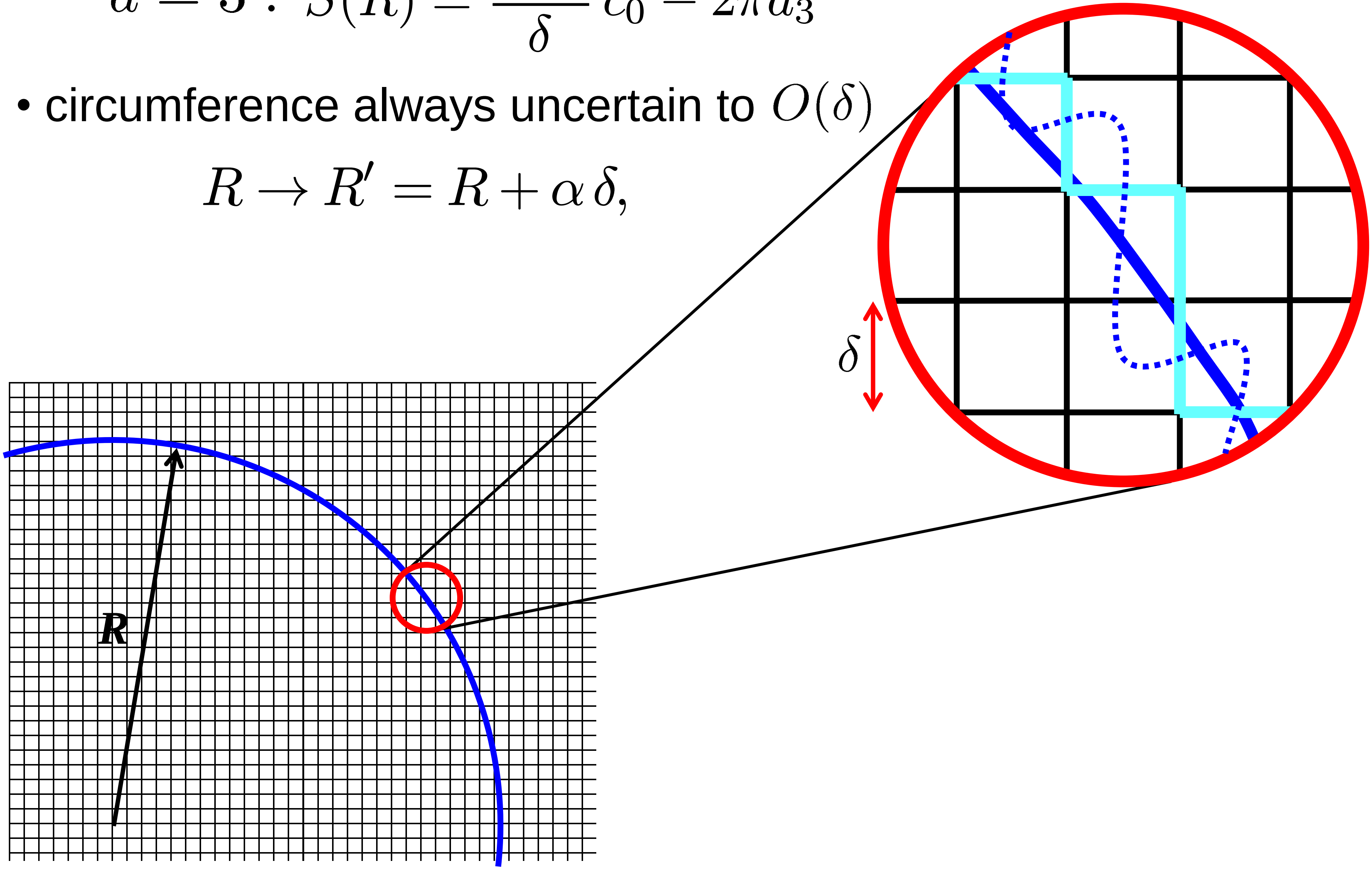

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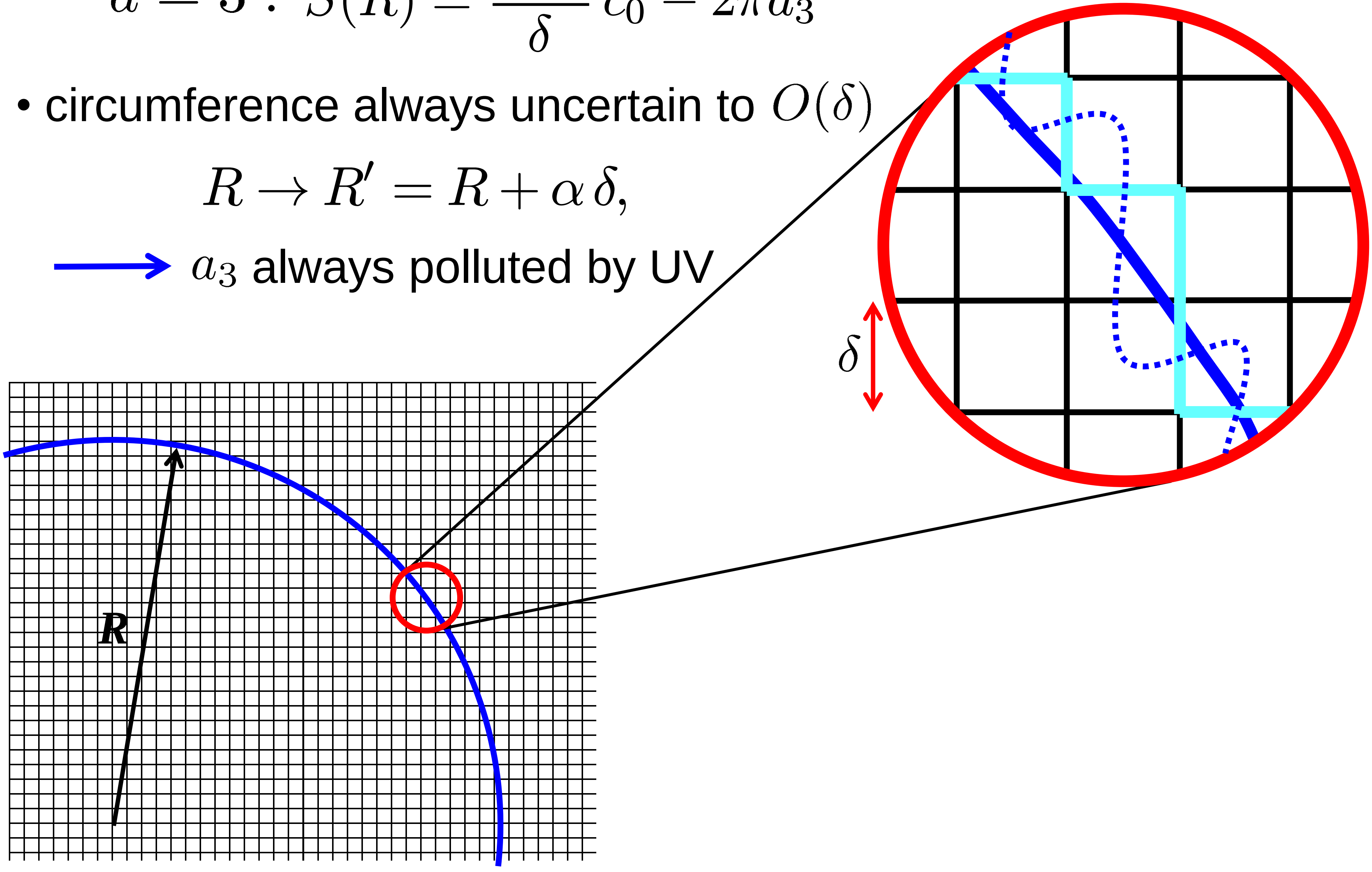

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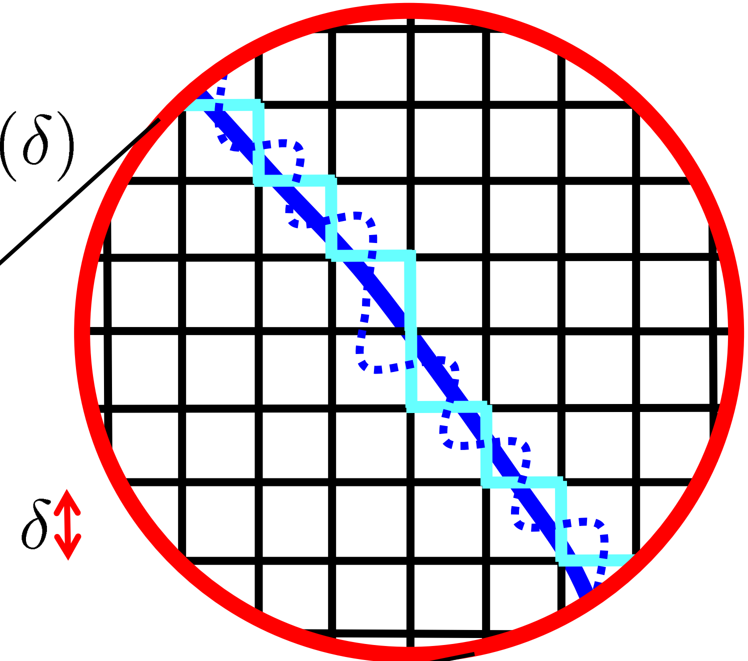


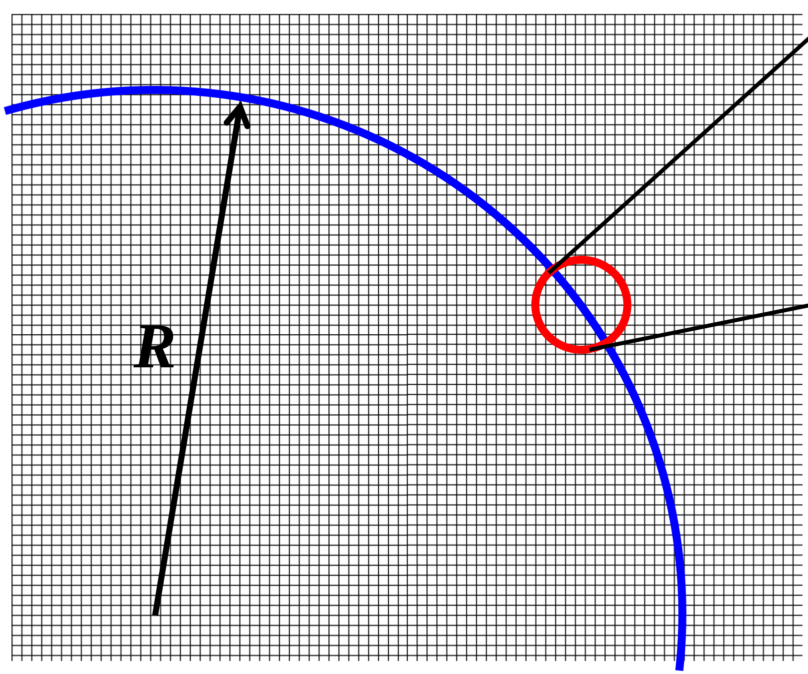


→ considering finer resolution, can **not** repair problem**!!**

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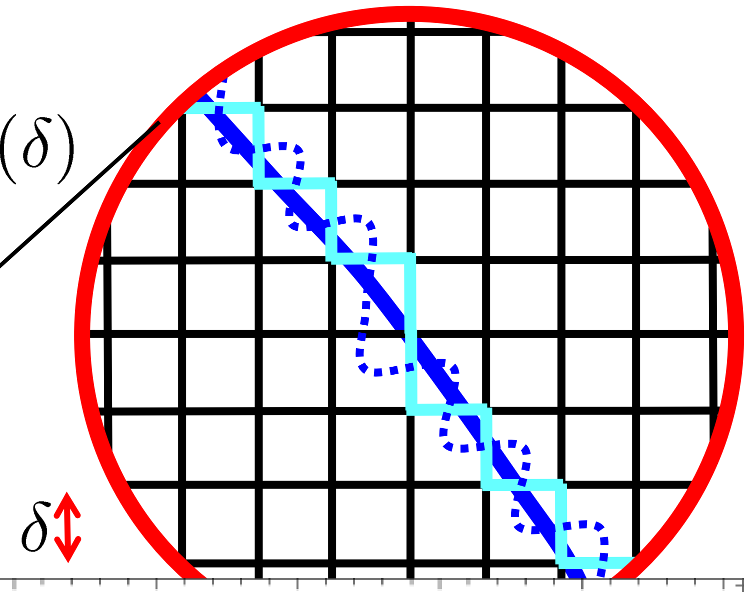


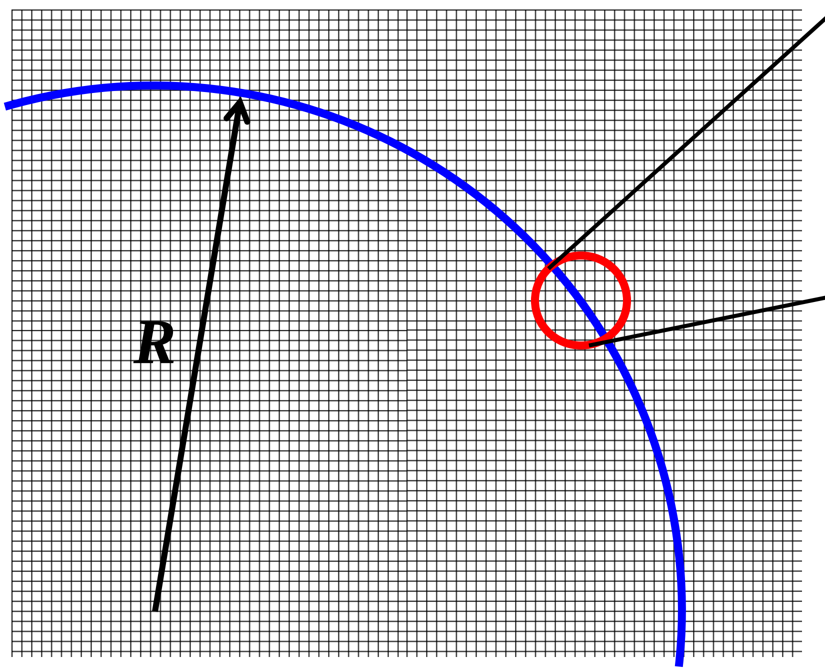


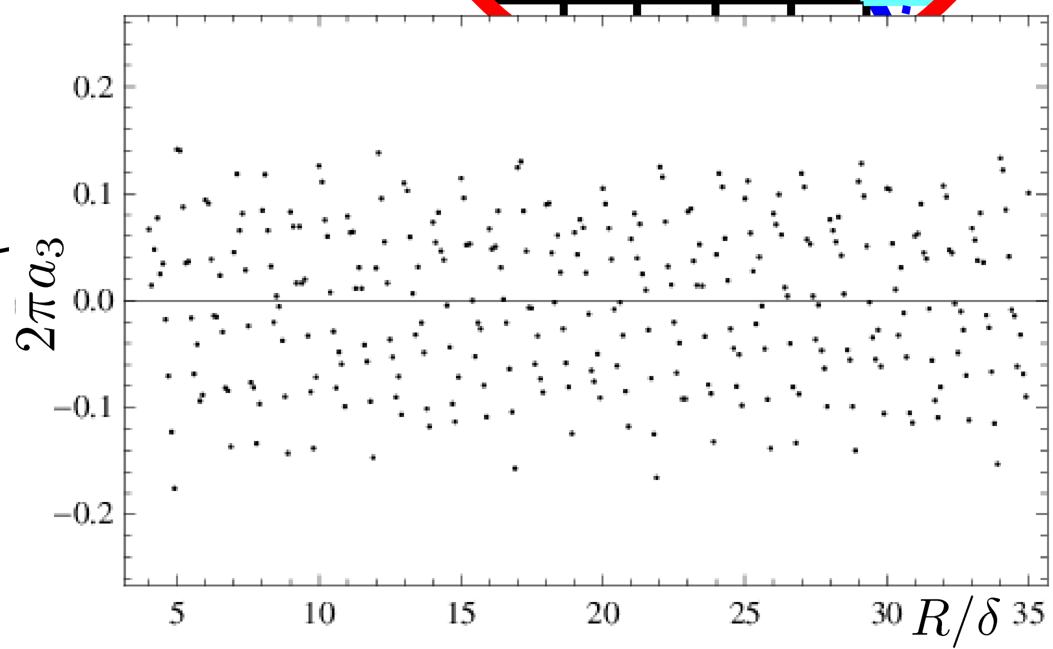

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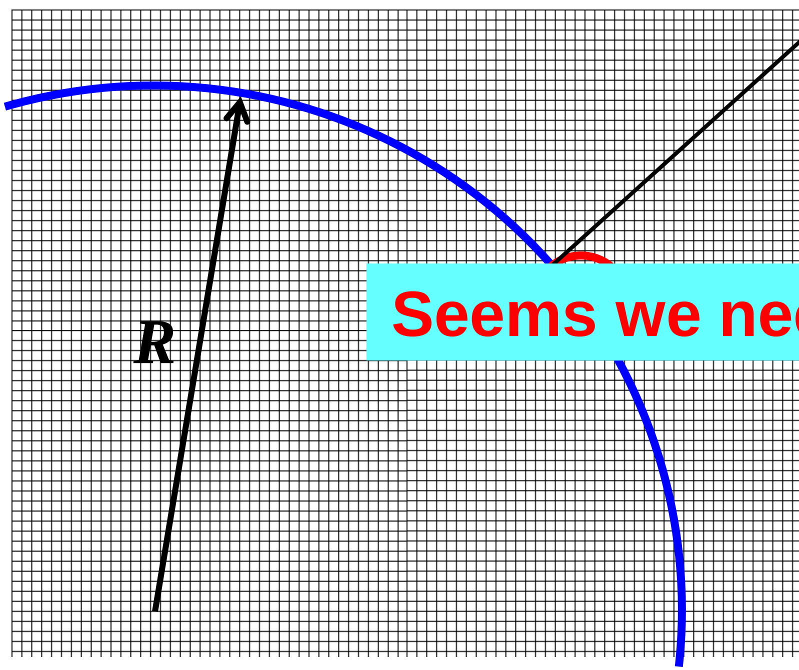


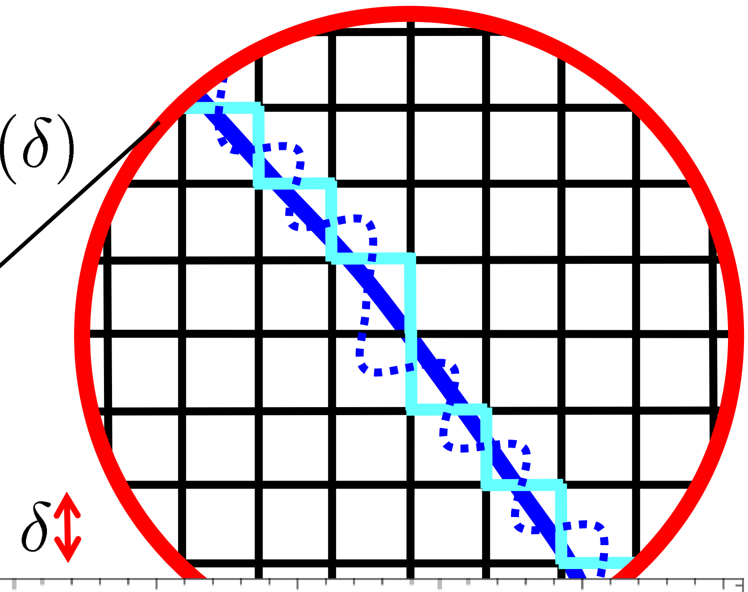


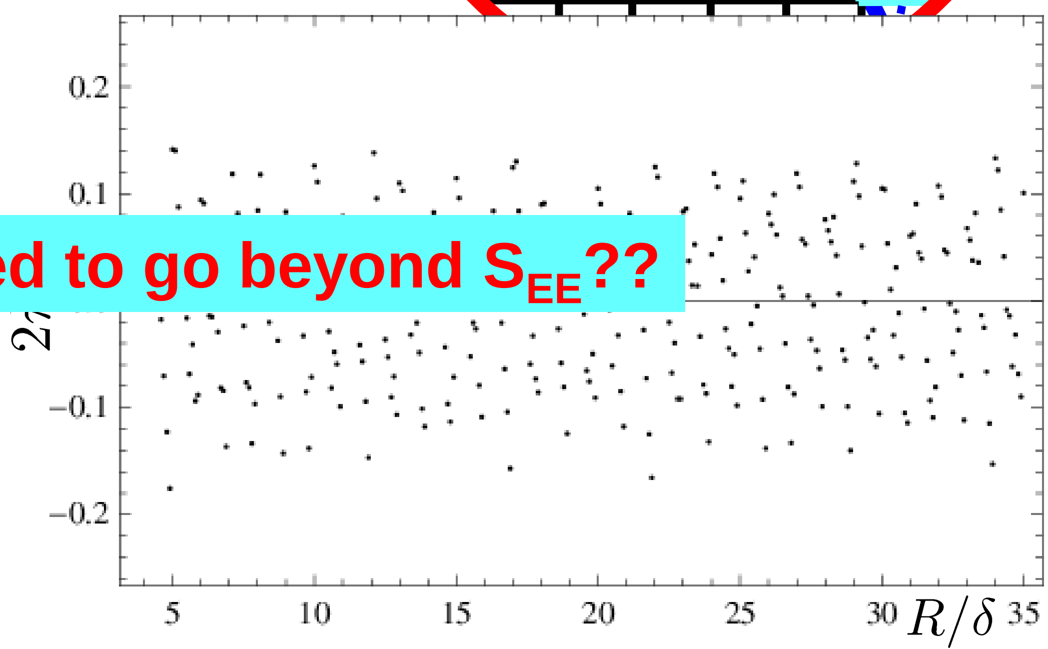


**Seems we need to go beyond $S_{EE}$??**

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- $S_{EE}$ seems to fail to satisfy criteria 1 & 2
- **alternate choice? alternate measure of entanglement?**

## Mutual Information:

- another measure of entanglement between two systems
- for non-intersecting regions A and B:

$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

- can be defined without reference to $S_{EE}$ (Araki; Narnhofer)
- bounds correlators between A and B (Wolf, Verstraete, Hastings & Cirac)

$$I(A,B) \geq \frac{|\langle \mathcal{O}_A \, \mathcal{O}_B \rangle_c|^2}{2\|\mathcal{O}_A\|^2 \, \|\mathcal{O}_B\|^2}$$

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$\longrightarrow$ criterion 1 will automatically be satisfied

$\longrightarrow$ criterion 2 & 3 will be satisfied with further care

## C-function from Mutual Information:

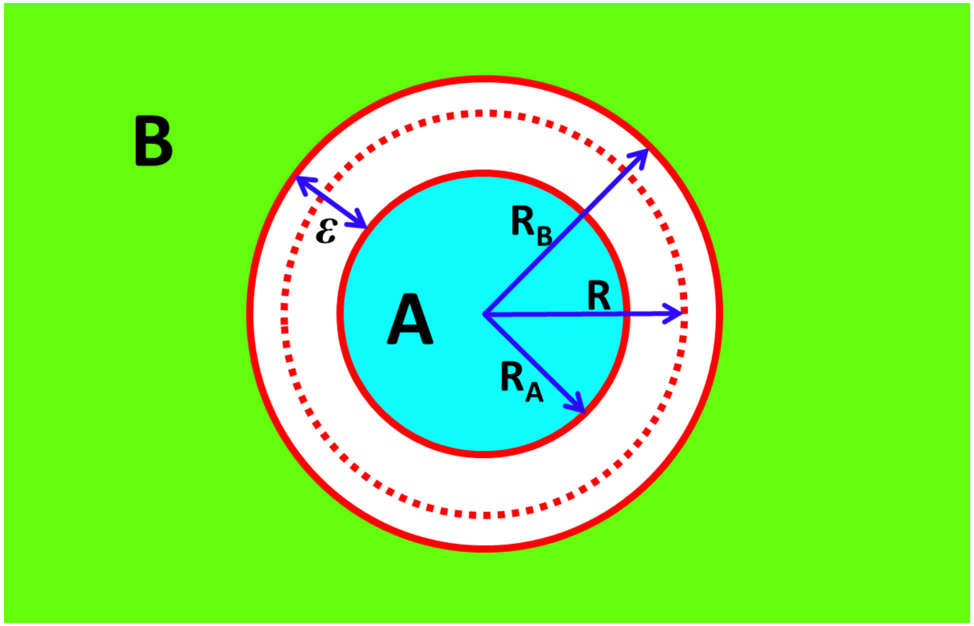
$$I(A,B) = S(A) + S(B) - S(A \cup B)$$

- consider following geometry:

$$R_A = R - \left(\frac{1}{2} - \alpha\right)\varepsilon$$

$$R_B = R + \left(\frac{1}{2} + \alpha\right)\varepsilon$$

or $$R = \frac{R_A + R_B}{2} - \alpha\,\varepsilon$$




- using $S(A) = S(\bar{A})$ for pure state:

$$I(A,B) = S(A) + S(\bar{B}) - S(\overline{A \cup B})$$

two disks ~ R

narrow annulus

- work in continuum limit: $R \gg \varepsilon \ggg \delta$

($R$ and $\varepsilon$ are macroscopic scales)

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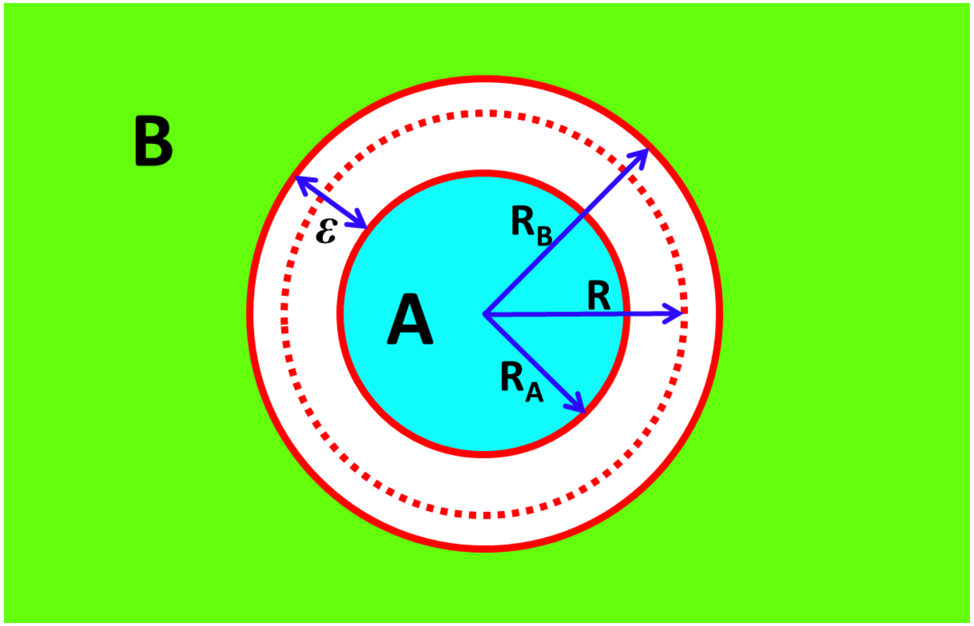


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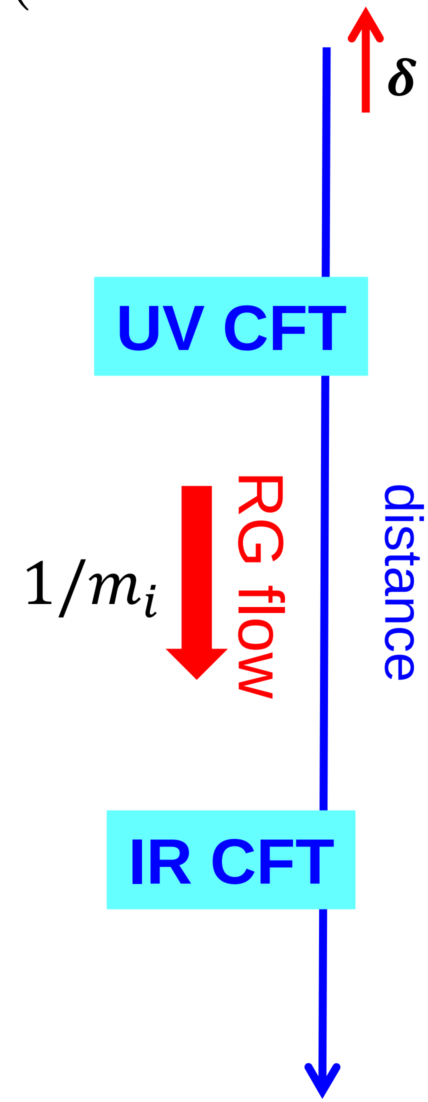

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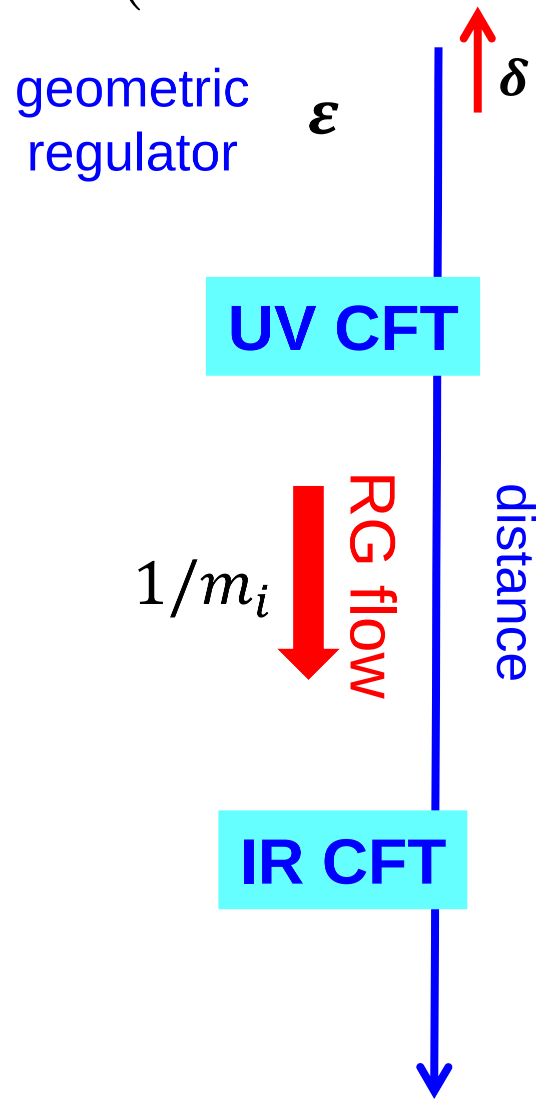


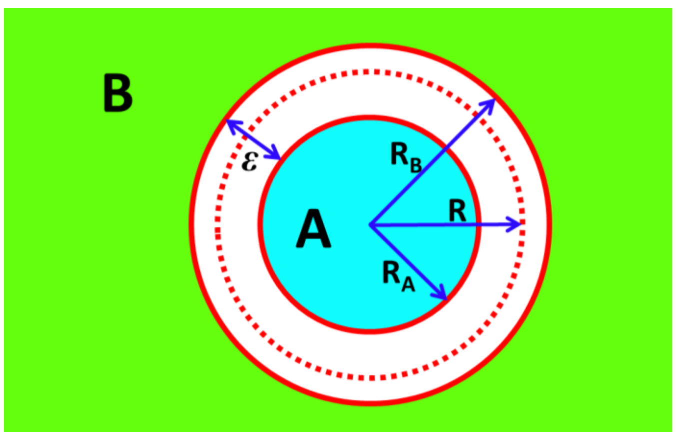

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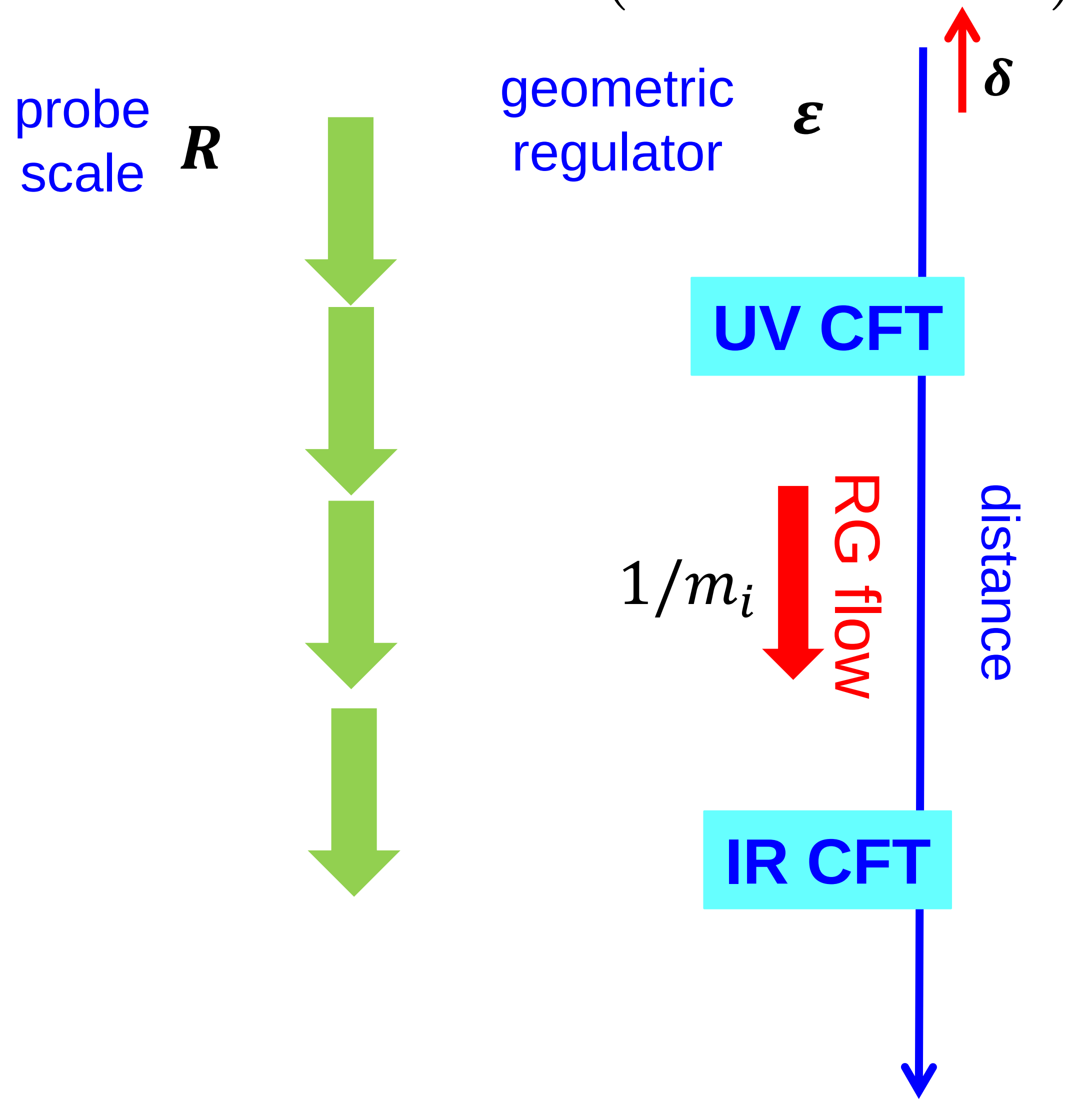


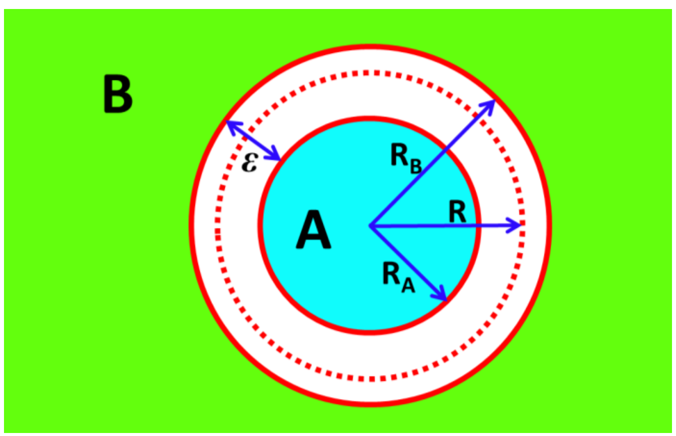

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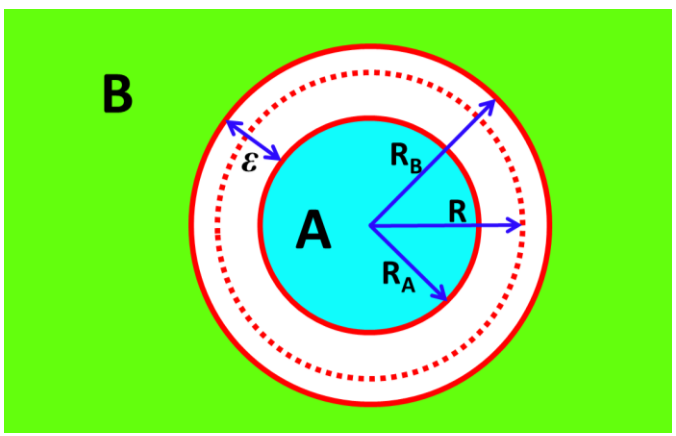


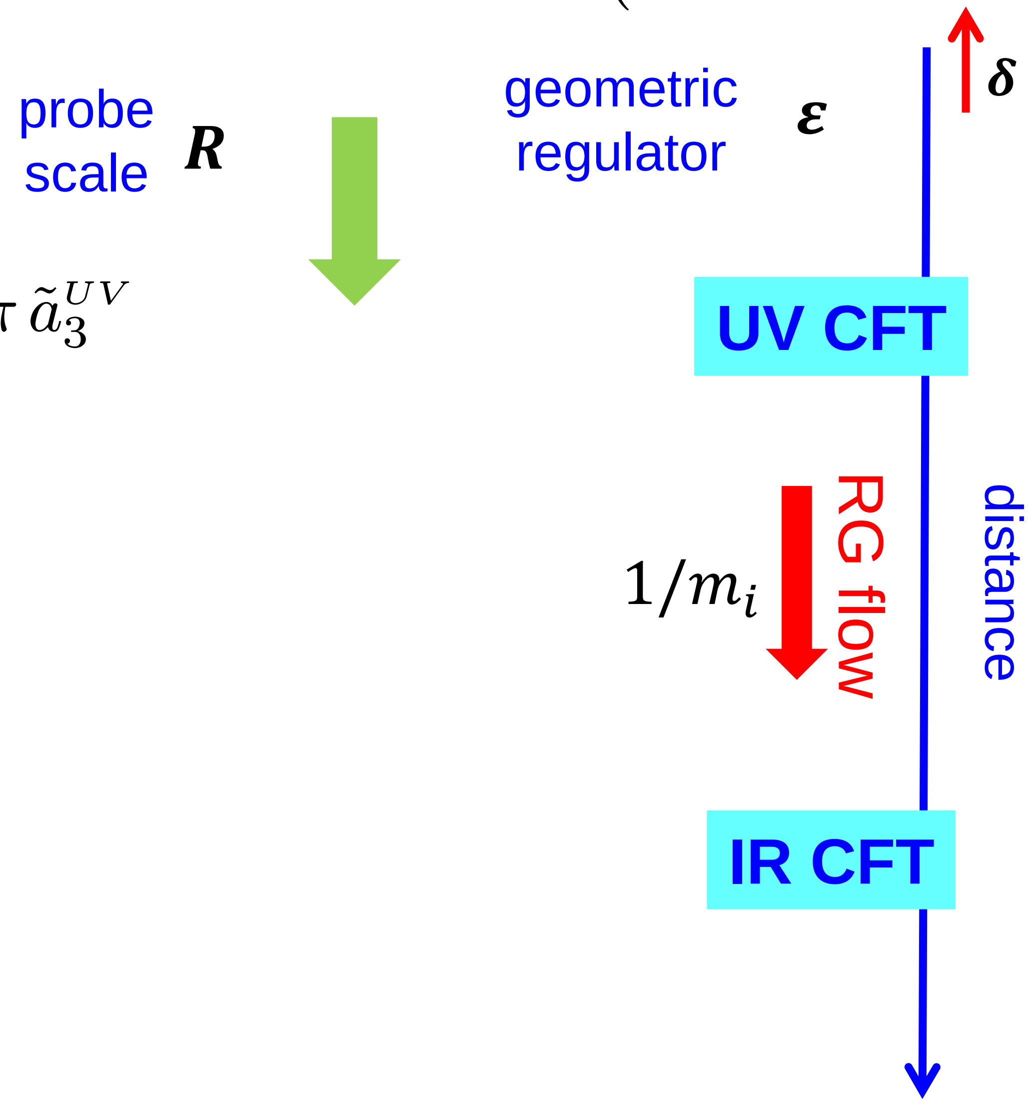

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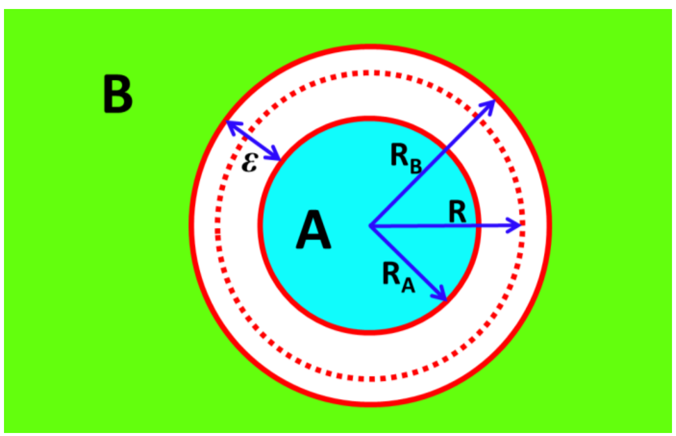


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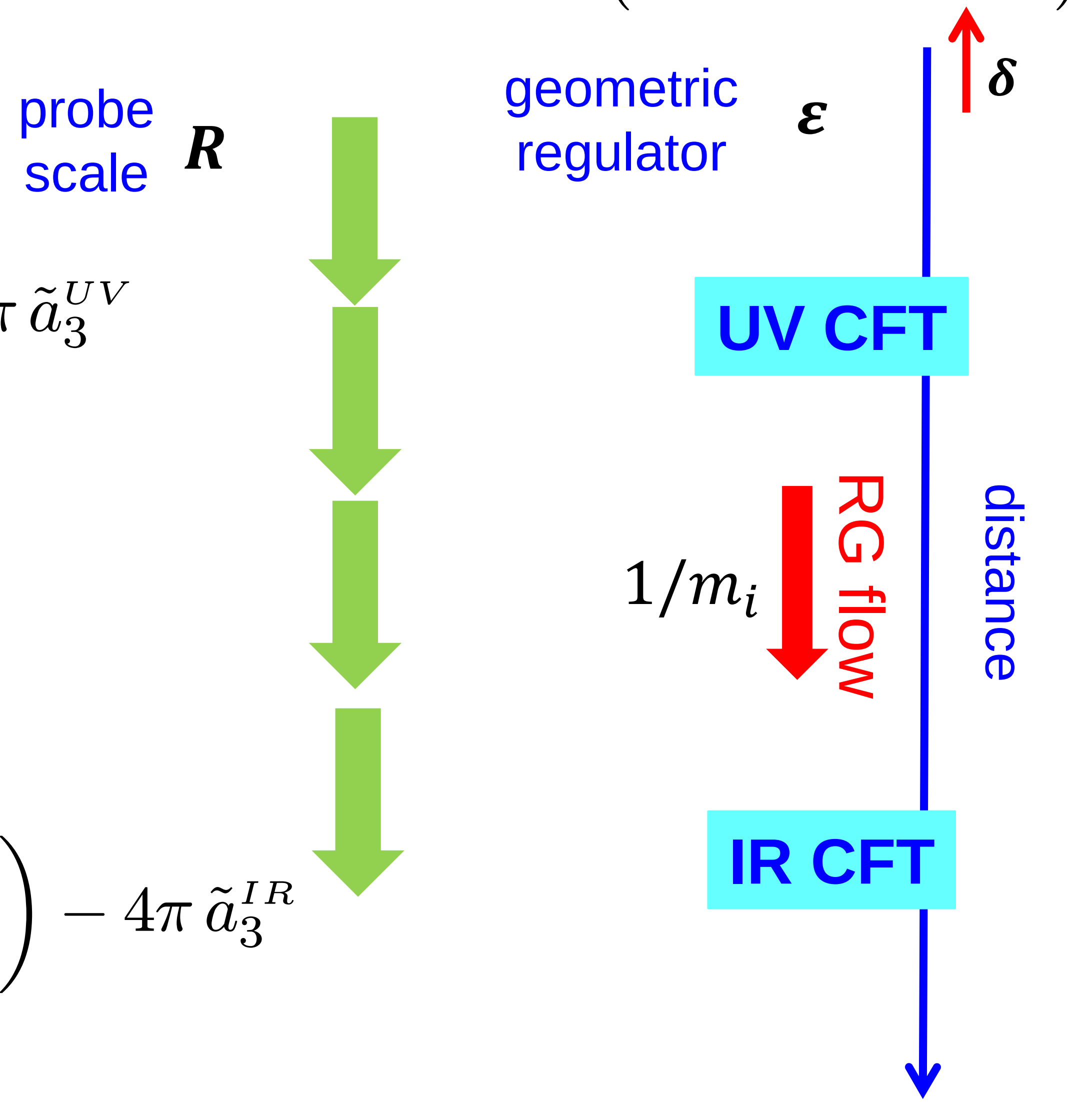

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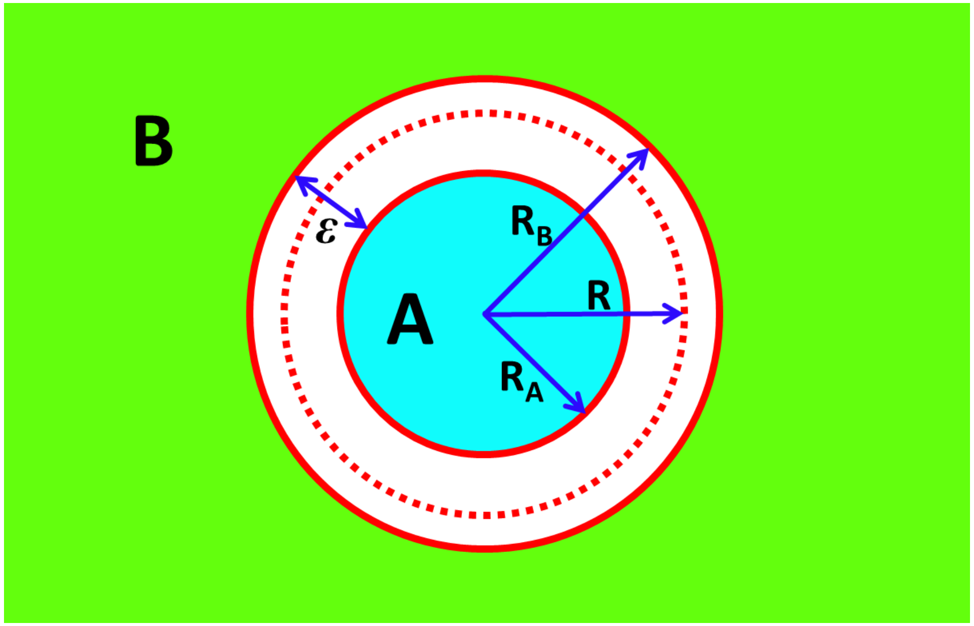


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- criterion 2? is $\tilde{a}_3$ intrinsic to fixed point??

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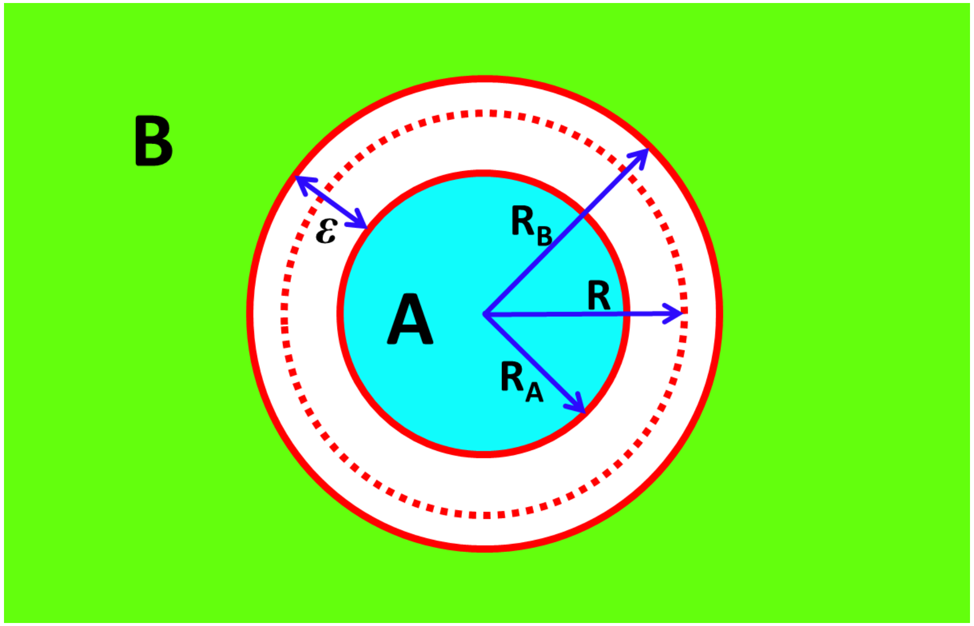


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- ambiguity: $\alpha \to \alpha' = \alpha + \delta\alpha\,,\;\; \tilde{a}_3 \to \tilde{a}_3' = \tilde{a}_3 + \tilde{c}_0\,\delta\alpha$

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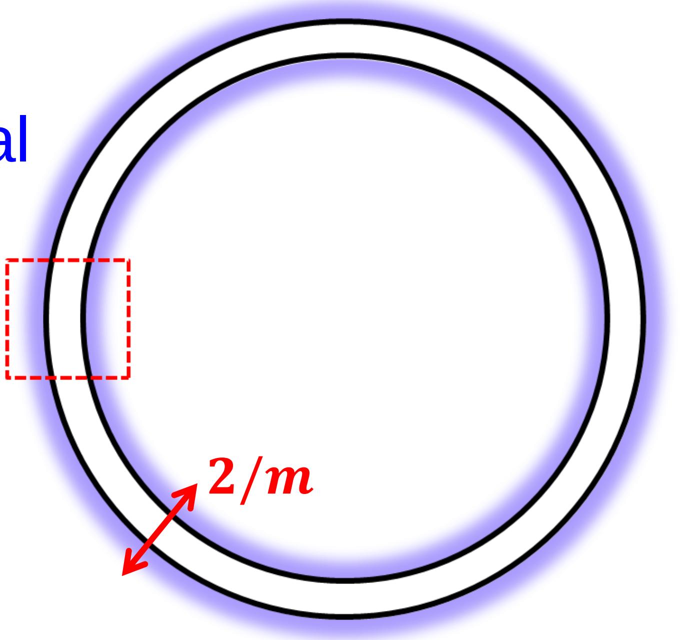

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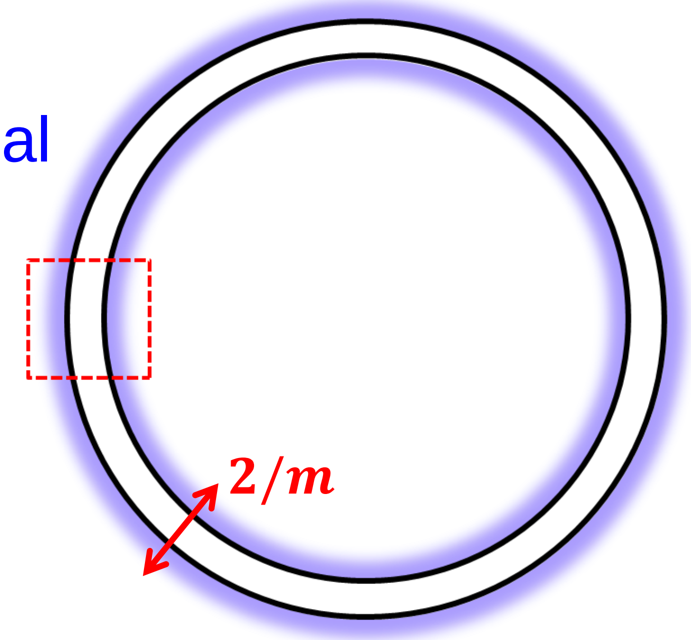


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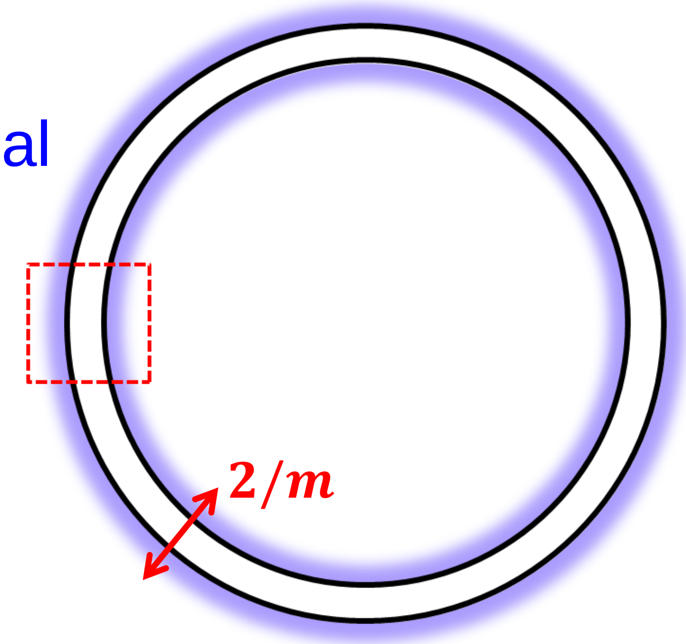


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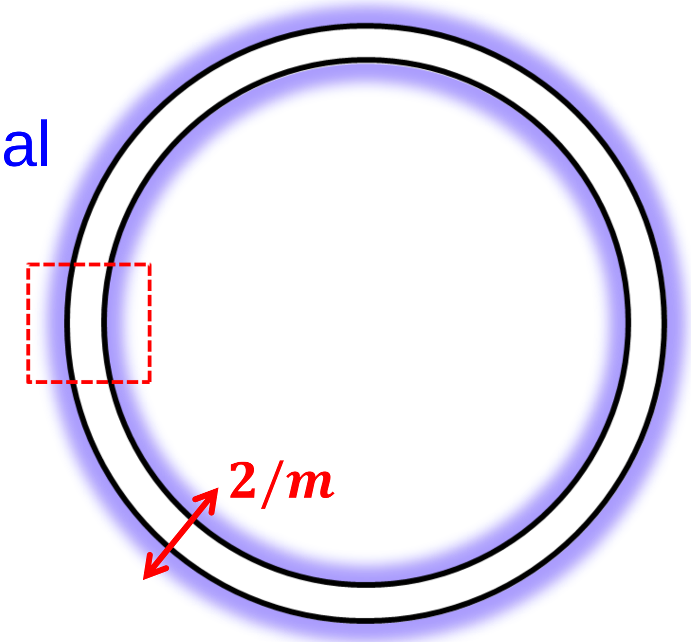


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- $\sigma_1$ must vanish if reflection symmetry $\longrightarrow$ $\alpha = 0$

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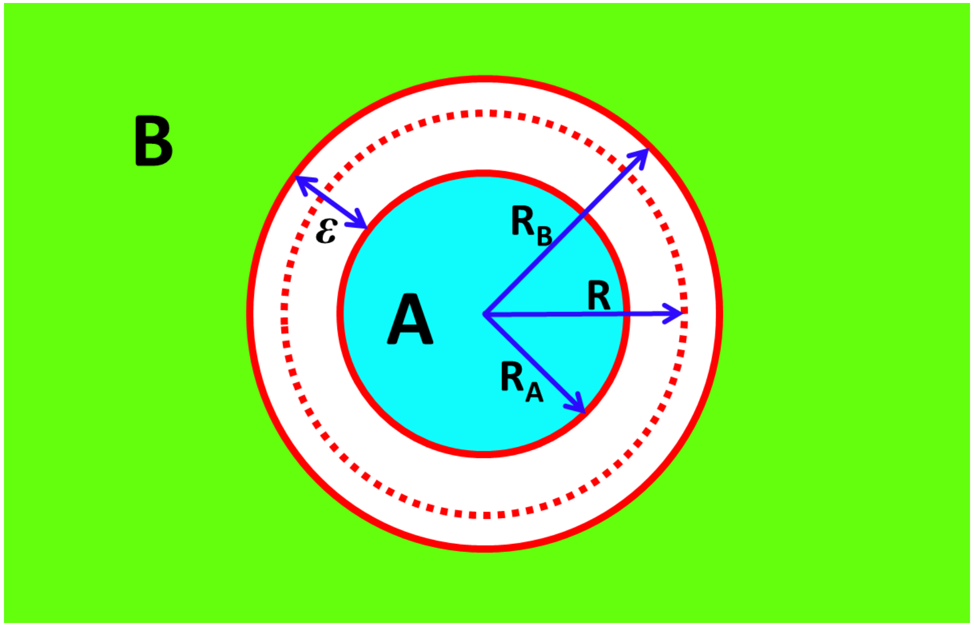


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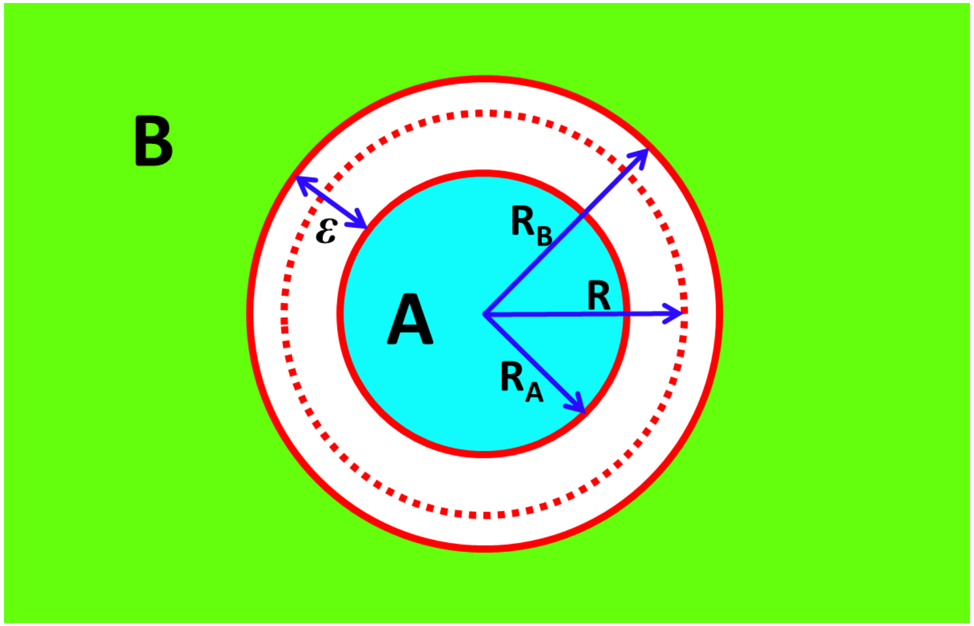



- work in continuum limit:

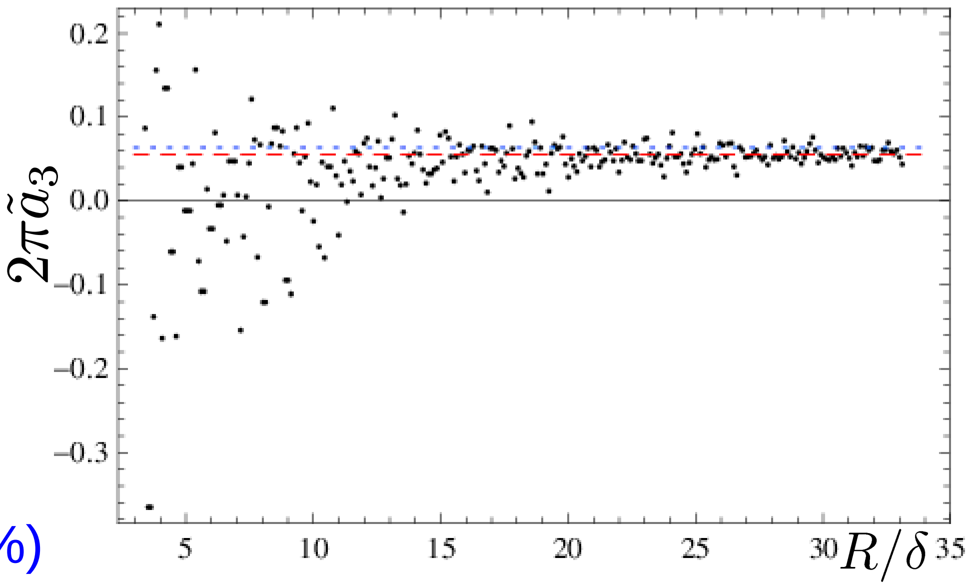
$$R \gg \varepsilon \ggg \delta$$

- calculate for a free scalar on a square lattice:

$$4\pi\tilde{a}_3 \simeq 0.110$$

$$(4\pi a_3)^{scalar} = \frac{1}{4}\left(\log 2 - \frac{3\zeta(3)}{2\pi^2}\right)$$

$$\simeq 0.127$$




($R : \varepsilon : \delta = 33 : 6 : 1$, result good to 15%)

## Criteria to properly establish c-theorem:

1. C-function must be dimensionless, well-defined quantity, which is independent of the regularization scheme

   $\longrightarrow$ computable with any regulator

2. C-function must be intrinsic to fixed point of interest

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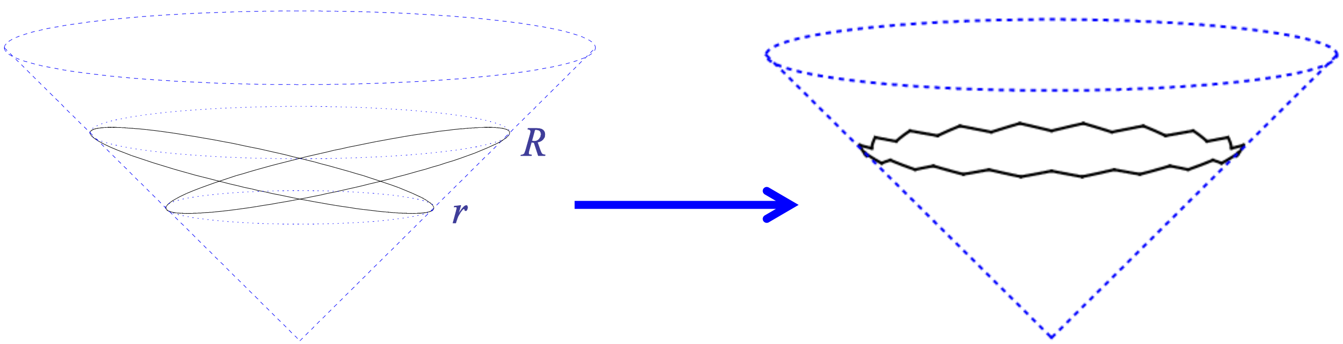
(Casini & Huerta '12)

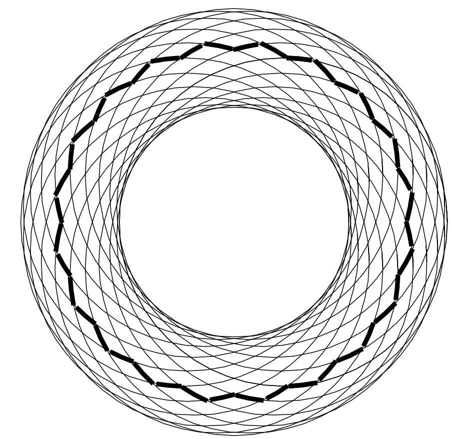
# Entanglement proof of F-theorem:

- F-theorem for d=3 RG flows established using unitarity, Lorentz invariance and **strong subadditivity**

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + \ldots + S(\cap_i X_i)$$

- geometry more complex than d=2: consider many circles intersecting on **null** cone



- no corner contribution from intersection in null plane
- define: $C(R) = RS'(R) - S(R)$
- for d=3 CFT: $S(R) = \frac{2\pi R}{\delta}c_0 - 2\pi a_3 \longrightarrow C_{\mathrm{CFT}}(R) = 2\pi a_3$
- with SSA and "continuum" limit $\longrightarrow \partial_R C(R) \leq 0$
- hence $C(R)$ decreases monotonically and $[a_3]_{\mathrm{UV}} > [a_3]_{\mathrm{IR}}$

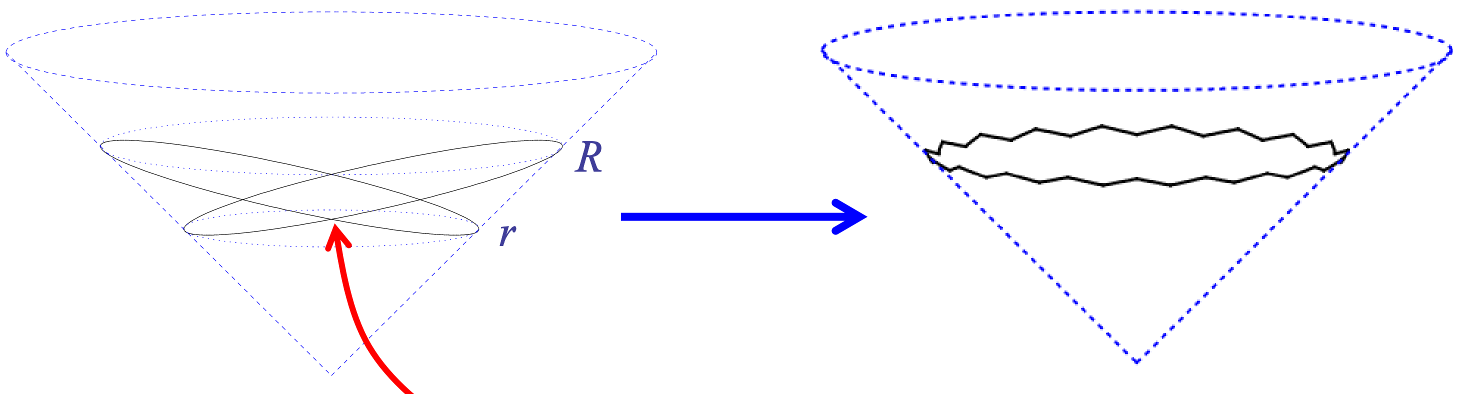
## Entanglement proof of F-theorem:

- key ingredients:

a) unitary & Lorentz invariant regularization of EE defined on regions with smooth boundaries except for "null cusps"

b) regulated EE satisfies strong subaddivity for sets whose union and intersection only generates more "null cusps"

c) wiggly circles have EE which approaches that of circle with same perimeter as the number of null cusps goes to $\infty$




null cusp: corner lying in null plane

$$\vec{t}_1 - \vec{t}_2 = \vec{v} \quad \text{with } \vec{v}\cdot\vec{v} = 0$$

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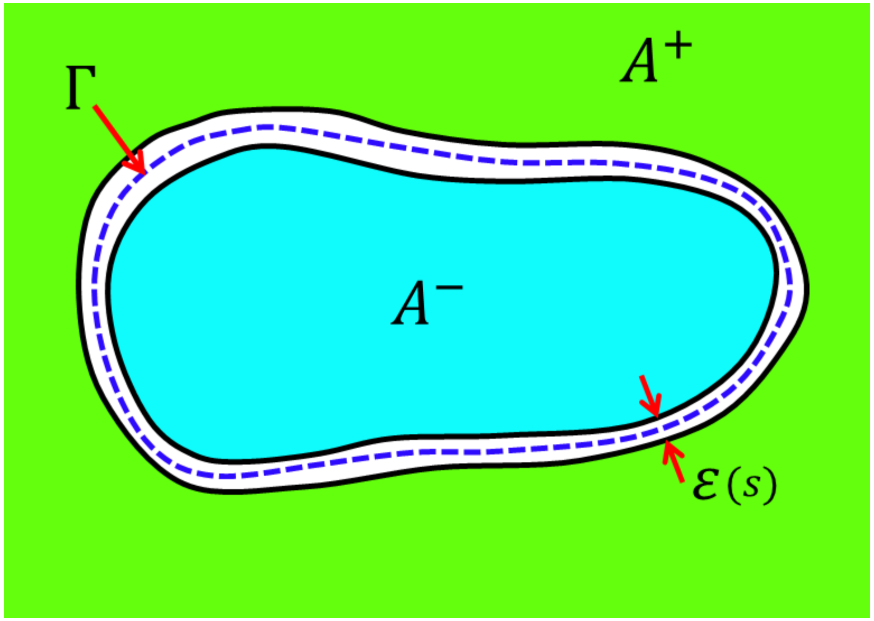
$$I(A^+,A^-) = \tilde{c}_0 \oint_\Gamma {}^{ds}\!/_{\varepsilon(s)} + I_0(A) + O(\varepsilon)$$



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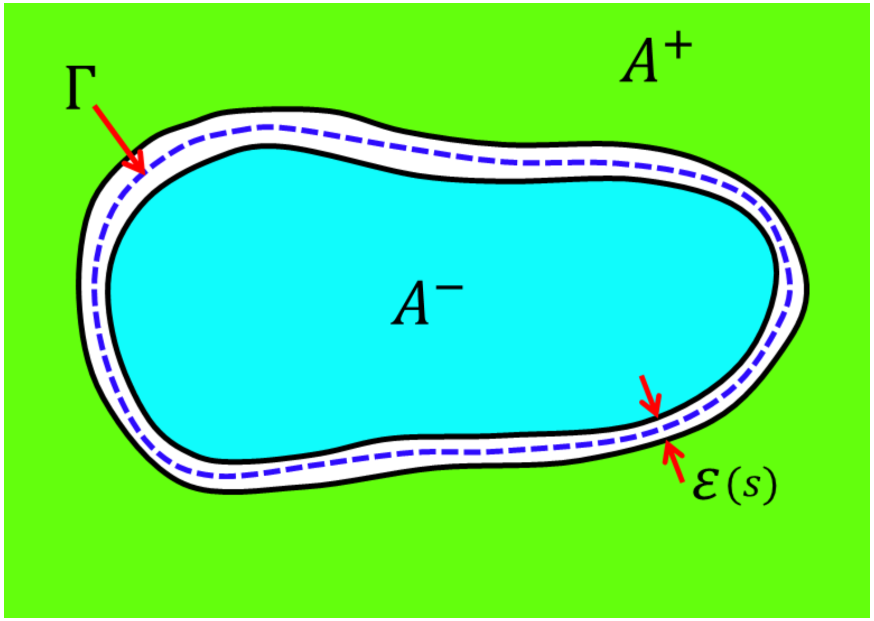
- regulated EE: property of A;
  independent of framing

eg, for circle

$$I_0(A) = 2\pi R\,\tilde{c}_1(m_i) - 4\pi\,\tilde{a}_3$$



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- consider region $A$ with ~~smooth~~ boundary $\Gamma$ **with null cusps**
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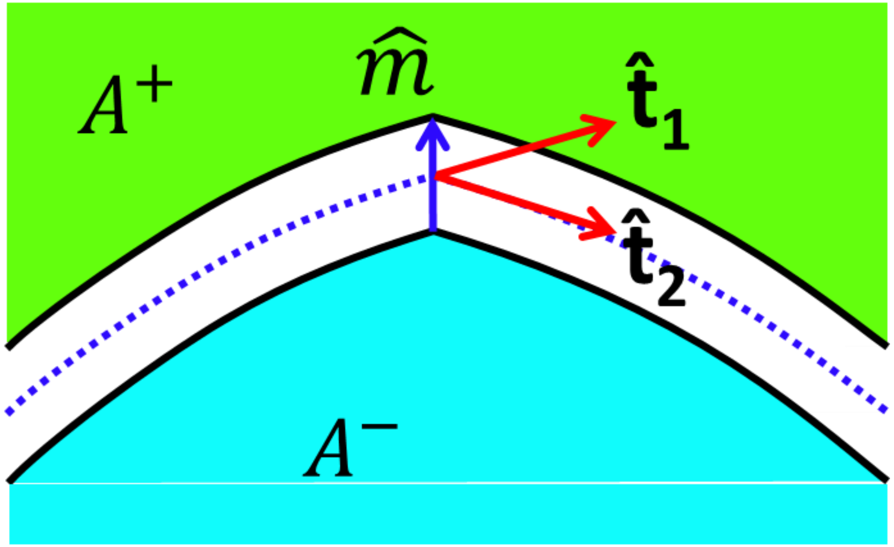
$$I(A^+,A^-) = \tilde{c}_0 \oint_\Gamma {}^{ds}/_{\varepsilon(s)} + I_0(A) + \sum f(q_{1i},q_{2i}) + O(\varepsilon)$$

- additional contributions for null cusps characterized by two local invariants:

$\boldsymbol{q_1 = \hat{m}\cdot\hat{t}_1} \quad \boldsymbol{q_2 = \hat{m}\cdot\hat{t}_2}$



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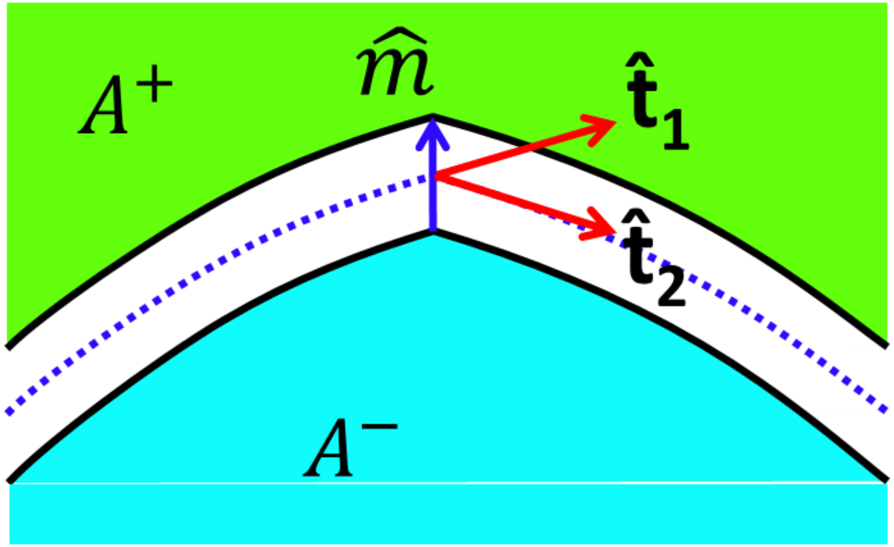


- $I_0(A)$ still satisfies SSA:

$$\mathbf{I_0(A) + I_0(B) \geq} \; \boldsymbol{I_0(A \cup B) + I_0(A \cap B)}$$

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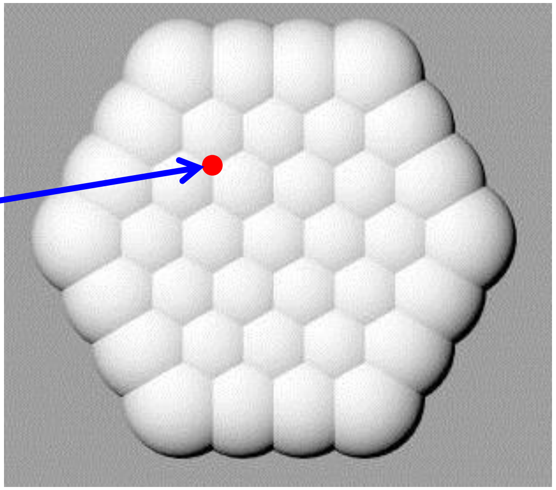
  → have properly established F-theorem in d=3

# Beyond d=3:

• is there entropic proof of c-theorem in higher dimensions?

$\longrightarrow$ need a new idea?

higher dim. intersections lead to subleading divergences which trivialize SSA inequality



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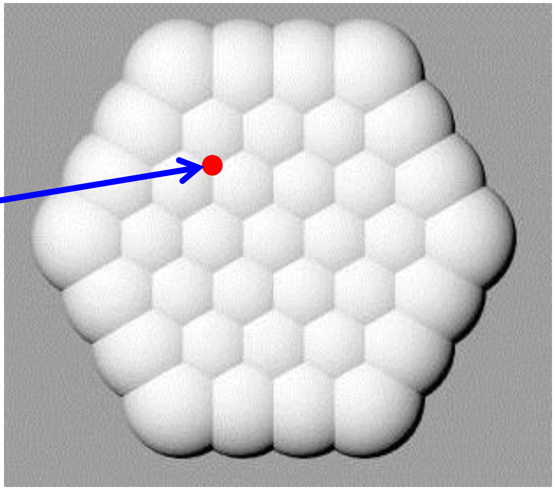

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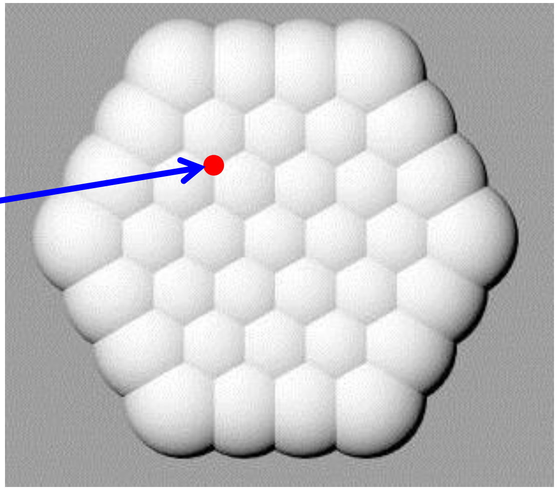

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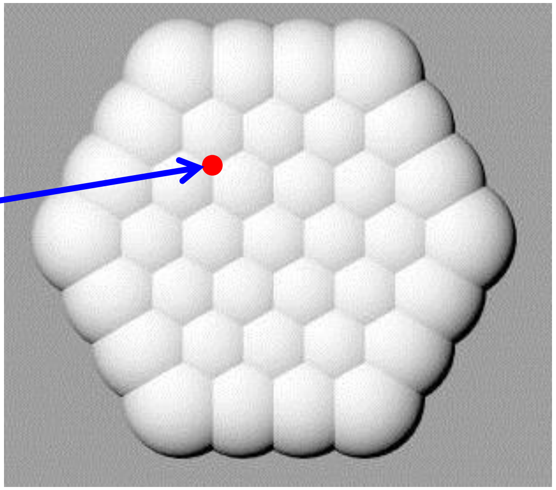

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• can c-theorems be proved for higher dimensions? eg, d=5 or 6

→ again, entropic approach needs a new idea

→ dilaton-effective-action approach requires refinement for d=6

(Elvang, Freedman, Hung, Kiermaier, RM & Theisen; Elvang & Olson)

# Conclusions and Questions:

- entanglement lends new insights into c-theorems
- using mutual information, properly established d=3 F-theorem
- how much of Zamolodchikov's structure for d=2 RG flows extends higher dimensions?

$\longrightarrow$ d=3 entropic C-function not always stationary at fixed points

(Klebanov, Nishioka, Pufu & Safdi)

$\longrightarrow$ same already observed for d=2; special case or generic? need a better C-function?

- does scale invariance imply conformal invariance beyond d=2?

$\longrightarrow$ "more or less" in d=4 (Luty, Polchinski & Rattazzi; Dymarsky, Komargodski, Schwimmer & Theisen)

- further lessons: RG flows and entanglement $\longleftrightarrow$ holography?

SSA $\longrightarrow$ NEC (Bhattacharya etal; Lashkari et al; Lin etal)

- what more entanglement/quantum information have to teach us about RG flows, holography or nonperturbative QFT?