Polarization Tensor of a Photon in an Electric Field

V.M Katkov
Budker Institute of Nuclear Physics,
Novosibirsk State University

1 General expression

We consider the case of pure electric field **E** in the sistem, where the photon momentum **k** is orthogonal to **E.** Our analysisis based on the expression for the polarization operator in electric field [Baier and Katkov, 2010] in the diagonal form:

$$\begin{aligned}\Pi^{mn} &= \sum_{i=2,3} \kappa_i b_i^m b_i^n, \qquad (1)\\ \mathbf{b}_i\mathbf{b}_j &= \ \delta_{ij}, \quad \mathbf{b}_i\mathbf{k} = 0;\\ \mathbf{b}_2 &= \mathbf{e}\times\mathbf{n}, \quad \mathbf{b}_3 = \mathbf{e}\\ \mathbf{e} &= \mathbf{E}/E, \quad \mathbf{n} = \mathbf{k}/\omega,\\ r &= \omega^2/4m^2,\end{aligned}$$

where we use the sistem $h = c = 1$,

$$\kappa_i = \frac{\alpha}{\pi}m^2 r\int\limits_{-1}^{1} dv \int\limits_{0}^{\infty-\mathrm{i}0} f_i(v,x)\exp[-\mathrm{i}\psi(v,x)]dx. \qquad (2)$$

The functions f_i has the form

$$\begin{aligned}f_2(v,x) &= 2\frac{\cosh(vx)-\cos x}{\sinh^3 x} + \frac{\cosh(vx)}{\sinh x} - v\frac{\cosh x\sinh(vx)}{\sinh^2 x},\\ f_3(v,x) &= v\frac{\cosh x\sin(vx)}{\sinh^2 x} - \frac{\cosh(vx)}{\sinh x} + (1-v^2)\coth x,\\ -\nu\psi(v,x) &= 2r\frac{\cosh x - \cosh(vx)}{\sinh x} - [r(1-v^2)+1]x;\\ \nu &= E/E_0, \quad E_0 = m^2/e. \qquad (3)\end{aligned}$$

The real part of κ_i defines the refruction index n_i of a photon with the polarization $\mathbf{e}_i = \mathbf{b}_i$:

$$n_i = 1 - \frac{\mathrm{Re}\kappa_i}{2\omega^2}. \tag{4}$$

And the imaginary part of κ_i determines the lifetime of a photon with the polarization $\mathbf{b}_i$:

$$W_i = -\frac{1}{\omega}\mathrm{Im}\kappa_i \tag{5}$$

2 Quasiclassical approximation

The standard quasiclassical approximation is valid for relativistic created particles ($r \gg 1, \nu \ll 1$) and can be derived from Eq. (3) by expanding the function $f_i(v,x)$, $\psi(v,x)$ over x power. Taking into account the higher powers of x in the exponent one gets:

$$f_2(v,x) = \frac{1-v^2}{12}(3+v^2)x, \quad f_3(v,x) = \frac{1-v^2}{6}(3-v^2)x,$$
$$\psi(v,x) = \frac{r(1-v^2)^2}{12\nu}\left(x^3 - \frac{3-v^2}{30}x^5\right) + \frac{x}{\nu} \tag{6}$$

Here the term $\propto x^5$ is correction to SQA. Expanding the term with x^5 in the exponent and making substitution $x = \nu t$ one finds:

$$\kappa_i = -\frac{\alpha}{4\pi} m^2\kappa^2 \int\limits_{-1}^{1} dv \int\limits_{0}^{\infty - \mathrm{i}0} g_i(v,t) \exp\left[-\mathrm{i}\left(t + \xi\frac{t^3}{3}\right)\right] tdt, \quad (7)$$

$$g_2(v,x) = \frac{1-v^2}{12}\left(3 + v^2 + \mathrm{i}\frac{9-v^4}{90}\xi\nu^2 t^5\right), \quad (8)$$

$$g_3(v,x) = \frac{1-v^2}{6}\left(3 - v^2 + \mathrm{i}\frac{(3-v^2)^2}{90}\xi\nu^2 t^5\right) \quad (9)$$

where

$$\xi = \frac{1}{16}(1-v^2)^2\kappa^2, \quad \kappa^2 = 4r\nu^2. \quad (10)$$

At $\kappa \ll 1$ one has

$$\mathrm{Re}\kappa_i = \frac{4\alpha}{45\pi} m^2\kappa^2 a_i, \quad a_2 = 1 + \kappa^2, \quad a_3 = \frac{7}{4} + \frac{13}{14}\kappa^2 \ ; \quad (11)$$

$$-\mathrm{Im}\kappa_i = -\mathrm{i}\sqrt{\frac{3}{32}}\alpha m^2\kappa \exp\left(-\frac{8}{3\kappa}\right) c_i,$$

$$c_3 = 2c_2, \quad c_2 = 1 + \frac{32\nu^2}{15\kappa^3} = 1 + \frac{4}{15r^{3/2}\nu}. \quad (12)$$

As the corrections should be small and $\nu \ll 1$, SQA already broken at relativistic energies when $r \lesssim \nu^{-2/3}$.

3 Region $\nu^2 \ll r \ll 1/\nu^2$.

Let's move the integration contour over x in Eq. (2) to the lower axis in point x_0,

$$x_0(r) = -\mathrm{i}a(r), \qquad a(r) = 2\arctan\frac{1}{\sqrt{r}}. \tag{13}$$

As a result we have the following expression for κ_i:

$$\kappa_i = \frac{\alpha}{\pi} m^2 r\left(\alpha_i + \beta_i\right),$$

where

$$\alpha_i = \int\limits_{-1}^{1} dv \int\limits_{0}^{x_0(r)} dx f_i(v,x)\exp[\mathrm{i}\psi(v,x)], \tag{14}$$

$$\beta_i = \int\limits_{-1}^{1} dv \int\limits_{x_0(r)}^{\infty} dz f_i(v,x)\exp[\mathrm{i}\psi(v,x_0)]. \tag{15}$$

In the integral α_i small values $x \sim \nu$ contribute. This integral we calculate expanding the entering functions over x. Taking into account that in the region under consideration the condition $r\nu^2 \ll 1$ is fulfilled we extend the integration over x to infinity. As a result we have Eq. (11). In the integral β_i the small values v

contribute. Expanding entering functions over v and extending the integration over v to infinity we have:

$$\beta_i = \int_{-\infty}^{\infty} dv \int_{x_0(r)}^{\infty} dx f_i(v=0,x) \exp[-\frac{\mathrm{i}}{\nu}(\varphi(x) + v^2\chi(x))]. \quad (16)$$

where

$$\varphi(x) = (r+1)x - 2r \tanh\frac{x}{2}, \quad (17)$$
$$\chi(x) = rx\left(\frac{x}{\sinh x} - 1\right) \quad (18)$$

From the aquation $\varphi'(x_0) = 0$ we find that x_0 is the stationary phase point. Due to the fact $\nu \ll 1$, we can use the stationary phase method. In x_0 we have:

$$\begin{aligned} \mathrm{i}\varphi(x_0) &\equiv b(r) = (r+1)a(r) - 2\sqrt{r}, \\ \mathrm{i}\varphi''(x_0) &= \frac{r+1}{r}, \quad \mathrm{i}\chi(x_0) = \sqrt{r}a(r)b(r), \\ \mathrm{i}f_2(v=0,x_0) &= \frac{r+1}{2r\sqrt{r}}, \ \mathrm{i}f_3(v=0,x_0) = \frac{1}{\sqrt{r}}. \end{aligned} \quad (19)$$

Performing integration in Eq.(16) over v and using the standard procedure of the stationary phase method, one obtains for the imaginary part of κ_i:

$$
\begin{aligned}
\operatorname{Im}\kappa_2 &= -\frac{\alpha m^2 \nu}{2}\sqrt{\frac{r+1}{ra(r)b(r)}}\exp\left(-\frac{b(r)}{\nu}\right), \\
\operatorname{Im}\kappa_3 &= \frac{2}{r+1}\left(r+\frac{\nu}{4\pi}\right)\operatorname{Im}\kappa_2,
\end{aligned} \tag{20}
$$

For $r \gg 1$, the first two term of the decomposition $b(r)/\nu$ over power of $1/r$ are

$$
-\frac{b(r)}{\nu} \simeq -\frac{4}{3\nu\sqrt{r}} + \frac{4}{15\nu r\sqrt{r}} = -\frac{8}{3\kappa} + \frac{32\nu^2}{15\kappa^3}. \tag{21}
$$

It follows from this formula that, applicability of Eq. (20) is limited by the condition $r \ll 1/\nu^2$. If the secon term is small, it can be droped down in Eq. (20). Under these conditions $a(r) \simeq 2/\sqrt{r}$, and we have

$$
\begin{aligned}
\operatorname{Im}\kappa_2 &= -\frac{\alpha m^2 \nu}{4}\sqrt{\frac{3r}{2}}\exp\left(-\frac{4}{3\nu\sqrt{r}}\right)\left(1+\frac{4}{15\nu r\sqrt{r}}\right), \\
\operatorname{Im}\kappa_3 &= 2\operatorname{Im}\kappa_2
\end{aligned} \tag{22}
$$

This formula coinside with Eq. (12). So at $r \gg 1$, the overlapping energy region exists where both the formulated here and SQA are valid. At low photon energy ($\nu^2 \ll r \ll \nu^{2/3}$) the probability Eq.(20) has a form

$$\operatorname{Im}\kappa_2 \simeq -\frac{\alpha m^2\nu}{2\pi\sqrt{r}}\exp\left(-\frac{\pi}{\nu}(1+r)+\frac{4\sqrt{r}}{\nu}\right) \times\left(1+\frac{3\sqrt{r}}{\pi}+\frac{4r\sqrt{r}}{3\nu}\right), \tag{23}$$

$$\operatorname{Im}\kappa_3 = \left(2r+\frac{\nu}{2\pi}\right)\operatorname{Im}\kappa_2. \tag{24}$$

At $r \ll 1$, the term $\propto v^2$ in the exponent Eq. (16) has the factor $\mathrm{i}\chi(x_0)/\nu \ \simeq \pi^2\sqrt{r}/\nu$. For this reson, the proposed procedure for integration over v is violated when this factor is comparable or smaller than unity.

4 Very low photon energy

At $r \simeq \nu^2$ for the imaginary part of κ_i in Eq.(2) we can integrate over x from $-\infty$ and divide the result by two. Then we close the integration contour in the lower half-plane by the following way

$$\operatorname{Im}\kappa_i = \mathrm{i}\frac{\alpha}{\pi}m^2 r\int\limits_{-1}^{1}dv\sum_{n=1}^{\infty}\oint f_i(v,x)\exp[-\mathrm{i}\psi(v,x)]dx, \tag{25}$$

where the path of integratin over x is any simple closed contour around the points $-\mathrm{i}n\pi$. Because of appearance of the factor

$\exp(-\pi n/\nu)$, the main contribution to the sum gives the term $n=1$. Expanding the function entering in Eq.(3) over variables $\xi = x+$ i$n\pi$ and keepin the main terms of the decomposition we find

$$f_2(v,x) = -\frac{4}{\xi^3}\cos^2(v\pi/2), \quad f_3(v,x) = \frac{\mathrm{i}v}{\xi^2}\sin(v\pi), \tag{26}$$

$$\psi(v,x) = \frac{4r}{\xi\nu}\ \cos^2(v\pi/2) - \frac{\xi}{\nu} + \frac{2\mathrm{i}}{\nu}rv\sin(v\pi) + \frac{\mathrm{i}\pi}{\nu}[1+r(1-v^2)]$$

Using the integrals Eq.(7.3.1) and Eq.(7.7.1) in [Bateman and Erdelyi 1953] we find

$$\mathrm{Im}\,\kappa_2 = -2\alpha m^2\ \mathrm{e}^{-\pi/\nu}\mathrm{I}_1^2\left(\frac{2\sqrt{r}}{\nu}\right), \tag{27}$$

$$\mathrm{Im}\,\kappa_3 = -\alpha m^2 \mathrm{e}^{-\pi/\nu}\frac{\nu}{\pi}\left[\mathrm{I}_0^2\left(\frac{2\sqrt{r}}{\nu}\right) - 1\right]. \tag{28}$$

where $\mathrm{I}_n(z)$ is the Bessel function of imaginary argumant.

For $r \gg \nu^2$, the asymptotic representation $\mathrm{I}_n(z) \simeq \mathrm{e}^z/\sqrt{2\pi z}$ can be used. As a result one obtains the probability Eqs.(23), (24). For $r \ll \nu^2$,we have

$$\mathrm{Im}\,\kappa_2 = -\frac{2r}{\nu^2}\alpha m^2\ \mathrm{e}^{-\pi/\nu}, \quad \mathrm{Im}\,\kappa_3 = -\frac{2r}{\nu\pi}\alpha m^2\ \mathrm{e}^{-\pi/\nu}. \tag{29}$$

The above analizysis is not complet. In this region of energy, we have to consider the probability of direct pair creation by an electric field (vacuum probability). The distribution of the pair density n_p over transverse momentum has a form

$$dn_p = \frac{m^2 \nu t}{4\pi^3} \exp\left(-\frac{\pi}{\nu} - \frac{\pi p_\perp^2}{\nu m^2}\right) d^2 p_\perp \tag{30}$$

From this equation, it's clear that the produced particles are non-relativistic. In turn, the cross-section of the absorption of a soft photon by a non-relativistic electron (positron) in an electric field is

$$\sigma = \frac{\alpha}{\pi^2 m^2 \nu^2} \mathrm{K}_1^2\left(\frac{2\sqrt{r}}{\nu}\right). \tag{31}$$

Here $\mathrm{K}_n(z)$ is the McDonald function. Multiplying the density and cross-section and integrating over the transverse momentum, we have for the vacuum probability per unit time of the absorption of a soft photon:

$$W = \frac{\alpha m^2 t}{4\pi^5} \exp\left(-\frac{\pi}{\nu}\right) \mathrm{K}_1^2\left(\frac{2\sqrt{r}}{\nu}\right) \tag{32}$$

For $r \gg \nu^2$, the asymptotic representation $\mathrm{K}_n\left(z\right) \simeq \mathrm{e}^{-z}/\sqrt{2\pi z}$ can be used. As a result one has

$$W = \frac{\alpha m^2}{4\pi^6 \omega} \exp\left(-\frac{\pi}{\nu} - \frac{4\sqrt{r}}{\nu}\right) (tm\nu) \tag{33}$$

The term $(tm\nu) = t/t_f$, $t_f = 1/m\nu$ in this case. Comparing this probability with the Eq. (23), we have

$$\frac{W}{W_2} = \frac{\nu^2}{8\pi^6 r} \exp\left(-\frac{8\sqrt{r}}{\nu}\right) \frac{t}{t_f} \tag{34}$$

In any case the value of t/t_f should be large. For $r \ll \nu^2$ $(z \ll 1)$, $\mathrm{K}_1^2(z) \simeq z^{-2}$ and the vacuum probability per unit time of the absorption of a soft photon has a form

$$W = \frac{\alpha m^2 \nu^2}{4\pi^5 r\omega} \exp\left(-\frac{\pi}{\nu}\right) \frac{t}{t_f}, \tag{35}$$

were $t_f = 1/\omega$. During the time t_f, the electron is accelerated to relativistic energies $eE/\omega \gg m$. Comparing this probability with the Eq. (29), we have

$$\frac{W}{W_2} = \frac{\nu^4}{32\pi^5 r^2} \frac{t}{t_f} = \frac{1}{2\pi} \left(\frac{eE}{\pi m\omega}\right)^4 \frac{t}{t_f} \tag{36}$$