Reaction Theory

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Re-writing Nuclear Physics textbooks:
30 years of radioactive ion-beam physics

Outline

- **Introduction**

  notation, interactions and reactions, scales, kinematics, conservation laws
- **General Reaction Theory**
  - cross sections and theoretical methods
  - stationary scattering theory
  - partial-wave expansion
  - operator formalism
- **Applications**
  - astrophysics and indirect methods
  - transfer reactions
- **Summary**

Aim of Lecture

- introduction to basic methods in reaction theory with particularities for exotic nuclei
- no need to follow every step in derivations, only for completeness
- not a review of recent results or details of specific approaches/individual cases

Introduction

General Remarks

- **definition**
  reactions = all processes that occur if two (or more) particles collide within an interaction zone (depends on range of interaction)

- **types of interactions**
  - electromagnetic interaction (long range)
  - strong interaction (medium range)
  - weak interaction (short range)

  $\Rightarrow$ possible competition or interference

- **notation**
  - general form: $X_1 + X_2 \to X_3 + X_4 + \ldots + X_n$
  - alternatively:

    $X_2(X_1, X_4, \ldots, X_n)X_3$ with projectile X_1, target X_2, and ejectiles $X_4, \ldots, X_n$

    inverse kinematics, e.g. if X_2 is unstable nucleus

    $X_1(X_2, X_4, \ldots, X_n)X_3$ with projectile X_2, target X_1, and ejectiles $X_4, \ldots, X_n$

Types of Reactions

- **reactions with two particles in the initial state**
  - elastic scattering $a + A \to A + a$
  - inelastic scattering $a + A \to A^* + a'$
  - rearrangement reactions $a + A \to B + b$ with $b \neq a$, $B \neq A$
  - radiative capture reactions $a + A \to C + \gamma$ with photon in final state
  - many-body reactions $a + A \to B + b_1 + b_2 + \ldots$
- example:

$$p + {}^7_3\text{Li} \to \begin{cases} {}^7_3\text{Li} + p \\ {}^7_3\text{Li}^* + p \\ {}^7_4\text{Be} + n \\ \alpha + \alpha \\ {}^8_4\text{Be} + \gamma \quad (\to \alpha + \alpha + \gamma) \\ \alpha + t + p \end{cases}$$

Length and Time Scales

- radius of stable nucleus $R = r_0 A^{1/3}$ with $r_0 \approx 1.25$ fm, mass number A
- corresponding area of circle $S = \pi R^2$
- time for light to traverse nucleus $t = 2R/c$
- example ^{208}Pb:
  $R \approx 7.4$ fm, $S \approx 172$ fm$^2 = 1.72$ b, $t \approx 14.8$ fm$/c = 4.9 \cdot 10^{-23}$ s
- unit for area in reaction theory: barn
  (1 b $= 10^{-28}$ m^2, 1 fm$^2 = 10$ mb)
- time scales for reactions:
  - fast, $t \approx 10^{-22}$ s: direct reactions (few nucleons involved, one-step processes)
  - intermediate, $t > 10^{-22}$ s: multi-step reactions (complicated)
  - slow, $t \gg 10^{-22}$ s: compound nucleus reactions (many nucleons involved, collective excitations, no memory of initial state, statistical features)

Energy Scales and Kinematics

- **typical energies**
  - nuclear structure: binding and excitation energies $\Rightarrow$ a few MeV
  - potentials: Coulomb barrier $\Rightarrow E_C = \frac{Z_a Z_A e^2}{R_C} \quad R_C \approx R_a + R_A \quad e^2 = 1.44$ MeV fm
  - reactions: very large range of energies, e.g.
    - thermal neutrons $\Rightarrow \approx 25$ meV
    - astrophysical reactions $\Rightarrow$ a few 10 or 100 keV
    - direct nucleon transfer reactions $\Rightarrow$ a few MeV per nucleon
    - Coulomb excitation reactions $\Rightarrow$ a few 10 MeV or 100 MeV per nucleon
    - relativistic heavy-ion collisions $\Rightarrow$ a few GeV or TeV per nucleon

- **Q value**

  $Q = (m_a + m_A - m_b - m_B)c^2$ for reaction $A(a,b)B$

  - $Q > 0$ exothermic reaction (release of kinetic energy due to larger binding)
  - $Q = 0$ elastic reaction
  - $Q < 0$ endothermic reaction (reaction not possible below threshold)

Conservation Laws

reaction $A(a,b)B$

- conservation of energy

$$E(a+A) = T_a + T_A + (m_a + m_A)c^2 = T_b + T_B + (m_b + m_B)c^2 = E(b+B)$$

with kinetic energies T_a, T_A, T_b, T_B

- conservation of momentum

$$\vec{P}(a+A) = \vec{p}_a + \vec{p}_A = \vec{p}_b + \vec{p}_B = \vec{P}(b+B)$$

- conservation of total angular momentum

$$\vec{J}(a+A) = \vec{J}_a + \vec{J}_A + \vec{J}_{aA} = \vec{J}_b + \vec{J}_B + \vec{J}_{bB} = \vec{J}(b+B)$$

- conservation of parity

$$P(a+A) = P_a \cdot P_A \cdot (-1)^{l_{aA}} = P_b \cdot P_B \cdot (-1)^{l_{bB}} = P(b+B)$$

- conservation of isospin

$$\vec{T}(a+A) = \vec{T}_a + \vec{T}_A = \vec{T}_b + \vec{T}_B = \vec{T}(b+B)$$

- conservation of charge, baryon number and lepton number

General Reaction Theory

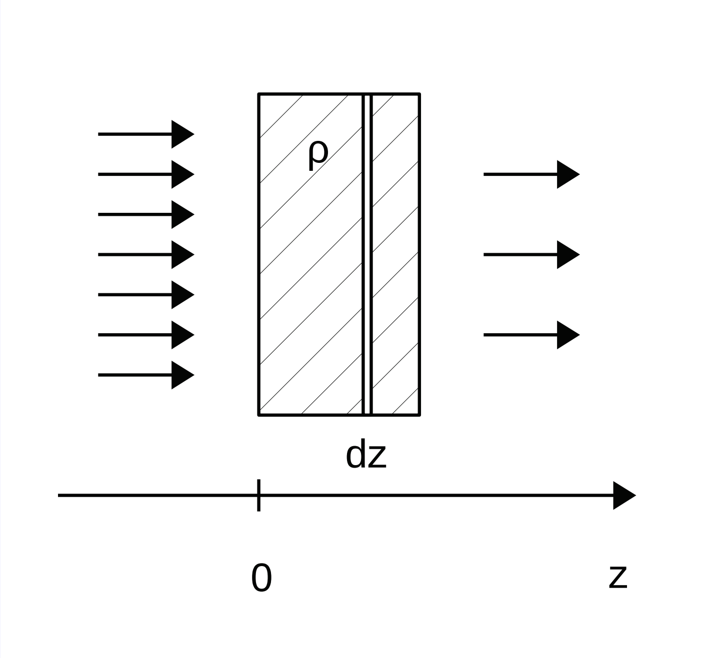
Theoretical Approaches

- **large variety of reactions**
  - numerous theoretical approaches
  - here only a selection, mainly for direct reactions
  - many topics not covered: e.g. compound nucleus reactions, heavy-ion collisions
- **theoretical description**
  - classical methods
  - semiclassical methods
  - quantal methods

  $\Rightarrow$ observables: cross sections
- **general conditions in the following**
  - nonrelativistic kinematics
  - no explicit consideration of antisymmetrization
  - no weak interaction processes
  - spins of particles mostly neglected

Definition of Cross Sections I

quantifying the strength of reactions



- uniform beam of particles in z-direction hitting a thick target
  $\Rightarrow$ reduction of current J due to reactions:

$$\frac{dJ}{dz} = -\sigma J(z) \rho \quad \Rightarrow \quad J(z) = J(0) \exp\left(-\sigma \rho z\right)$$

  - proportial to current $J(z)$ and density of target nuclei ρ
  - proportionality constant $=$ (interaction) cross section σ
  - dimensions: $[J] = L^{-2}T^{-1}$, $[\rho] = L^{-3}$, $[\sigma] = L^{2}$ with length L and time T

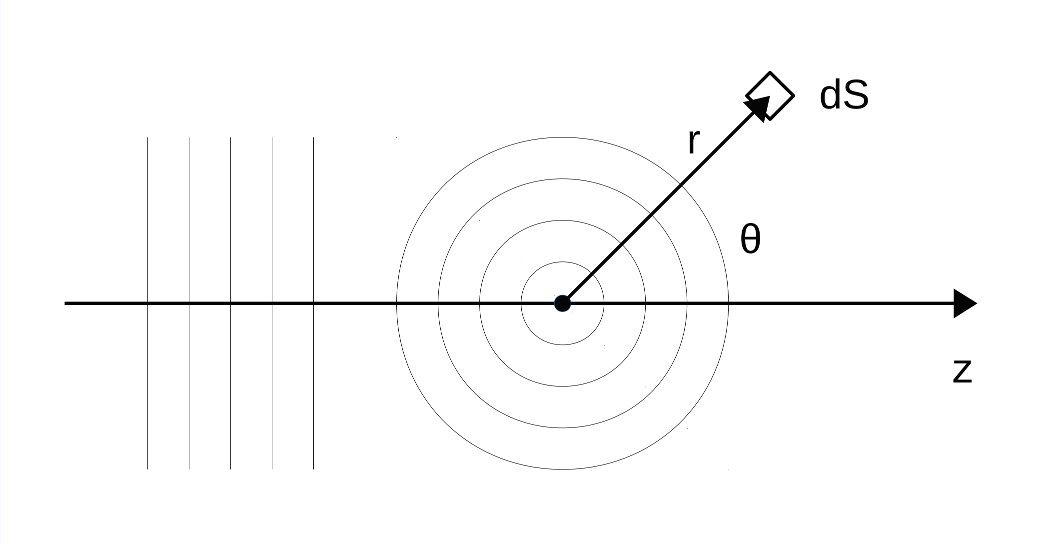
- cross section σ depends on energy E of beam $\Rightarrow$ excitation function $\sigma(E)$

- more detailed information on reaction with dependence on scattering angle θ

  $\Rightarrow$ differential cross section $\dfrac{d\sigma}{d\Omega}$

Definition of Cross Sections II

differential cross section



- initial state: uniform beam of particles in z direction with current J_i
- scattering on a single target nucleus, detection of scattered particles at distance r from target in area $dS = r^2 d\Omega$ at scattering angle θ
- final state: current of particles J_f in radial direction

$$\Rightarrow \quad d\sigma_{fi} = \frac{J_f dS}{J_i} \quad \text{or} \quad \frac{d\sigma_{fi}}{d\Omega} = \frac{J_f r^2}{J_i}$$

- in classical physics: $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{d\theta}{db}\right|^{-1}$ (elastic scattering $i = f$)

  with deflection function $\theta(b)$ depending on impact parameter b assuming azimuthal symmetry (ϕ independence) of scattering

Classical Description of Scattering

determine trajectories of particles

- solve Newtonian equations of motion $m_i \ddot{\vec{r}}_i = \vec{F}_i$

  (or use conservation laws for energy, momentum, angular momentum)

  with given initial conditions (position, velocity)

  $\Rightarrow$ ordinary time-dependent differential equations

- example: elastic Coulomb scattering of particle a with energy E and impact parameter b on target A

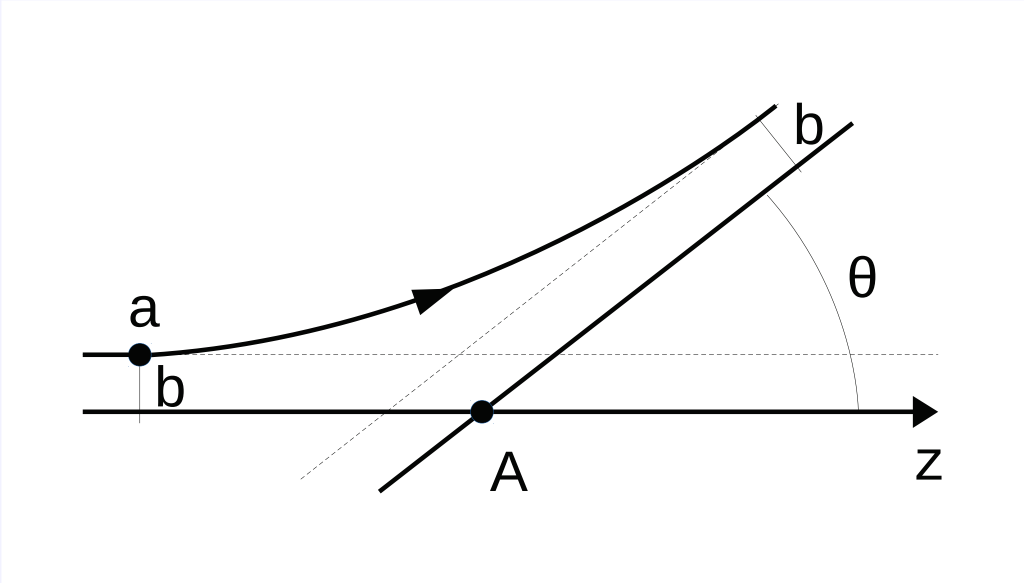
$$\vec{F}_a = \frac{Z_a Z_A e^2}{r^2} \frac{\vec{r}}{r}$$

$\Rightarrow$ deflection function

$$\theta(b) = 2 \ \mathrm{arccot} \left(\frac{2bE}{Z_a Z_A e^2} \right)$$

$\Rightarrow$ Rutherford cross section

$$\frac{d\sigma_R}{d\Omega} = \left(\frac{Z_a Z_A e^2}{4E} \right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}$$



Semiclassical Description

combination of classical and quantal methods

- example: Coulomb excitation of nucleus a from ground state $|i\rangle$ to excited state $|f\rangle$ with excitation energy $E = \hbar\omega$ in time-dependent Coulomb potential

  $V(\vec{x},t) = \frac{Z_a Z_A e^2}{|\vec{r}(t)+\vec{x}|}$ of target nucleus A

  - excitation cross section

    $$\frac{d\sigma_{fi}}{d\Omega} = \frac{d\sigma_R}{d\Omega} \times P_{fi}$$

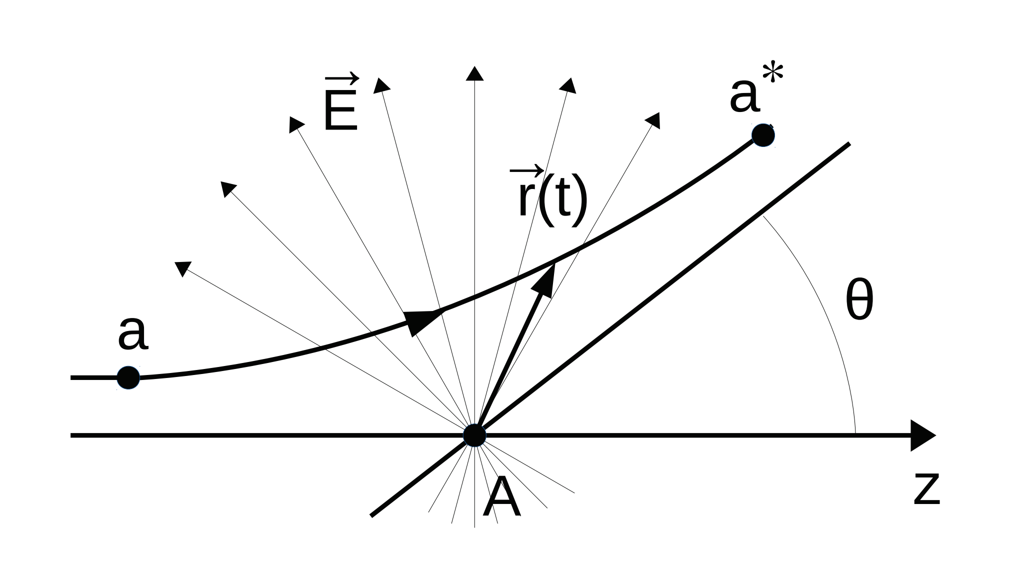
  - Rutherford cross section $\frac{d\sigma_R}{d\Omega}$
  - excitation probability $P_{fi} = |a_{fi}|^2$

    in first order time-dependent perturbation theory with amplitude

    $$a_{fi} = \frac{1}{i\hbar}\int_{-\infty}^{\infty} dt\, e^{-i\omega t}\langle f|V(\vec{x},t)|i\rangle$$

  - application to exotic nuclei:

    $\Rightarrow$ excitation to continuum states/breakup



Quantal Description

determine scattering wavefunction

- solve Schrödinger equation with given boundary conditions
  - time-dependent $i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi$ $\Rightarrow$ time evolution of wave packet

    (not considered in the following)
  - stationary $E\psi = \hat{H}\psi$ $\Rightarrow$ fixed energy E

    two formulations:

    – partial differental equations

    – integral equations
  - boundary conditions:

    "plane wave + outgoing (ingoing) spherical waves"
- define channels $c = i, f$ that characterize asymptotic states
  - partition, e.g. $A + a$, $B + b$, $C + \gamma$, . . .
  - additional quantum numbers, e.g. for particular states of nuclei A, a, . . .
  - energy, momentum

General Reaction Theory

- Stationary Scattering Theory

Stationary Scattering Theory I

theoretical formulation for reaction with nuclei a and A in initial channel

- total Hamiltonian $\hat{H} = \hat{H}_a + \hat{H}_A + \hat{T}_{aA} + \hat{V}_{aA}$ with
  - Hamiltonians $\hat{H}_a$, $\hat{H}_A$ of nuclei with wave functions ϕ_a, ϕ_A

    $\hat{H}_a \phi_a = E_a \phi_a \quad \hat{H}_A \phi_A = E_A \phi_A$
  - kinetic energy operator of relative motion

    $\hat{T}_{aA} = -\frac{\hbar^2}{2\mu_{aA}} \Delta_{\vec{r}_{aA}}$ with reduced mass $\mu_{aA} = \frac{m_a m_A}{m_a + m_A}$ and $\vec{r}_{aA} = \vec{r}_a - \vec{r}_A$
  - interaction potential $\hat{V}_{aA}$
- Hamiltonian without aA interaction $\hat{H}_0^{(i)} = \hat{H}_a + \hat{H}_A + \hat{T}_{aA}$

  $\Rightarrow$ wave function $\Phi_i = \phi_i \exp\left(i\vec{k}_i \cdot \vec{r}_i\right)$ $\quad \vec{r}_i = \vec{r}_{aA} \quad \vec{k}_i = \mu_{aA}\left(\frac{\vec{k}_a}{m_a} - \frac{\vec{k}_A}{m_A}\right)$

  with $\phi_i = \phi_a \phi_A$, $\hat{H}_0^{(i)} \Phi_i = \left(E_a + E_A + E_{aA}\right) \Phi_i$ and $E_{aA} = \frac{\hbar^2 k_i^2}{2\mu_{aA}}$

Stationary Scattering Theory II

- full solution with total Hamiltonian $\hat{H}\Psi_i^{(\pm)} = E\Psi_i^{(\pm)}$
- boundary condition: asymptotic form for large radii

$$\Psi_i^{(\pm)} \to \Phi_i + \sum_f \phi_f f_{fi}^{(\pm)} \frac{\exp\left(\pm i k_f r_f\right)}{r_f}$$

with $\Phi_i = \phi_i \exp\left(i\vec{k}_i \cdot \vec{r}_i\right)$

and scattering amplitude $f_{fi}^{(\pm)}$ in final channels f

◦ "$+$" solution: outgoing spherical waves

◦ "$-$" solution: ingoing spherical waves

- differential cross section for reaction from initial channel i to final channel f

$$\frac{d\sigma_{fi}}{d\Omega} = \frac{J_f r^2}{J_i}$$

with currents J_i, J_f of relative motion

◦ current in nonrelativistic quantum mechanics for wavefunction ψ

for particle with mass m $\vec{J} = \frac{\hbar}{2mi}\left[\psi^* \left(\vec{\nabla}\psi\right) - \left(\vec{\nabla}\psi^*\right)\psi\right]$

Stationary Scattering Theory III

- initial state: current for wave function of relative motion

$$\psi = \exp\left(i\vec{k}_i \cdot \vec{r}_i\right) \Rightarrow \vec{J}_i = \frac{\hbar\vec{k}_i}{\mu_i} = \vec{v}_i = \vec{v}_{aA}$$

- final state: current for wave function of relative motion

$$\psi = f_{fi}^{(+)} \frac{\exp\left(\pm i k_f r_f\right)}{r_f} \Rightarrow \vec{J}_f \to \frac{\left|f_{fi}^{(+)}\right|^2}{r_f^2} \frac{\hbar k_f}{\mu_f} \frac{\vec{r}_f}{r_f} \quad \text{for } r_f \to \infty$$

- cross section for reaction from initial state i to final state f

$$\frac{d\sigma_{fi}}{d\Omega} = \frac{J_f r_f^2}{J_i} = \frac{k_f}{k_i}\left|f_{fi}^{(+)}\right|^2 \Rightarrow \text{determine scattering amplitude } f_{fi}^{(+)}$$

- methods to find $f_{fi}^{(+)}$:
  - partial-wave expansion of wave function $\Psi_i^{(+)}$
  - formulation with integral equation $\Rightarrow$ operator formalism

General Reaction Theory

- **Partial-Wave Expansion**

Partial-Wave Expansion I

elastic scattering $A(a,a)A$ on a spherically symmetric short-range potential $V_{aA}(r)$ for nonrelativistic energies (single channel)

- expansion of full wave function $$\Psi_i^{(+)} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\varphi_l^{(+)}(r)}{r} Y_{lm}(\hat{r}) \phi_a \phi_A$$

  with radial wave functions $\varphi_l^{(+)}$ and spherical harmonics Y_{lm}

- $\hat{H}\Psi_i^{(+)} = E_i \Psi_i^{(+)} = (E_a + E_A + E_{aA})\Psi_i^{(+)}$ $\Rightarrow$ radial Schrödinger equation

  $$\left[-\frac{\hbar^2}{2\mu_{aA}} \frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + V_{aA}(r) \right] \varphi_l^{(+)}(r) = E_{aA} \varphi_l^{(+)}(r)$$ with energy $E_{aA} = \frac{\hbar^2 k^2}{2\mu_{aA}}$

- boundary conditions:
  - $r = 0 \Rightarrow \varphi_l^{(+)}(r) = 0$
  - $r \to \infty \Rightarrow \varphi_l^{(+)}(r) \to ?$, but short-range potential $V_{aA}(r) \to 0$ for $r \to \infty$

    $\Rightarrow$ linear combination of regular and irregular spherical Bessel functions (modifications for Coulomb potentials)

Partial-Wave Expansion II

- wavefunction without potential $\Phi_i = \phi_a \phi_A \exp\left(i\vec{k}_i \cdot \vec{r}\right)$ with $\vec{k}_i = k_i \vec{e}_z$
- partial-wave expansion $$\Phi_i = 4\pi \sum_{l,m} i^l j_l(k_i r) Y_{lm}(\hat{r}) Y^*_{lm}(\hat{k}_i) \phi_a \phi_A$$

  with spherical Bessel functions j_l, spherical harmonics Y_{lm}, $\hat{r} = \vec{r}/r$, $\hat{k}_i = \vec{k}_i / k_i$
- use properties of spherical Bessel functions

$$j_0(z) = \frac{\sin z}{z} = \frac{e^{iz} - e^{-iz}}{2iz} \quad j_l = z^l \left(-\frac{1}{z}\frac{d}{dz}\right)^l j_0(z) = \frac{1}{2iz}\left[u_l^{(+)}(z) - u_l^{(-)}(z)\right]$$

with in/outgoing spherical waves $u_l^{(\pm)}(z) \to \exp\left[\pm i \left(z - l\frac{\pi}{2}\right)\right]$ for $z \to \infty$

$$\Rightarrow \quad \Phi_i = 4\pi \sum_{l,m} i^l \frac{1}{2ik_i r}\left[u_l^{(+)}(k_i r) - u_l^{(-)}(k_i r)\right] Y_{lm}(\hat{r}) Y^*_{lm}(\hat{k}_i) \phi_a \phi_A$$

Partial-Wave Expansion III

- scattering can only affect outgoing spherical waves $u_l^{(+)}$
  $\Rightarrow$ introduce S-matrix elements $S_l(k_i)$ (complex numbers)
  $\Rightarrow$ asymptotics of solution of Schrödinger equation

$$\Psi_i^{(+)} \to 4\pi \sum_{l,m} i^l \frac{1}{2ik_i r} \left[S_l(k_i) u_l^{(+)}(k_i r) - u_l^{(-)}(k_i r) \right] Y_{lm}(\hat{r}) Y_{lm}^*(\hat{k}_i) \phi_a \phi_A$$

- asymptotics of scattering part

$$\Psi_i^{(+)} - \Phi_i \to 4\pi \sum_{l,m} i^l \frac{1}{2ik_i r} \left[S_l(k_i) - 1 \right] u_l^{(+)}(k_i r) Y_{lm}(\hat{r}) Y_{lm}^*(\hat{k}_i) \phi_a \phi_A$$

$\Rightarrow$ elastic scattering amplitude
$$f_{ii}^{(+)}(\theta) = \sum_l \frac{2l+1}{2ik_i} \left[S_l(k_i) - 1 \right] P_l(\cos\theta)$$

with Legendre polynomials P_l, argument $\cos\theta = \hat{r} \cdot \hat{k}_i$ and
$P_l(\cos\theta) = 4\pi \sum_m Y_{lm}(\hat{r}) Y_{lm}^*(\hat{k}_i)$

Partial-Wave Expansion IV

- differential elastic scattering cross section

$$\frac{d\sigma_{ii}}{d\Omega} = \left| f_{ii}^{(+)} \right|^2 = \left| \sum_l \frac{2l+1}{2ik_i} \left[S_l(k_i) - 1 \right] P_l(\cos\theta) \right|^2$$

- total elastic scattering cross section

$$\sigma_{\mathrm{el}} = \int d\Omega \, \frac{d\sigma_{ii}}{d\Omega} = 2\pi \int_{-1}^{1} d\cos\theta \, \left| f_{ii}^{(+)}(\theta) \right|^2 = \frac{\pi}{k_i^2} \sum_l (2l+1) \left| S_l(k_i) - 1 \right|^2$$

with orthogonality relation of Legendre polynomials $\int_{-1}^{1} dz \, P_l(z) P_{l'}(z) = \frac{2}{2l+1} \delta_{ll'}$

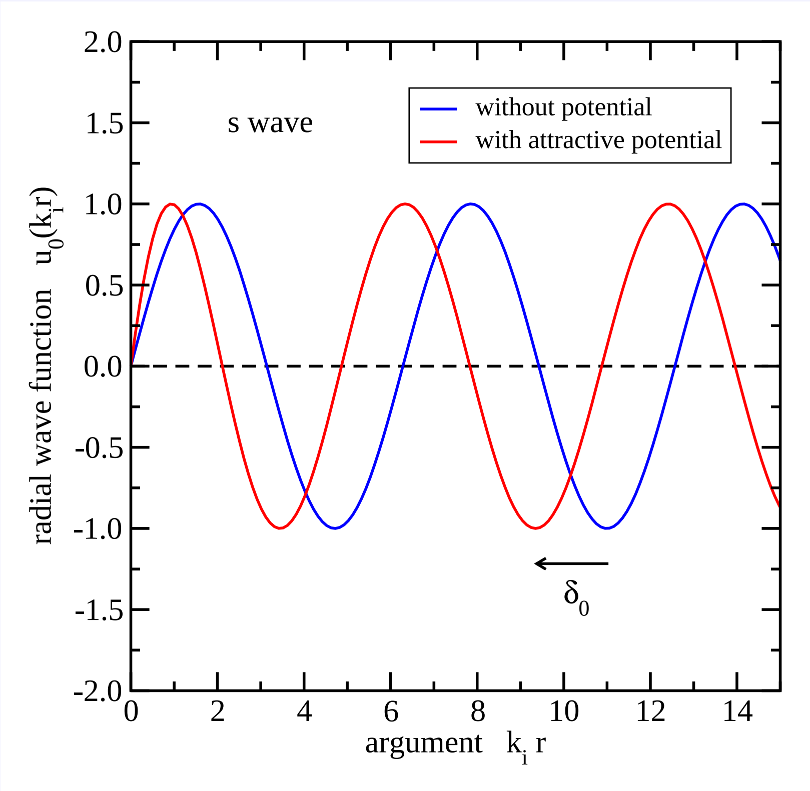
$\Rightarrow$ knowledge of S-matrix elements S_l sufficient to calculate cross sections

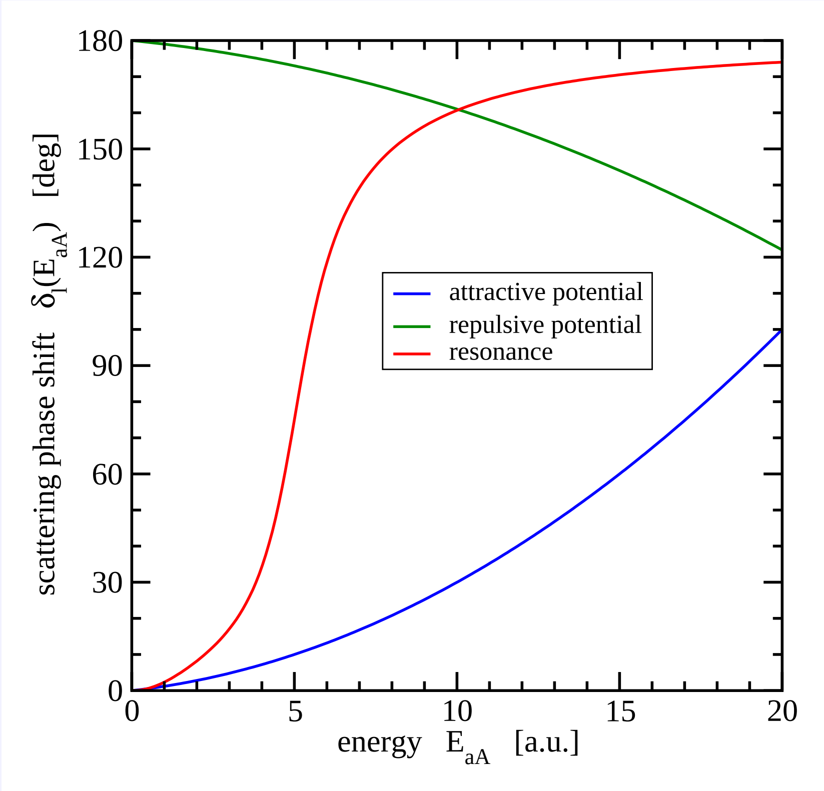
Partial-Wave Expansion V

- parametrization $S_l = \exp(2i\delta_l)$ with scattering phase shifts $\delta_l \in [0, \pi]$

  $\Rightarrow$ asymptotics of radial wave functions

  $$u_l = \frac{1}{2i}\left[S_l u_l^{(+)} - u_l^{(-)}\right] \to \exp(i\delta_l)\sin\left(k_i r + \delta_l - l\frac{\pi}{2}\right) \quad \text{for } r \to \infty$$





Partial-Wave Expansion VI

- calculation of cross section with current of full wave function $\Psi_i^{(+)}$

  $\Rightarrow$ absorption cross section $\sigma_{\text{abs}} = \frac{\pi}{k_i^2} \sum_l (2l+1) \left[1 - |S_l(k_i)|^2\right]$

- total reaction cross section

$$\sigma_{\text{tot}} = \sigma_{\text{el}} + \sigma_{\text{abs}} = \frac{2\pi}{k_i^2} \sum_l (2l+1) \text{Re}\left[1 - S_l(k_i)\right]$$

- scattering phase shifts δ_l real $\Rightarrow$ $|S_l(k_i)| = |\exp(2i\delta_l)| = 1$

  $\Rightarrow$ $\sigma_{\text{abs}} = 0$ only elastic scattering

- scattering phase shifts δ_l complex with $\text{Im}(\delta_l) > 0 \Rightarrow$ $|S_l(k_i)| = \exp[-2\text{Im}(\delta_l)] < 1$

  $\Rightarrow$ $\sigma_{\text{abs}} > 0$ reactions with removal of flux from elastic scattering channel

  - can be described phenomenologically by optical potential
    $U = V + iW$ with real and imaginary contributions (V, W real)

Partial-Wave Expansion VII

isolated narrow resonance in partial wave l

- resonance energy E_r • resonance width $\Gamma \ll E_r$

- different parametrisation of S-matrix element

$$S_l(E) = \frac{E - E_r - i\frac{\Gamma}{2}}{E - E_r + i\frac{\Gamma}{2}} \quad \Rightarrow \quad |S_l(E)| = 1 \quad \Rightarrow \quad S_l(E) - 1 = -i\frac{\Gamma}{E - E_r + i\frac{\Gamma}{2}}$$

$\Rightarrow$ elastic scattering cross section

$$\sigma_{\text{el}} = \frac{\pi}{k_i^2}(2l+1)\frac{\Gamma^2}{(E-E_r)^2 + \frac{\Gamma^2}{4}}$$

Breit-Wigner form

general formulation with many resonances and many channels including Coulomb potential

$\Rightarrow$ **R-matrix theory** with parameters: resonance energies and reduced widths

General Reaction Theory

- Operator Formalism

Operator Formalism I

- simplified notation: point-like nuclei without internal structure
  $\Rightarrow \phi_a = \phi_A = 1,\ E_a = E_A = 0,\ E = E_i = E_{aA} = \frac{\hbar^2 k_i^2}{2\mu_{aA}}$
- solve Schrödinger equation $\hat{H}\Psi = \left(\hat{T} + \hat{V}\right)\Psi = E\Psi$ with $\hat{T} = \hat{H}_0 = -\frac{\hbar^2}{2\mu_{aA}}\Delta$
- $\Phi_0(\vec{k}_i) = \exp\left(i\vec{k}_i \cdot \vec{r}\right)$ is solution of Schrödinger equation $\hat{H}_0\Phi_0 = E\Phi_0$
- rewrite full Schrödinger equation $\left(\hat{H}_0 + \hat{V}\right)\Psi = E\Psi$ as $\hat{V}\Psi = \left(E - \hat{H}_0\right)\Psi$

  and introduce operator $\mathcal{G}_0^{(\pm)} = \left(E - \hat{H}_0 \pm i\epsilon\right)^{-1}$

  $\Rightarrow$ integral (Lippmann-Schwinger) equation $\Psi^{(\pm)} = \Phi_0(\vec{k}_i) + \mathcal{G}_0^{(\pm)}\hat{V}\Psi^{(\pm)}$

  - term with $\Phi_0 \Rightarrow$ correct solution for $\hat{V} = 0$
  - $(\pm)$ forms for different asymptotics

Operator Formalism II

- explicit form $$\Psi^{(\pm)}(\vec{r}) = \Phi_0(\vec{k}_i, \vec{r}) + \int d^3r' \, G_0^{(\pm)}(\vec{r}, \vec{r}') \hat{V}(\vec{r}') \Psi^{(\pm)}(\vec{r}')$$

  with Green's function $$G_0^{(\pm)}(\vec{r}, \vec{r}') = -\frac{2\mu_{aA}}{\hbar^2} \frac{\exp\left(\pm i k_i |\vec{r} - \vec{r}'|\right)}{4\pi |\vec{r} - \vec{r}'|}$$

- formal solution: Born series
$$\Psi^{(\pm)} = \Phi_0(\vec{k}_i) + \mathcal{G}_0^{(\pm)} \hat{V} \Psi^{(\pm)} = \Phi_0(\vec{k}_i) + \mathcal{G}_0^{(\pm)} \hat{V} \Phi_0 + \mathcal{G}_0^{(\pm)} \hat{V} \mathcal{G}_0^{(\pm)} \hat{V} \Phi_0 + \dots$$

  with integral operator $$\mathcal{G}_0^{(\pm)} [\dots] = \int d^3r' \, G_0^{(\pm)}(\vec{r}, \vec{r}') [\dots]$$

- integration range of coordinate $\vec{r}'$ limited by extension potential V
  - for $r \gg r'$ use approximation $k_i |\vec{r} - \vec{r}'| \approx k_i r - \vec{k}_f \vec{r}' + \dots$ with $\vec{k}_f = k_i \frac{\vec{r}}{r}$

  $$\Rightarrow \quad \Psi^{(\pm)}(\vec{r}) = \Phi_0(\vec{k}_i, \vec{r}) - \frac{2\mu_{aA}}{\hbar^2} \frac{\exp(\pm i k_i r)}{4\pi r} \int d^3r' \, \exp\left(-i\vec{k}_f \cdot \vec{r}'\right) \hat{V}(\vec{r}') \Psi^{(\pm)}(\vec{r}')$$

Operator Formalism III

- scattering amplitude

$$f_{fi}^{(\pm)} = -\frac{\mu_{aA}}{2\pi\hbar^2} \int d^3r' \, \exp\left(-i\vec{k}_f \cdot \vec{r}'\right) \hat{V}(\vec{r}')\Psi^{(\pm)}(\vec{r}') = -\frac{\mu_{aA}}{2\pi\hbar^2} T_{fi}$$

  with T-matrix element $T_{fi} = \langle \Phi_0(\vec{k}_f)|\hat{V}|\Psi^{(+)}(\vec{k}_i)\rangle$

  $\Rightarrow$ knowledge of T-matrix elements T_{fi} sufficient to calculate cross sections

- reformulation
  - ◦ introduce potential U with known solutions $\chi^{(\pm)}$ (distorted waves)

    of Schrödinger equation $\left(\hat{H}_0 + \hat{U}\right)\chi^{(\pm)} = E\chi^{(\pm)}$
  - ◦ use operator identity $\frac{1}{A} - \frac{1}{B} = \frac{1}{B}(B-A)\frac{1}{A}$

  $\Rightarrow$ two-potential formula

$$T_{fi} = \langle \Phi_0(\vec{k}_f)|\hat{U}|\chi^{(+)}(\vec{k}_i)\rangle + \langle \chi^{(-)}(\vec{k}_f)|\hat{V} - \hat{U}|\Psi^{(+)}(\vec{k}_i)\rangle$$

Operator Formalism IV

- generalisation with Gell-Mann–Goldberger relation for reaction $A(a,b)B$

$$T_{fi} = \langle \phi_b \phi_B \Phi_0(\vec{k}_f) | \hat{U}_{aA} | \phi_a \phi_A \chi_{aA}^{(+)}(\vec{k}_i) \rangle + \langle \phi_b \phi_B \chi_{bB}^{(-)}(\vec{k}_f) | \hat{V}_{bB} - \hat{U}_{bB} | \Psi_{aA}^{(+)}(\vec{k}_i) \rangle$$

- potential U_{aA} acting only on coordinates of relative motion, not internal coordinates

$\Rightarrow$ $\langle \phi_b \phi_B \Phi_0(\vec{k}_f) | \hat{U}_{aA} | \phi_a \phi_A \chi_{aA}^{(+)}(\vec{k}_i) \rangle = 0$ if $aA \neq bB$

- for rearrangement reactions $aA \neq bB$
  - exact results: – "post form" $T_{fi} = \langle \phi_b \phi_B \chi_{bB}^{(-)}(\vec{k}_f) | \hat{V}_{bB} - \hat{U}_{bB} | \Psi_{aA}^{(+)}(\vec{k}_i) \rangle$

    – "prior form" $T_{fi} = \langle \Psi_{bB}^{(-)}(\vec{k}_f) | \hat{V}_{aA} - \hat{U}_{aA} | \phi_a \phi_A \chi_{aA}^{(-)}(\vec{k}_i) \rangle$

  $\Rightarrow$ exact scattering wave fuctions $\Psi_{aA}^{(+)}$ or $\Psi_{bB}^{(-)}$ still needed

- potentials $\hat{V}_{aA}$ and $\hat{V}_{bB}$ depend on coordinates of all nucleons in the nuclei, should be consistent with those used for microscopic description of nuclei itself
$\Rightarrow$ challenge: combination of structure and reaction calculations

Cross Sections I

- general form for two-body reaction $A(a,b)B$ in c.m. system
  - energies $E_i = E_a + E_A + \frac{p_{aA}^2}{2\mu_{aA}}$, $E_f = E_b + E_B + \frac{p_{bB}^2}{2\mu_{bB}}$
  - with spins $\Rightarrow$ averaging over initial states, summation over final states

$$d\sigma(a + A \to b + B) = \frac{2\pi}{\hbar} \frac{\mu_{aA}}{p_{aA}} \frac{1}{(2J_a + 1)(2J_A + 1)} \sum_{m_a, m_A} \sum_{m_b, m_B} \times \int \frac{d^3 p_{bB}}{(2\pi\hbar)^3} \left| T_{bBaA} \right|^2 \delta(E_i - E_f + Q_{a+A \to B+b})$$

with $d^3 p_{bB} = p_{bB}^2 dp_{bB} d\Omega_{bB}$, $dE_f / dp_{bB} = p_{bB}/\mu_{bB}$ and integration over p_{bB}

$$\Rightarrow \frac{d\sigma}{d\Omega_{bB}}(a + A \to b + B) = \frac{\mu_{aA}\mu_{bB}}{(2\pi)^2 \hbar^4} \frac{p_{bB}}{p_{aA}} \frac{1}{(2J_a + 1)(2J_A + 1)} \sum_{m_a, m_A} \sum_{m_b, m_B} \left| T_{bBaA} \right|^2$$

  - similar expression for inverse reaction $B(b,a)A$
- result can be generalized to reactions with three or more particles in the final state

Cross Sections II

- cross sections for reactions $A(a,b)B$ and $B(b,a)A$

$$\frac{d\sigma}{d\Omega_{bB}}(a+A\to b+B) = \frac{\mu_{aA}\mu_{bB}}{(2\pi)^2\hbar^4}\frac{p_{bB}}{p_{aA}}\frac{1}{(2J_a+1)(2J_A+1)}\sum_{m_a,m_A}\sum_{m_b,m_B}\left|T_{bBaA}\right|^2$$

$$\frac{d\sigma}{d\Omega_{aA}}(b+B\to a+A) = \frac{\mu_{bB}\mu_{aA}}{(2\pi)^2\hbar^4}\frac{p_{aA}}{p_{bB}}\frac{1}{(2J_b+1)(2J_B+1)}\sum_{m_b,m_B}\sum_{m_a,m_A}\left|T_{aAbB}\right|^2$$

- time-reversal symmetry $\left|T_{bBaA}\right|^2 = \left|T_{aAbB}\right|^2$

  $\Rightarrow$ theorem of detailed balance (for two-body reactions)

$$(2J_a+1)(2J_A+1)\, p_{aA}^2 \frac{d\sigma}{d\Omega}(a+A\to b+B)$$

$$= (2J_b+1)(2J_B+1)\, p_{bB}^2 \frac{d\sigma}{d\Omega}(b+B\to a+A)$$

  ◦ application: indirect methods (e.g. Coulomb dissociation method)

Applications

Relevance of Reaction Theory

applications

- direct interest in relevant reaction cross sections, e.g.
  - prediction of production rates of exotic nuclei (not considered here)
  - astrophysics: nucleosynthesis
- reactions to study of nuclear structure, e.g.
  - gross properties: radii, density distributions, . . .
    ⇒ elastic scattering (with electrons, protons, α-particles, . . .), absorption reactions, . . . (not considered here)
  - detailed structure:
    - excitation of specific states with electromagnetic or nuclear interaction
    - study of single-particle structure

challenges with exotic nuclei

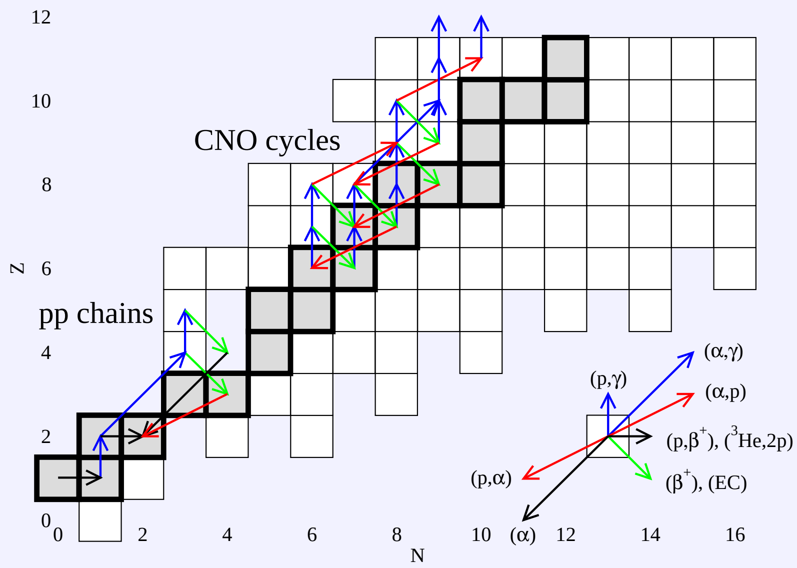
- correct treatment of continuum states
- combination of reaction theory with modern structure models
- choice of nuclear interaction

Applications

- **Astrophysics and Indirect Methods**

Reactions of Astrophysical Interest

- nuclear reactions rates
  - input for astrophysical models
  - various processes (pp-chain, CNO cycles, s- , r-, p-, rp-process)
  - often unstable nuclei involved
  - cross sections at low energies needed
    $\Rightarrow$ direct measurement practically impossible
  - alternative: indirect methods



- nuclei in hot plasma
  $\Rightarrow$ Maxwellian-averaged reaction rate

$$r_{aA} = \frac{\rho_a \rho_A}{1+\delta_{aA}} \langle \sigma v \rangle \quad \text{with} \quad \langle \sigma v \rangle = \sqrt{\frac{8}{\pi \mu_{aA}}} \int \frac{dE}{(kT)^{3/2}} E \sigma(E) \exp\left(-\frac{E}{kT}\right)$$

- reactions with charged particles
  $\Rightarrow$ cross section needed in Gamov window around effective energy

$$E_{\text{eff}} = 0.1220 \, \mu_{aA}^{1/3} (Z_a Z_Z T_9)^{2/3} \text{ MeV}$$

with effective mass μ_{aA} in amu and temperature $T_9 = T/(10^9 \text{ K})$

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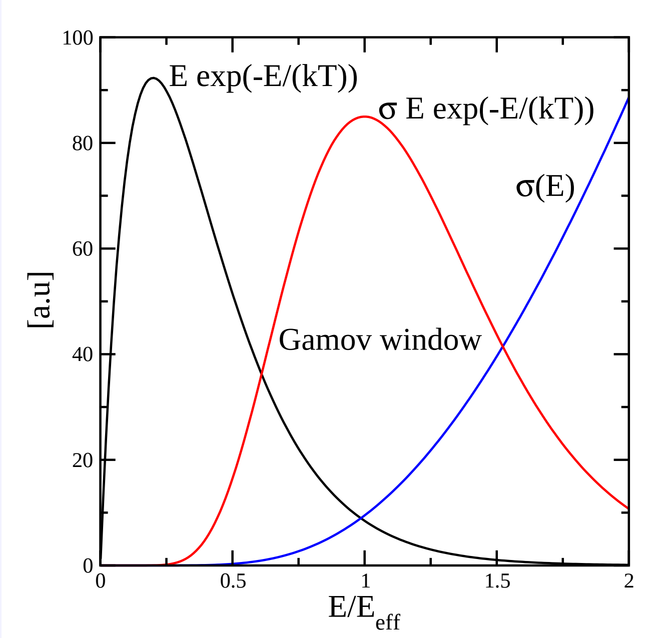

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Scattering with Coulomb Potential I

- modification with Coulomb potential $V_{\mathrm{Coul}} = \frac{Z_a Z_A e^2}{r}$ in initial state

  $\Rightarrow$ replacement of plane wave in initial state
  with exact solution for Coulomb scattering (analytically known)

  $\Rightarrow$ asymptotics of radial wave functions with additional nuclear interaction

  $$u_l \to \frac{\exp(2i\sigma_l)}{2i} \left[S_l u_l^{(+)} - u_l^{(-)} \right]$$ for $r \to \infty$

  with Coulomb scattering phase shift $\sigma_l = \arg \Gamma(l + 1 + i\eta)$

  depending on the Sommerfeld parameter $\eta = \frac{Z_a Z_A e^2}{\hbar v_{aA}}, \; v_{aA} = \frac{p_{aA}}{\mu_{aA}}$

  and $u_l^{(\pm)}(k_i r) = \exp(\mp i\sigma_l) \left[G_l \pm i F_l \right] \to \exp\left[\pm i \left(k_i r - 2\eta \ln(k_i r) + \sigma_l - l\frac{\pi}{2} \right) \right]$

  with regular and irregular Coulomb wave functions F_l and G_l

Scattering with Coulomb Potential II

- elastic scattering amplitude

$$f_{ii}^{(+)}(\theta) = \sum_l \frac{2l+1}{2ik_i} \left[\exp\left(2i\sigma_l\right) S_l(k_i) - 1\right] P_l(\cos\theta) = f_C^{(+)}(\theta) + f_N^{(+)}(\theta)$$

with Coulomb scattering amplitude (analytically known)

$$f_C^{(+)}(\theta) = \sum_l \frac{2l+1}{2ik_i} \left[\exp\left(2i\sigma_l\right) - 1\right] P_l(\cos\theta)$$

and nuclear scattering amplitude

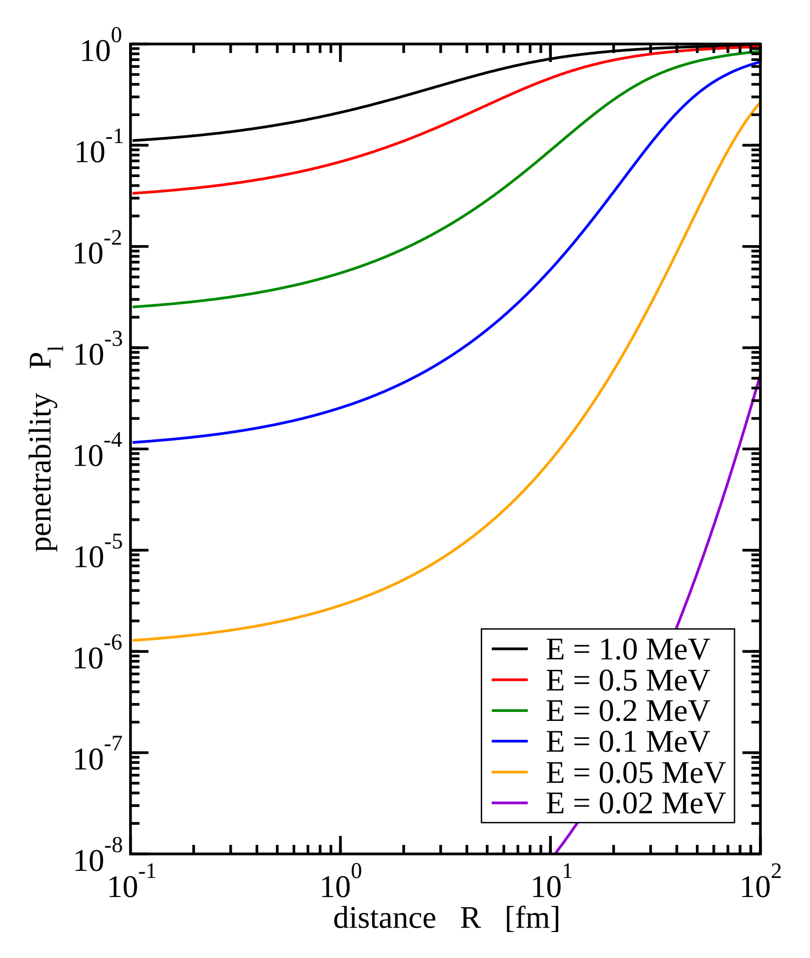
$$f_N^{(+)}(\theta) = \sum_l \frac{2l+1}{2ik_i} \exp\left(2i\sigma_l\right) \left[S_l(k_i) - 1\right] P_l(\cos\theta)$$

$\Rightarrow$ interference in cross sections!

Scattering with Coulomb Potential III

effect of Coulomb barrier (and centrifugal barrier)
$\Rightarrow$ reduced probability of finding
the particles at small distance R

example: p + ^{7}Be scattering
solid lines: $l = 0$



- introduce penetrability factor

$$P_l(R) = \frac{\lim_{r\to\infty} \left| u_l^{(\pm)}(\eta;kr)\right|^2}{\left| u_l^{(\pm)}(\eta;kR)\right|^2} = \frac{1}{F_l^2(\eta,kR)+G_l^2(\eta,kR)}$$

- s-wave scattering ($l = 0$):

$$\lim_{R\to 0} P_0(R) = \frac{2\pi\eta}{\exp(2\pi\eta) - 1}$$

- define astrophysical S factor

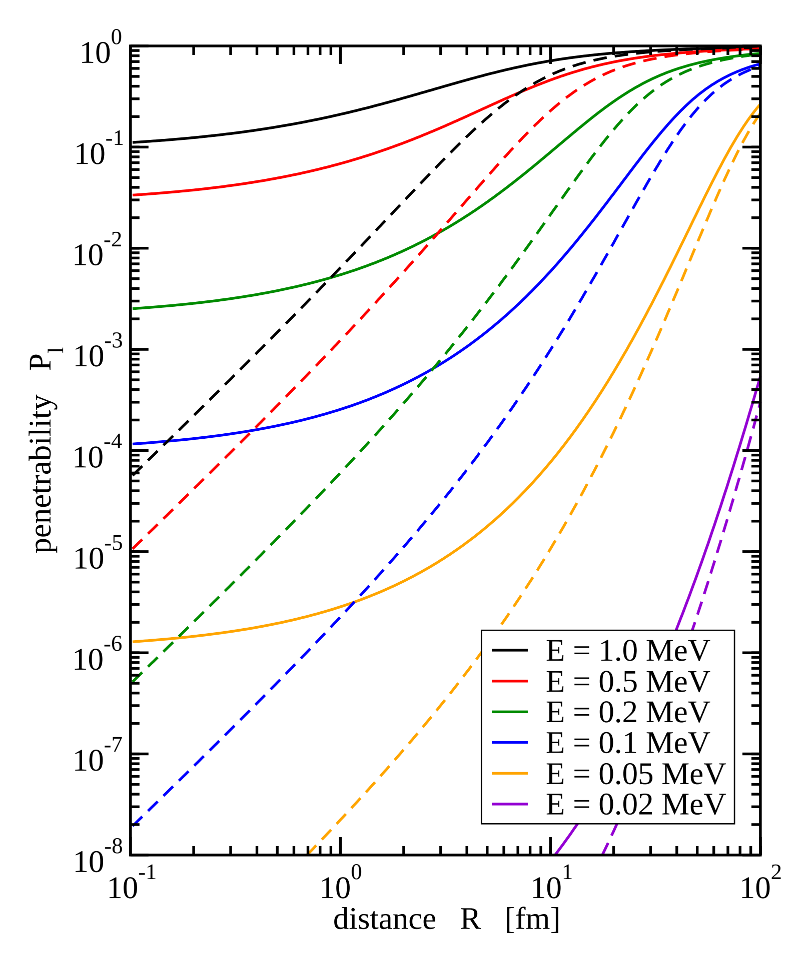
$$S(E) = \sigma(E)\, E\, \exp\left(2\pi\eta\right)$$

weak energy dependence

Scattering with Coulomb Potential III

effect of Coulomb barrier (and centrifugal barrier)
⇒ reduced probability of finding the particles at small distance R

example: p + ^{7}Be scattering
solid lines: $l = 0$
dashed lines: $l = 1$



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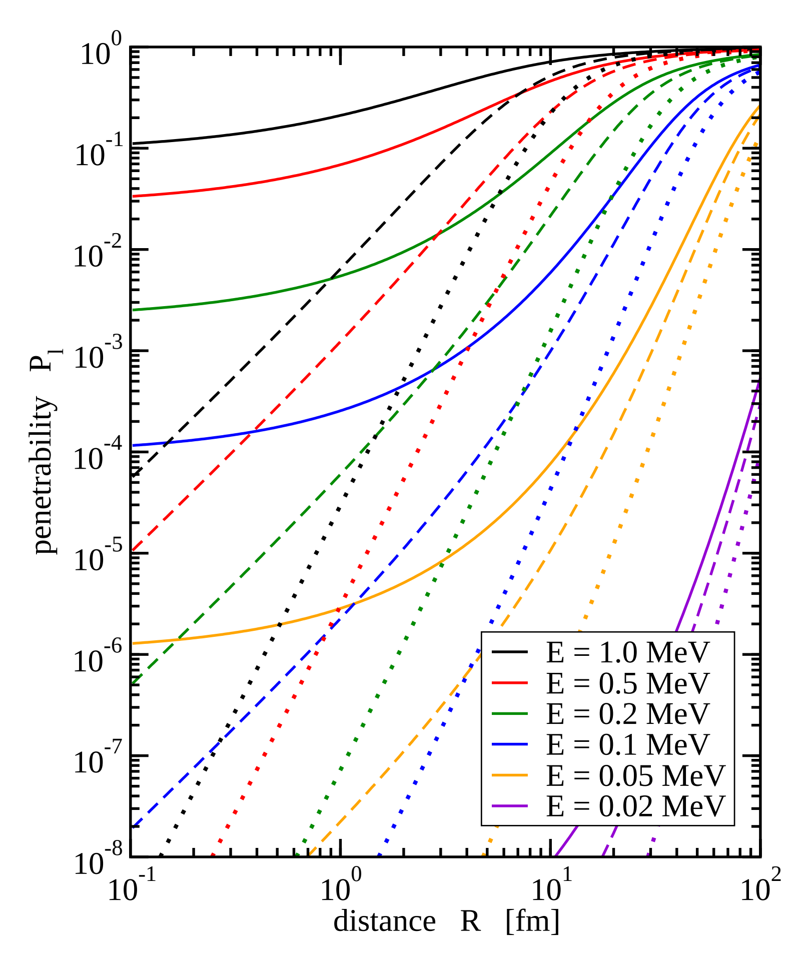
Scattering with Coulomb Potential III

effect of Coulomb barrier (and centrifugal barrier)
$\Rightarrow$ reduced probability of finding
the particles at small distance R

example: p + ^{7}Be scattering
solid lines: $l = 0$
dashed lines: $l = 1$
dotted lines: $l = 2$



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- define astrophysical S factor

$$S(E) = \sigma(E)\, E\, \exp(2\pi\eta)$$

weak energy dependence

Indirect Methods I

Coulomb dissociation

- study inverse of radiative capture reaction $b(x, \gamma)a \;\Leftrightarrow\; a(\gamma, x)b$
- use Coulomb field of target nucleus A as source of photons $a(\gamma, x)b \;\Leftrightarrow\; A(a, bx)A$

$\Downarrow$

absolute S factors as a function of energy

ANC method

- extract asymptotic normalization coefficient of ground state wave function of nucleus a from transfer reactions
- calculate matrix elements for radiative capture reaction $b(x, \gamma)a$

$\Downarrow$

S factor at zero energy

Trojan-Horse method

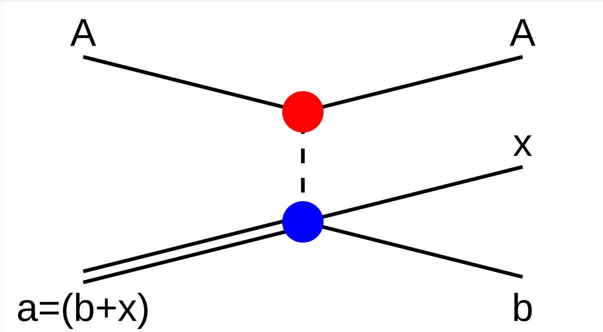
- study three-body reaction $A + a \rightarrow C + c + b$ with Trojan horse $a = b + x$ and spectator b
- extract cross section of two-body reaction $A + x \rightarrow C + c$

$\Downarrow$

energy dependence of S factor

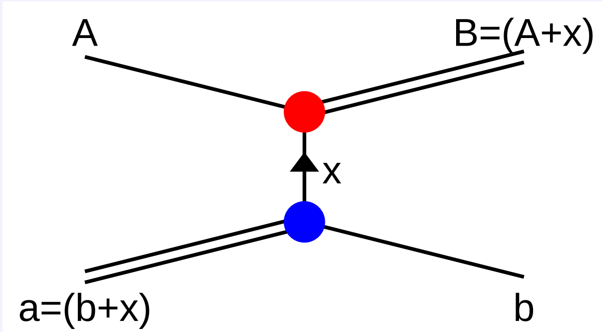
Indirect Methods II

Coulomb dissociation



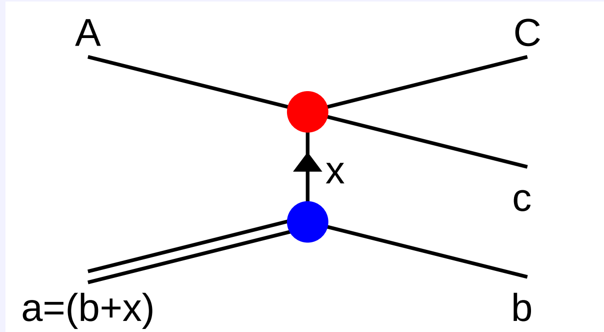
photon exchange

ANC method



transfer of particle to bound state

Trojan-Horse method



transfer of particle to continuum state

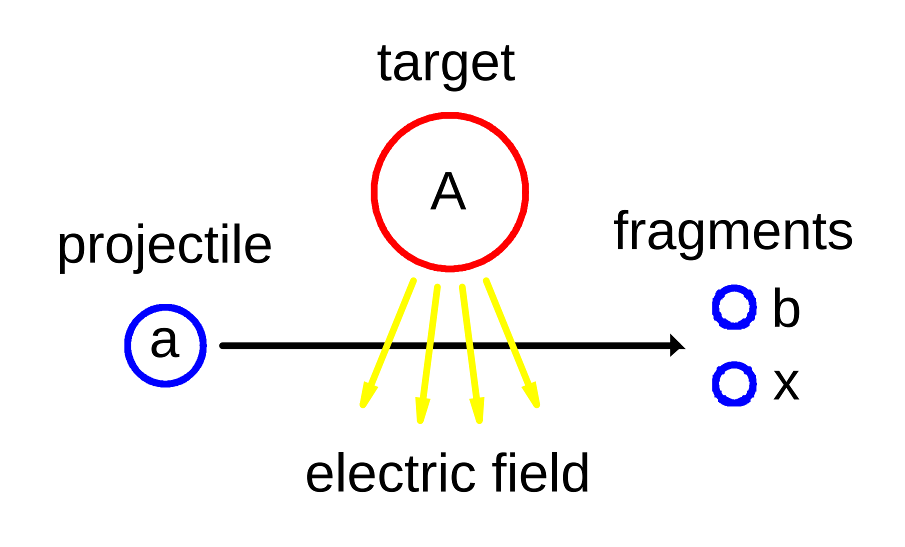
- similar reaction mechanisms: transfer of virtual particle
- final state with three particles (bound/continuum states)
- theoretical descripton with direct reaction theory

Indirect Methods III

general characteristics:

- two-body reaction at low-energy is replaced by three-body reaction at "high-energy" with large cross section
  - ◦ Coulomb dissociation $b(x,\gamma)a \Rightarrow A(a,bx)A$
  - ◦ ANC method $b(x,\gamma)a \Rightarrow A(a,B)b$ $\quad a = (b+x) \quad B = (A+x)$
  - ◦ Trojan-horse method $A(x,c)C \Rightarrow A(a,Cc)b$
- transfer of virtual particle (photon γ or nucleus x)
- relation of cross sections is found with the help of nuclear direct reaction theory
- theoretical approximations essential
- study of peripheral reactions
  - – asymptotics of wave functions relevant
  - – selection of suitable kinematical conditions important

Coulomb Dissociation Method I



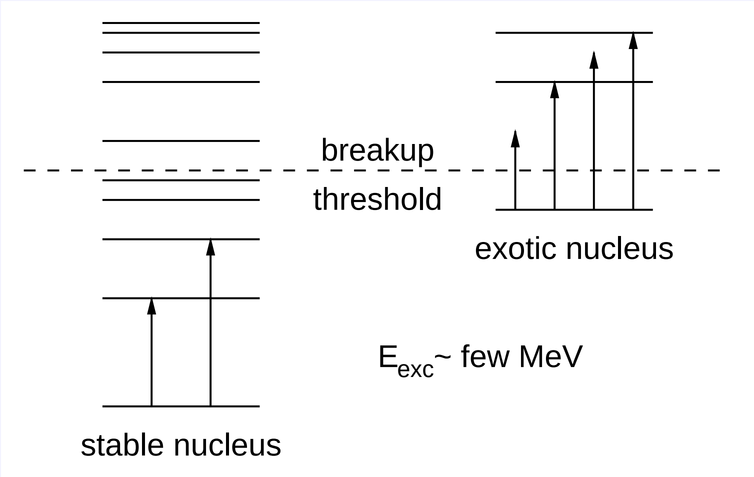
radiative capture $b(x,\gamma)a$

detailed balance $\Updownarrow$

photo absorption $a(\gamma,x)b$

equivalent photons in Coulomb field of target A $\Updownarrow$

Coulomb dissociation $A(a,bx)A$



only ground state transitions !

correspondence

(Fermi 1924, Weizsäcker-Williams 1932)

time-dependent electromagnetic field
of highly-charged nucleus A
during scattering of projectile a
$\Updownarrow$
spectrum of (virtual, equivalent) photons

- further application:
  study structure of nucleus a

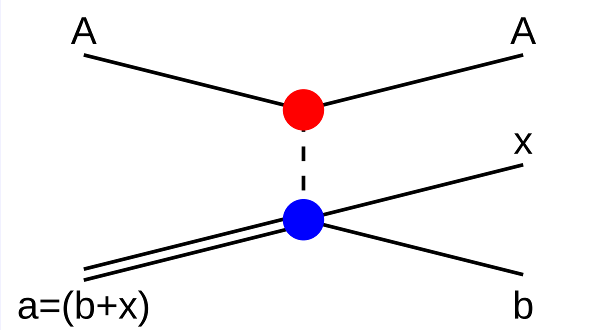
Coulomb Dissociation Method II

breakup cross section for reaction $A(a,bx)A$ (in first-order approximation, with angular integration over relative momentum between fragments)

$$\Rightarrow \quad \frac{d^2\sigma}{dE_{bx}d\Omega_{aA}} = \frac{1}{E_\gamma}\sum_{\pi\lambda}\sigma_{\pi\lambda}(a+\gamma\to b+x)\,\frac{dn_{\pi\lambda}}{d\Omega_{aA}} \qquad \pi = E,M \qquad \lambda = 1,2,\ldots$$

- photo absorption cross section $\sigma_{\pi\lambda}(a+\gamma\to b+x)$
- virtual photon numbers $\frac{dn_{\pi\lambda}}{d\Omega_{aA}}$ depend on kinematics: scattering angle ϑ_{aA} or impact parameter b, projectile velocity v, excitation energy $E_\gamma = \hbar\omega$
- calculation
  - in semiclassical approximation with trajectories
  - with quantal methods using scattering wave functions in partial-wave expansion or eikonal approximation
- final state: usually three charged particles A, b, x
  $\Rightarrow$ higher-order effects = multi-step transitions/Coulomb post-acceleration?
  $\Rightarrow$ better approximation of full scattering wave function needed

Coulomb Dissociation Method III



- Coulomb dissociation cross section

$$\frac{d^2\sigma}{dE_{bx}d\Omega_{Aa}} = \frac{1}{E_\gamma}\sum_{\pi\lambda}\sigma_{\pi\lambda}(a+\gamma\to b+x)\frac{dn_{\pi\lambda}}{d\Omega_{Aa}}$$

- theorem of detailed balance

$$\sigma_{\pi\lambda}(a+\gamma\to b+x) = \frac{(2J_b+1)(2J_x+1)}{2(2J_a+1)}\frac{k_{bx}^2}{k_\gamma^2}\sigma_{\pi\lambda}(b+x\to a+\gamma)$$

  with photo absorpton and radiative capture cross sections

- phase space factor $\frac{k_{bx}^2}{k_\gamma^2} = \frac{2\mu_{bx}c^2E_{bx}}{(E_{bx}+S_{bx})^2} \gg 1$ for not too small E_{bx}

- virtual photon numbers $\frac{dn_{\pi\lambda}}{d\Omega_{Aa}} \gg 1$ for large Z_A and for not too high E_{bx}

$\Rightarrow$ **large Coulomb dissociation cross sections**

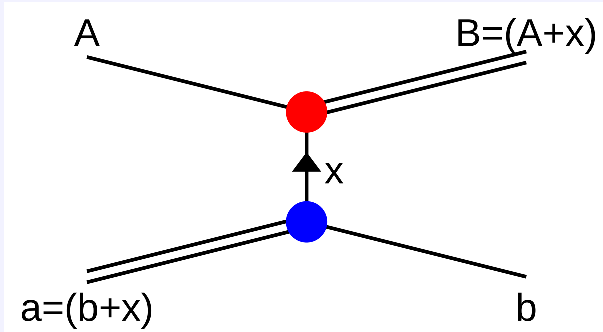
Applications

- Transfer Reactions

Transfer Reactions I

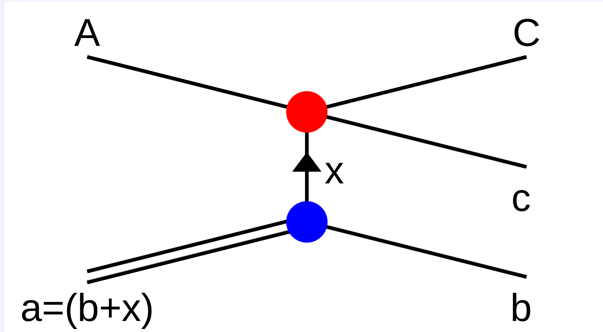
application of transfer reactions

- indirect methods for astrophysics:
  Coulomb breakup, ANC method, Trojan Horse method
- study of nuclear structure:
  - ○ pickup reaction, e.g. (p,d), (d,t), (d,^{3}He), (d,^{6}Li), . . .
  - ○ stripping reaction, e.g. (d,p), (d,n), (^{3}He,p), . . .
  - ○ knockout/breakup reactions, e.g. (p,pn), (p,pα), . . .



theoretical description

- information on reaction process in T-matrix elements
  $\Rightarrow$ full scattering wave function $\Psi^{(\pm)}$ needed,
  in general complicated many-body wave function
  $\Rightarrow$ choose appropriate approximations
- reactions with stable nuclei: two particles in final state,
  $\Rightarrow$ transfer to bound states



- reaction with exotic nuclei: in most cases three (or more) particles in final state
  $\Rightarrow$ transfer to continuum states, study of correlations

Transfer Reactions II

coordinates in two-body system $a + A$

- relative and center-of-mass coordinates

$$\vec{r}_{aA} = \vec{r}_a - \vec{r}_A \qquad \vec{R} = \frac{m_a \vec{r}_a + m_A \vec{r}_A}{m_a + m_A}$$

$$\vec{p}_{aA} = \mu_{aA} \left(\frac{\vec{p}_a}{m_a} - \frac{\vec{p}_A}{m_A} \right) \qquad \vec{P} = \vec{p}_a + \vec{p}_A$$

with reduced mass $\mu_{aA} = \frac{m_a m_A}{m_a + m_A}$

coordinates in three-body system $b + c + C = b + B$

- Jacobi coordinates $\Rightarrow$ three possibilities, e.g.

$$\vec{r}_{cC} = \vec{r}_c - \vec{r}_C \qquad \vec{r}_{b(cC)} = \vec{r}_b - \vec{r}_B \qquad \vec{R} = \frac{m_b \vec{r}_b + m_c \vec{r}_c + m_C \vec{r}_C}{m_b + m_c + m_C}$$

$$\vec{p}_{cC} = \mu_{cC} \left(\frac{\vec{p}_c}{m_c} - \frac{\vec{p}_C}{m_C} \right) \qquad \vec{p}_{b(cC)} = \mu_{bB} \left(\frac{\vec{p}_b}{m_b} - \frac{\vec{p}_B}{m_B} \right) \qquad \vec{P} = \vec{p}_b + \vec{p}_c + \vec{p}_C$$

with $\vec{r}_B = \frac{m_c \vec{r}_c + m_C \vec{r}_C}{m_c + m_C}$, $\vec{p}_B = \vec{p}_c + \vec{p}_C$, $m_B = m_c + m_C$, $\mu_{bB} = \frac{m_b m_B}{m_b + m_B}$

Transfer Reactions III

- general form of cross section for three-body reaction $A(a, cC)b$ in c.m. system
  - energies $E_i = E_a + E_A + \frac{p_{aA}^2}{2\mu_{aA}}$, $E_f = E_b + E_c + E_C + \frac{p_{cC}^2}{2\mu_{cC}} + \frac{p_{bB}^2}{2\mu_{bB}}$

$$d\sigma(a + A \to b + c + C) = \frac{2\pi}{\hbar} \frac{\mu_{aA}}{p_{aA}} \frac{1}{(2J_a + 1)(2J_A + 1)} \sum_{m_a, m_A} \sum_{m_b, m_c, m_C}$$

$$\times \int \frac{d^3 p_{bB}}{(2\pi\hbar)^3} \frac{d^3 p_{cC}}{(2\pi\hbar)^3} \left| T_{(bcC)(aA)} \right|^2 \delta(E_i - E_f + Q_{a+A \to C+c+b})$$

with $d^3 p_{bB} = p_{bB}^2 dp_{bB} d\Omega_{bB}$, $d^3 p_{cC} = p_{cC}^2 dp_{cC} d\Omega_{cC}$, $dE_f / dp_{bB} = p_{bB}/\mu_{bB}$
$E_{cC} = p_{cC}^2/(2\mu_{cC})$, $dE_{cC}/dp_{cC} = p_{cC}/\mu_{cC}$ and integration over p_{bB}

$$\Rightarrow \quad \frac{d^3\sigma}{dE_{cC} d\Omega_{cC} d\Omega_{cB}} (a + A \to b + c + C)$$

$$= \frac{\mu_{aA} \mu_{bB} \mu_{cC}}{(2\pi)^5 \hbar^7} \frac{p_{bB} p_{cC}}{p_{aA}} \frac{1}{(2J_a + 1)(2J_A + 1)} \sum_{m_a, m_A} \sum_{m_b, m_c, m_C} \left| T_{(bcC)(aA)} \right|^2$$

  - integration over unobserved quantities $\Rightarrow$ less detailed information

Transfer Reactions IV

T-matrix elements for transfer reactions $A + a \rightarrow B + b$
with $a = b + x$, $B = A + x$

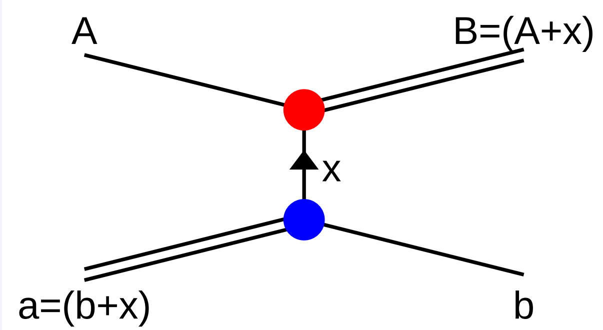
- introduce optical potentials U_{ij} $(ij = Aa, Bb)$

  and distorted waves $\chi_{ij}^{(\pm)}$ with $(T_{ij} + U_{ij})\,\chi_{ij}^{(\pm)} = E_{ij}\chi_{ij}^{(\pm)}$

  - post form: $T_{(Bb)(Aa)} = \langle \phi_B \phi_b \chi_{Bb}^{(-)} | \hat{V}_{Bb} - \hat{U}_{Bb} | \Psi_{Aa}^{(+)} \rangle$ exact!
  - prior form: $T_{(Bb)(Aa)} = \langle \Psi_{Bb}^{(-)} | \hat{V}_{Aa} - \hat{U}_{Aa} | \phi_A \phi_A \chi_{Aa}^{(+)} \rangle$ exact!

- approximations for exact scattering wave functions:
  - distorted-wave Born approximation $\Psi_{Aa}^{(+)} \rightarrow \phi_a \phi_A \chi_{Aa}^{(+)}$ or $\Psi_{Bb}^{(-)} \rightarrow \phi_B \phi_b \chi_{Bb}^{(-)}$
  - better: $\Psi_{Aa}^{(+)}$ or $\Psi_{Bb}^{(-)}$ from coupled-channel calculation
  - other methods

Transfer Reactions V



introduction of spectroscopic amplitudes/factors

- introduce overlap functions
  $\hat{=}$ wave function of transferred particle

  $$\Phi^a_{bx} = \langle \phi_b | \phi_a \rangle \qquad \Phi^B_{Ax} = \langle \phi_A | \phi_B \rangle$$

  (integration only over internal coordinates)

- approximation $\Phi^a_{bx} \approx \mathcal{A}^a_{bx} \varphi^a_{bx}(\vec{r}_{bx}) \phi_x \quad \Phi^B_{Ax} \approx \mathcal{A}^B_{Ax} \varphi^B_{Ax}(\vec{r}_{Ax}) \phi_x$

  with single-particle wave functions φ^a_{bx}, φ^A_{Bx}
  generated from (standard) potentials (usually Woods-Saxon type)

  ◦ for bound states $\langle \varphi^a_{bx} | \varphi^a_{bx} \rangle = \langle \varphi^A_{Bx} | \varphi^A_{Bx} \rangle = 1$

  and spectroscopic amplitudes $\mathcal{A}^a_{bx}$, $\mathcal{A}^B_{Ax}$

  $\Rightarrow$ spectroscopic factors $\mathcal{S}^a_{bx} = \left| \mathcal{A}^a_{bx} \right|^2 \qquad \mathcal{S}^B_{Ax} = \left| \mathcal{A}^B_{Ax} \right|^2$

  ◦ model dependent quantities!

Transfer Reactions VI

distorted-wave Born approximation (DWBA) and spectroscopic factors

- approximation for T-matrix elements (still expensive to calculate)
  - post form: $T_{(Bb)(Aa)} \approx \mathcal{A}^{B*}_{Ax}\mathcal{A}^{a}_{bx} \langle \varphi^{B}_{Ax}(\vec{r}_{Ax})\chi^{(-)}_{Bb}|\hat{V}_{Bb} - \hat{U}_{Bb}|\varphi^{a}_{bx}(\vec{r}_{bx})\chi^{(+)}_{Aa}\rangle$
  - prior form: $T_{(Bb)(Aa)} \approx \mathcal{A}^{B*}_{Ax}\mathcal{A}^{a}_{bx} \langle \varphi^{B}_{Ax}(\vec{r}_{Ax})\chi^{(-)}_{Bb}|\hat{V}_{Aa} - \hat{U}_{Aa}|\varphi^{a}_{bx}(\vec{r}_{bx})\chi^{(+)}_{Aa}\rangle$
- distorted waves $\chi^{(+)}_{Aa}$, $\chi^{(-)}_{Bb}$ from full calculation in partial-wave expansion or eikonal approximation at high energies
- cross sections $d\sigma \propto \left|T_{(Bb)(Aa)}\right|^2$ $\Rightarrow$ $d\sigma \approx \mathcal{S}^{a}_{bx}\mathcal{S}^{B}_{Ax} d\sigma_{\text{single particle}}$
- experimental spectroscopic factors $\mathcal{S}^{a}_{bx}$, $\mathcal{S}^{B}_{Ax}$ from comparison of measured cross sections with single-particle cross sections
- microscopic nuclear structure models: $\mathcal{S}^{a}_{bx} = \langle \Phi^{a}_{bx}|\Phi^{a}_{bx}\rangle$, $\mathcal{S}^{B}_{Ax} = \langle \Phi^{B}_{Ax}|\Phi^{B}_{Ax}\rangle$
- choice of potentials $\hat{V}_{aA}$, $\hat{V}_{bB}$, $\hat{U}_{aA}$, $\hat{U}_{bB}$? often approximations
- interpretation if $B = C + c$ is continuum state?

Applications

- Coupled-Channel Approach

Coupled-Channel Approach I

explicit calculation of full scattering wave function

$\Rightarrow$ expansion $\Psi_{aA}^{(+)} = \sum_c \psi_c$ with all channels $c = a + A, b + B, \ldots$

and correct asymptotics $\psi_c \to \phi_c f_{c(aA)}^{(+)} \frac{\exp(ik_c r_c)}{r_c}$ with $\phi_c = \phi_a \phi_A, \phi_b \phi_B, \ldots$

- Hamiltonian $\hat{H} = \hat{H}_c + \hat{T}_c + \hat{V}_c$

  with, e.g. $\hat{H}_c = \hat{H}_a + \hat{H}_A$, $\hat{T}_c = \hat{T}_{Aa}$, $\hat{V}_c = \hat{V}_{aA}$ for $c = a + A$

- stationary Schrödinger equation $\hat{H}\Psi_{aA}^{(+)} = E\Psi_{aA}^{(+)}$
  - projection on channel wave functions ϕ_c

    $\Rightarrow$ coupled equations

    $$\langle \phi_c | \hat{T}_c + \hat{V}_c + E_c - E | \psi_c \rangle = -\sum_{c'} \langle \phi_c | \hat{H} - E | \psi_{c'} \rangle$$

    with, e.g., $E_c = E_a + E_A$ for $c = a + A$ in diagonal part

Coupled-Channel Approach II

problems

- infinitely many channels (different excited states and partitions, partial waves)
  truncation needed $\Rightarrow$ choice of relevant channels
- asymptotic solution in channels with three particles
  $\Rightarrow$ different methods/approximations
  - use hyperspherical coordinates, but no exact solution for three charges particles
  - use product wave function $\phi_{bcC} \approx \phi_b \Psi_{cC}^{(+)}$

    with two-body scattering wave function $\Psi_{cC}^{(+)}$
    – continuum states depend on energy, not normalizable
    – discretize continuum by introducing energy bins

      $\phi_{bcC} = \phi_b \int_{E_{\min}}^{E_{\max}} dE\, w(E)\, \Psi_{cC}^{(+)}(E)$ with appropriate weight functions $w(E)$

      $\Rightarrow \phi_{bcC}$ normalizable

    $\Rightarrow$ continuum-discretized coupled channels (CDCC) approach

Coupled-Channel Approach III

CDCC approach

- single-particle states and collective states can be considered
- binning of continuum can be adapted to resonances
- numerical convergence can be tested
- computationable expensive, only for not too high energies

optical potentials used in calculation of T-matrix elements

- imaginary part considers loss of flux to open channels
  - consistency with explicit treatment of open channels?
- often systematic potentials used from fits of elastic scattering cross sections
  - mostly available for scattering of nucleons and light nuclei (d, α, . . .)
  - usually not available for exotic nuclei
- other approaches: e.g.
  - single-folding or double-folding potentials
  - dispersive methods

Summary

Summary

- information on reactions is contained in cross sections
  - depend only on asymptotics of scattering wave functions
  - knowledge of S-matrix elements or T-matrix elements sufficient
  - many different types of reactions and kinematical conditions
  - many methods: partial-wave expansion, R-matrix theory, DWBA, CDCC, . . .
- reactions with exotic nuclei
  - direct interest in reaction cross sections (e.g. astrophysics)
  - reactions as tool to study nuclear structure
  - major challenges:
    - adaption of standard methods to specific conditions
    - combination of reaction theory with modern nuclear structure models
    - treatment of (many-body) continuum states
    - choice and consistent application of potentials

$\Rightarrow$ **need for development and many exciting applications in the future**