(Two loop) Four dimensional formulation of dimensionally regularized amplitudes in quantum chromodynamics

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Main reference

The talk is based on the article

C. Gnendiger, A. Signer, D. Stockinger, A. Broggio, A. L. Cherchiglia, F. Driencourt-Mangin , A. R. Fazio, B. Hiller, P. Mastrolia, T. Peraro, R. Pittau, G. M. Pruna, G. Rodrigo, M. Sampaio, G. Sborlini, W. J. Torres Bobadilla, F. Tramontano, Y. Ulrich, A. Visconti

“To d, or not to d: Recent developments and comparisons of regularization schemes”

arXiv:1705.01827

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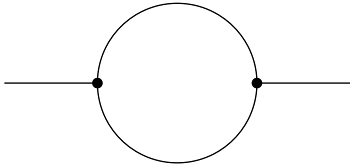
Quantum loop corrections

- Loop diagrams could be divergent



$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2)^2} = \pi^2 \int_0^\infty dk^2 \frac{1}{k^2} = \pi^2 \int_0^\infty \frac{dx}{x}$$

This integral diverges at

- $k^2 \to \infty$ (UV divergence) and at
- $k^2 \to 0$ (IR divergence).

Dimensional regularization

In dimensional regularization

$$\int \frac{d^4k}{(2\pi)^4} \rightarrow \mu_{\mathrm{DS}}^{4-d} \int \frac{d^dk}{(2\pi)^d}$$

Example

$$\int d^dk \frac{1}{(k^2)^2} = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \int_0^\infty dk^2 (k^2)^{\frac{d}{2}-3}$$

The continuation from 4 to $d = 4 - 2\epsilon$ dimensions makes all momentum integrals well defined and UV and IR singularities appear in the Laurent expansion of meromorphic functions as $\frac{1}{\epsilon^n}$ poles.

Physical cross section at Next-to-Leading Order

$$\sigma = \sigma_V + \sigma_R = \int d\Phi_V |M_V|^2 + \int d\Phi_R |M_R|^2$$

By choosing a complete dimensional scheme (DS)

$$\sigma^{DS} = \underbrace{\int d\Phi_V |M_V(....., [g], ...)|^2}_{\frac{a}{\epsilon^2}+\frac{b}{\epsilon}+c+d\epsilon+e\epsilon^2+...} + \underbrace{\int d\Phi_R |M_R(...., [g], ...)|^2}_{-\frac{a}{\epsilon^2}-\frac{b}{\epsilon}+l+m\epsilon+n\epsilon^2+...}$$

$$= \sigma_{\text{finite}} + \epsilon\sigma_1 + \epsilon^2\sigma_2 + \dots$$

The physical cross section is

$$\sigma = \lim_{\epsilon \to 0} \sigma^{DS} = \sigma_{\text{finite}}.$$

- A different (but consistent) treatment of the gluon metric in the amplitude will modify the scheme dependence in the virtual and real contribution, keeping however the physical cross-section invariant.
- ($\Longrightarrow$) The purely d-dim. treatment of all objects is conceptually simpler, but it breaks supersymmetry and gives ambiguities $\mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau)$ with chiral symmetries.
- The 4-dim. treatment of the gluons is better compatible with supersymmetry and it is more amenable to helicity methods

$$A_{\mathrm{MHV}}(1^+, ..., i^-, ..., j^-, ...n^+) =$$

$$i(-g)^{n-2}\frac{< ij >^4}{< 12 >< 23 > .... < n1 >}$$

Parke-Taylor formula for Maximally Helicity Violating amplitudes.

Schemes

To treat the different schemes in a single framework we distinguish three vector spaces:

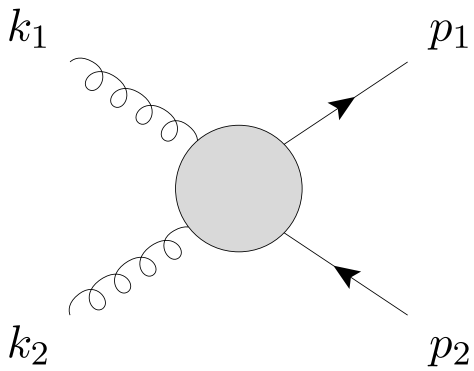
- the original four dimensional space ($4S$) with metric tensor $g^{\mu\nu}_{[4]}$,
- the "quasi- d- dimensional" space ($QS_{[d]}$) with metric tensor $g^{\mu\nu}_{[d]}$,
- the "quasi- d_s- dimensional" space ($QS_{[d_s]}$)with metric tensor $g^{\mu\nu}_{[d_s]}$.

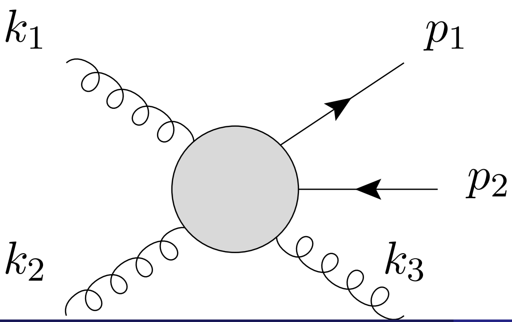
The "quasi-dimensionalities" of those infinite dimensional spaces are related by

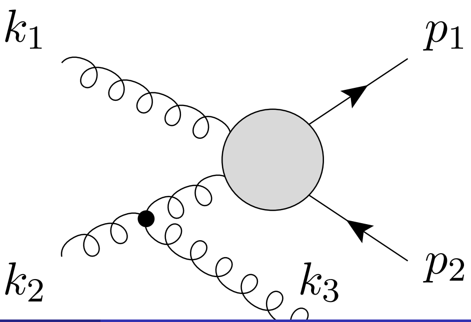
$$d_s = d + n_\epsilon = 4 - 2\epsilon + n_\epsilon$$

$$QS_{[d_s]} = QS_{[d]} \oplus QS_{[n_\epsilon]} \qquad S_4 \subset QS_{[d]} \subset QS_{[d_s]}.$$

Only gluons that appear inside a divergent loop or phase space integral need to be regularized, they are **singular** all others are **regular**.







Variants of dimensional regularization and dimensional reduction

Dimensional regularization

- CDR ("Conventional dimensional regularization"): Here singular and regular gluons (and other vector fields) are all treated as quasi-$d-$dimensional.
- HV (" 't Hooft Veltman"): Singular gluons are treated as quasi-$d-$dimensional but the regular ones are treated as $4-$dimensional.

Variants of dimensional regularization and dimensional reduction

Dimensional reduction

- DRED ("original/old dimensional reduction"): Regular and singular gluons are all treated as quasi-d_s-dimensional.
- FDH ("four-dimensional helicity scheme"): Singular gluons are treated as quasi-d_s-dimensional but external ones are treated as 4-dimensional.

Treatment of vector fields in the four different regularization schemes

Prescription on which metric tensor has to be used in propagator numerators and vectors polarization sums, usually $d_s \to 4$

	CDR	HV	FDH	DRED
singular VF	$g^{\mu\nu}_{[d]}$	$g^{\mu\nu}_{[d]}$	$g^{\mu\nu}_{[d_s]}$	$g^{\mu\nu}_{[d_s]}$
regular VF	$g^{\mu\nu}_{[d]}$	$g^{\mu\nu}_{[4]}$	$g^{\mu\nu}_{[4]}$	$g^{\mu\nu}_{[d_s]}$

Using DRED and FDH

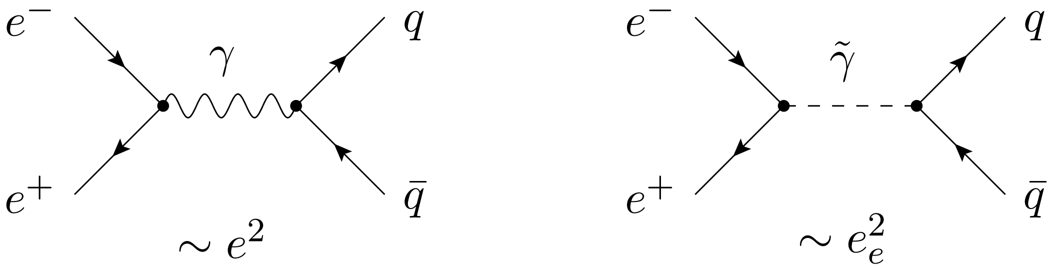
The crucial step is to split quasi-d_s-dimensional gluons into d-component gauge fields and $n_\epsilon \to 2\epsilon$ scalar fields, so called ϵ-scalars:

$$(g_{[d_s]})^{\mu\nu} = (g_{[d]})^{\mu\nu} + (g_{[n_\epsilon]})^{\mu\nu}$$

$$g^{\mu\nu}_{[d_s]}(g_{[d]})_{\nu\rho} = (g_{[d]})^{\mu}{}_{\rho}$$

$$g^{\mu\nu}_{[d_s]}(g_{[n_\epsilon]})_{\nu\rho} = (g_{[n_\epsilon]})^{\mu}{}_{\rho} \quad g^{\mu\nu}_{[d]}(g_{[n_\epsilon]})_{\nu\rho} = 0 \quad (g_{[n_\epsilon]})_{\nu\rho} g^{\rho\mu}_{[4]} = 0$$

($\Longrightarrow$) **During the renormalization process the couplings of the ϵ-scalars must be treated as independent, resulting in different renormalization constants and β-functions.**



In QED the gauge coupling e and the evanescent coupling e_e renormalize differently being not protected by the Lorentz and the gauge invariance on $QS_{[d]}$

$$\beta_e = \mu_{\mathrm{DS}}^2 \frac{d}{d\mu_{\mathrm{DS}}^2} \left(\frac{e}{4\pi}\right)^2 = \frac{4}{3} \left(\frac{e}{4\pi}\right)^4 + \dots$$

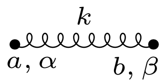
$$\beta_{e_e} = \mu_{\mathrm{DS}}^2 \frac{d}{d\mu_{\mathrm{DS}}^2} \left(\frac{e_e}{4\pi}\right)^2 = 6 \left(\frac{e}{4\pi}\right)^4 - 6 \left(\frac{e}{4\pi}\right)^2 \left(\frac{e_e}{4\pi}\right)^2 + \dots$$

Even by imposing the renormalization condition $e = e_e$ the flows of the two couplings is different.

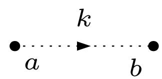
FDF: Four Dimensional Formulation of the FDH scheme

The external legs are treated as usual four dimensional states.

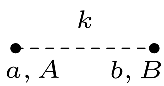
- Loop propagators in Feynman-'t Hooft gauge



$$= -i\,\delta^{ab}\,\frac{g_{[4]}^{\alpha\beta}}{k_{[4]}^2 - \mu^2 + i\varepsilon} \quad \text{(gluon)},$$



$$= i\,\delta^{ab}\,\frac{1}{k_{[4]}^2 - \mu^2 + i\varepsilon} \quad \text{(ghost)},$$



$$= -i\,\delta^{ab}\,\frac{G^{AB}}{k_{[4]}^2 - \mu^2 + i\varepsilon} \quad \text{(scalar)},$$

The scalars come from a dimensional reduction of $d_s = 4 + (-2\epsilon + n_\epsilon)$ dimensional gluons vector fields.

In $d=4-2\epsilon$ dimensions we perform the decomposition of the loop momentum $k_{[d]}^{\alpha}$ in a 4-dimensional part $k_{[4]}^{\alpha}$ and in its orthogonal complement the -2ϵ-dimensional **fixed** vector $k_{[-2\epsilon]}^{\alpha} \equiv \mu^{\alpha}$

$$k_{[d]}{}^{\alpha} = k_{[4]}^{\alpha} + \mu^{\alpha} \quad \mu^{\alpha}\mu_{\alpha} = -\mu^{2}$$

$$g_{[d_s]}^{\alpha\beta} = g_{[4]}^{\alpha\beta} + g_{[n_{\epsilon}-2\epsilon]}^{\alpha\beta} \quad g_{[n_{\epsilon}-2\epsilon]}{}^{\alpha\beta} \to G^{AB} \quad \mu^{\alpha} \to i\mu Q^{A}$$

where the A and B label the components of the complementary space of dimension $d_s - 4$.

The metric G^{AB} and the vector Q^{A} needed to reformulate the Feynman rules satisfy

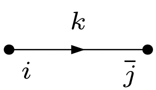
$$G^{AB}G^{BC} = G^{AC}, \qquad G^{AA} = 0, \qquad G^{AB} = G^{BA}$$

$$Q^{A}G^{AB} = Q^{B}, \qquad Q^{A}Q^{A} = 1$$

and reproduce the numerator of the Feynman diagrams of the Four Dimensional Helicity Scheme (FDH).

- **Fermionic propagator in a loop**

  Dirac matrices have the following splitting
$$\gamma^{\alpha}_{[d_s]} = \gamma^{\alpha}_{[4]} + \gamma^{\alpha}_{[n_\epsilon - 2\epsilon]}$$
  and satisfy in d_s dimensions the Clifford algebra
$$\{\gamma^{\alpha}_{[d_s]}, \gamma^{\beta}_{[d_s]}\} = 2 g^{\alpha\beta}_{[d_s]}.$$
  A possible 4-dimensional representation of $\gamma_{[n_\epsilon - 2\epsilon]}$ matrices is in terms of $\gamma^5_{[4]}$ by the replacement
$$\gamma^{\alpha}_{[n_\epsilon - 2\epsilon]} \to \gamma^5_{[4]} \Gamma^A.$$
  By imposing the rule $Q^A \Gamma^A = 1$ needed to recover $\not{\mu}\not{\mu} = -\mu^2$ and $\Gamma^A \Gamma_A = 0$ to reproduce the Breintenlohner-Maison prescription of γ_5



$$= i\delta^i_{\bar{j}} \frac{\not{k}_{[4]} + m - i\mu\gamma^5_{[4]}}{k^2_{[4]} - m^2 - \mu^2 + i\varepsilon}.$$

Generalized Internal legs

- Generalized subluminal Dirac equation.
  Given the ℓ **four** dimensional vector

$$\left(\not{\ell} - i\mu\gamma^5 - m\right) u_\lambda(\ell) = 0\,,$$
$$\left(\not{\ell} - i\mu\gamma^5 + m\right) v_\lambda(\ell) = 0\,,$$
$$\ell^\mu = \ell^{\flat\mu} + \frac{m^2+\mu^2}{2\ell\cdot q_\ell} {q^\mu}_\ell; \qquad (\ell^\flat)^2 = 0 = q_\ell^2.$$

- Solutions of the generalized Dirac equation

$$u_+(\ell) = \left|\ell^\flat\right\rangle - \frac{(m-i\mu)}{\left[\ell^\flat\, q_\ell\right]}\left|q_\ell\right]\,,\ u_-(\ell) = \left|\ell^\flat\right] - \frac{(m+i\mu)}{\left\langle\ell^\flat\, q_\ell\right\rangle}\left|q_\ell\right\rangle\,,$$
$$v_-(\ell) = \left|\ell^\flat\right\rangle + \frac{(m-i\mu)}{\left[\ell^\flat\, q_\ell\right]}\left|q_\ell\right]\,,\ v_+(\ell) = \left|\ell^\flat\right] + \frac{(m+i\mu)}{\left\langle\ell^\flat\, q_\ell\right\rangle}\left|q_\ell\right\rangle\,. \tag{3a}$$

- Polarization sum of the solutions of the generalized Dirac equation

$$\sum_{\lambda=\pm} u_\lambda(\ell)\, \bar{u}_\lambda(\ell) = \not{\ell} - i\mu\gamma^5 + m\,,$$

$$\sum_{\lambda=\pm} v_\lambda(\ell)\, \bar{v}_\lambda(\ell) = \not{\ell} - i\mu\gamma^5 - m\,.$$

- **Generalized Polarization Vectors**

Once again let us decompose the massive **four**-dimensional vector ($\ell^2 = \mu^2$)

$$\ell^\alpha = \ell^{\flat\,\alpha} + \hat{q}_\ell^\alpha$$

the μ-massive polarizations vectors are

$$\varepsilon_+^\alpha(\ell) = -\frac{\left[\ell^\flat\left|\gamma^\alpha\right|\hat{q}_\ell\right\rangle}{\sqrt{2}\mu}\,, \qquad \varepsilon_-^\alpha(\ell) = -\frac{\left\langle\ell^\flat\left|\gamma^\alpha\right|\hat{q}_\ell\right]}{\sqrt{2}\mu}\,,$$

$$\varepsilon_0^\alpha(\ell) = \frac{\ell^{\flat\alpha} - \hat{q}_\ell^\alpha}{\mu}$$

with the usual Proca's completeness relation

$$\sum_{\lambda=\pm,0} \varepsilon_\lambda^\alpha(\ell)\,\varepsilon_\lambda^{*\beta}(\ell) = -g_{[4]}^{\alpha\beta} + \frac{\ell^\alpha\ell^\beta}{\mu^2}$$

$$\varepsilon_\pm^2(\ell) = 0\,, \qquad \varepsilon_\pm(\ell)\cdot\varepsilon_\mp(\ell) = -1\,,$$

$$\varepsilon_0^2(\ell) = -1\,, \qquad \varepsilon_\pm(\ell)\cdot\varepsilon_0(\ell) = 0\,,$$

$$\varepsilon_\lambda(\ell)\cdot\ell = 0 \quad \lambda = \pm, 0.$$

Four point massless one-loop color ordered amplitudes A_4

In terms of the one-loop master integrals: boxes, triangles and bubbles

$$
\begin{aligned}
A_4 = \Big[& c_{1|2|3|4;\,0}\, I_{1|2|3|4} + \big(c_{12|3|4;\,0}\, I_{12|3|4} \\
& + c_{1|2|34;\,0}\, I_{1|2|34} + c_{1|23|4;\,0}\, I_{1|23|4} + c_{2|3|41;\,0}\, I_{2|3|41} \big) \\
& + \big(c_{12|34;\,0}\, I_{12|34} + c_{23|41;\,0}\, I_{23|41} \big) \Big] + \mathcal{R} + O(\epsilon)\,, \\
\mathcal{R} = \Big[& c_{1|2|3|4;\,4}\, I_{1|2|3|4}[\mu^4] + \big(c_{12|3|4;\,2}\, I_{12|34}[\mu^2] \\
& + c_{1|2|34;\,2}\, I_{1|2|34}[\mu^2] + c_{1|23|4;\,2}\, I_{1|23|4}[\mu^2] \\
& + c_{2|3|41;\,2}\, I_{2|3|41}[\mu^2] \big) \\
& + \big(c_{12|34;\,2}\, I_{12|34}[\mu^2] + c_{23|41;\,2}\, I_{23|41}[\mu^2] \big) \Big]\,.
\end{aligned}
$$

The coefficients c_i are just functions of the spinor variables: **NO** ϵ.

By the separation

$$\int \frac{d^D \ell_{[D]}}{(2\pi)^D} = \int \frac{d^{-\epsilon}(\mu^2)}{(2\pi)^{-2\epsilon}} \int \frac{d^4 \ell_{[4]}}{(2\pi)^4}.$$

and using polar coordinates in the -2ϵ dimensional Euclidean vector space, all the integrals in $\mathcal{R}$ can be computed. In particular

$$\lim_{\epsilon \to 0} I_{1|2|3|4}^{4-2\epsilon}[\mu^4] = \lim_{\epsilon \to 0} \left(\epsilon(\epsilon - 1) 16\pi^2 I_{1|2|3|4}^{8-2\epsilon} \right) = -\frac{1}{6}.$$

We found a way of computing the rational part of scattering amplitudes by four-dimensional unitarity cuts.

Two Loops - Reduction

- Tensor reduction as at one-loop is necessary and useful.
- But not sufficient: need additional technology to reduce powers of irreducible invariants.
- Integration by parts (IBP)

$$0 = \int d^d\ell_1 d^d\ell_2 \frac{\partial}{\partial \ell_i^\mu} \frac{v^\mu_{[d]}}{\text{Denominator}}$$

Gives linear relations between integrals.

$v^\mu_{[d]}$ is an IBP generating vector, algebraic geometry determines it by imposing the absence of doubled propagator.

- One of the terms in $v^\mu_{[d]}$ is of the form

$$\frac{\partial}{\partial \ell_i^\mu} \frac{\ell_i^\mu}{\text{Denominator}} = \frac{4-2\epsilon}{\text{Denominator}} + \ldots$$

- ϵ seems to be intrinsic to these reductions as i.e. in the one-loop bubble by Passarino-Veltman

$$B^{\mu\nu}(k^2) = \frac{2-\epsilon}{4(3-2\epsilon)} B_0(k^2) k^\mu_{[4]} k^\nu_{[4]} - \frac{1}{4(3-2\epsilon)} B_0(k^2) g^{\mu\nu}_{[4]}$$

  This is a problem in view of implementing the $4-2\epsilon$ dimensional generalized unitarity program.
- Suspicion for a μ-augmented basis (with μ_1^2, μ_2^2, $\mu_1 \cdot \mu_2$). The tree amplitudes needed for two-loop, and computed by FDF rules, are embedded in a **six dimensional space**.
- That augmented basis is found by 4-dimensional IBPs

$$0 = \int d^d\ell_1 d^d\ell_2 \frac{\partial}{\partial \ell^\mu_{[4]_i}} \frac{v^\mu_{[4]}}{\text{Denominator}}.$$

Conversion Back to Standard Integrals

Want to trade μ_i inside integrand for ϵ.

There are basically two techniques to be combined:

- Gram determinants $[G(\{p_i\},\{q_i\})] = \det_{i,j}(p_i \cdot q_j)$, do Feynman parametrization

$$P^{*,*}_{2,2}[(\mu_1^2)^2] = -\frac{\epsilon(1-\epsilon)}{(3-2\epsilon)(1-2\epsilon)G^2_{124}} P^{*,*}_{2,2}[G^2(\ell_1,1,2,4)].$$

- Standard (d-dimensional) IBPs with μ_i factors

$$0 = \int d^d\ell_1 d^d\ell_2 \frac{\partial}{\partial \ell^\mu_{i[d]}} \frac{\{1,\mu_1^2,.....\}(\text{irreducible})^j v^\mu_{[4]}}{\text{Denominator}}$$

Conclusions

- Alternative dimensional regularization schemes are available for higher order computations in perturbative gauge theories. However there is not a wide use of them. It is needed to provide more practical examples to show the efficiency of those schemes.
- The four dimensional formulation (FDF) of the four dimensional helicity scheme (FDH) is a proposal to get the cut constructible and the rational part of one-loop amplitudes by just four dimensional cuts.
- FDF is efficient to find contributions of evanescent operators in perturbative computations.
- The foundations for d-dimensional unitarity within FDF at two loops have been discussed.