A Study of the Top Mass Determination Using New NLO+PS generators

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Work done in collaboration with
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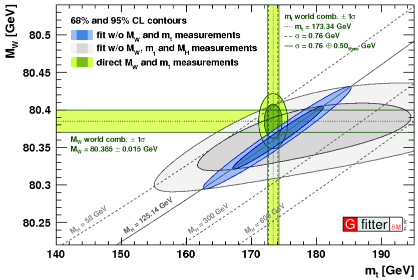
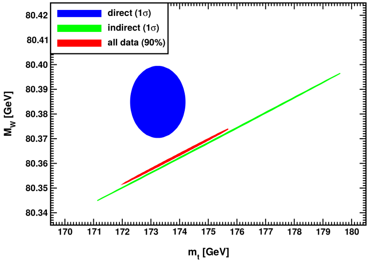
Outline

- Top, precision physics, vacuum stability
- Current measurements
- Theoretical issues on the top mass measurements: "which mass"; Pole mass and $\overline{\mathrm{MS}}$ mass
- How to determine the error
- New generators
- Error study using the old and new generators
- dependence upon the shower generator, i.e. Herwig7 vs. Pythia8
- A list of (unsuccessfull) attempts to resolve the issue.
- Conclusions
- More on the renormalon issue.

Top and precision physics



From PDG:



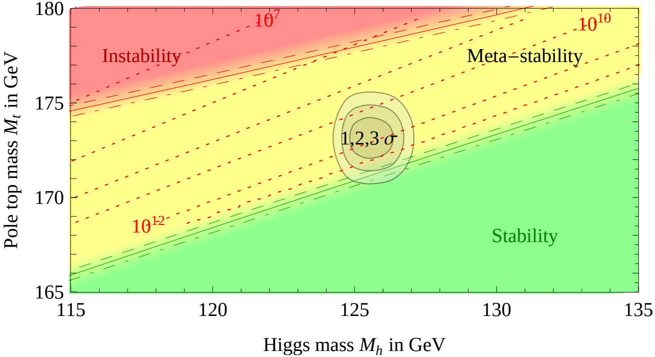
$$\Delta G_\mu / G_\mu = 5 \cdot 10^{-7}; \quad \Delta M_Z / M_Z = 2 \cdot 10^{-5};$$

$$\Delta\alpha(M_Z)/\alpha(M_Z) = \begin{cases} 1 \cdot 10^{-4} (\text{Davier et al.; PDG}) \\ 3.3 \cdot 10^{-4} (\text{Burkhardt, Pietrzyk}) \end{cases}$$

M_W can be predicted from the above with high precision, provided M_H and M_T (entering radiative corrections) are also known (and depending on how aggressive is the error on $\alpha(M_Z)$).

Top and vacuum stability

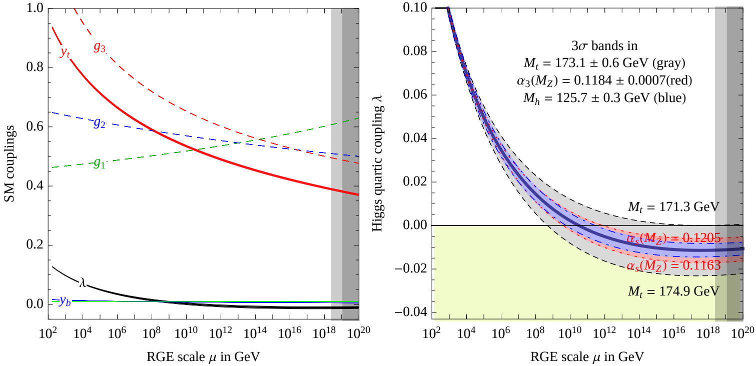
Degrassi et al. 2012



With current value of M_t and M_H the vacuum is metastable.
No indication of new physics up to the Plank scale from this.

Top and vacuum stability

Degrassi et al. 2012

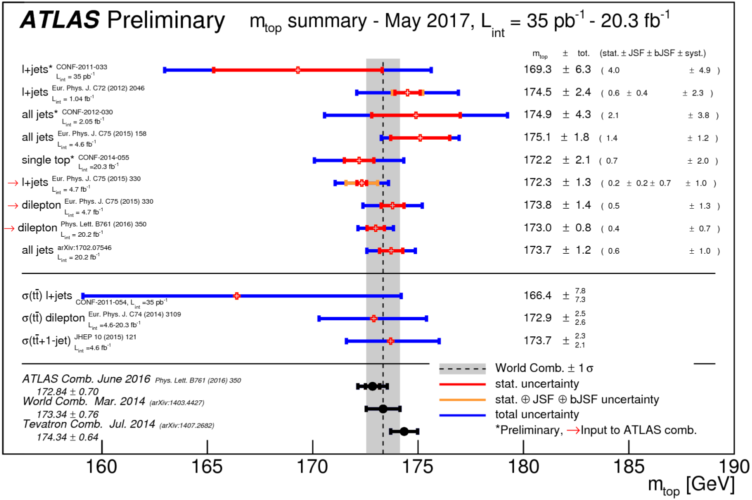


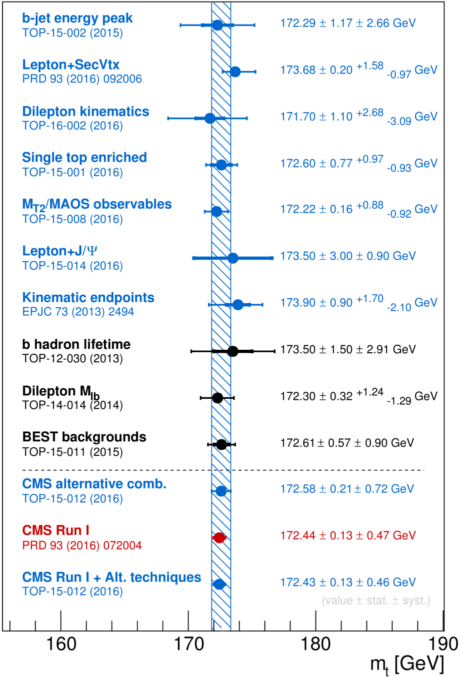
The quartic coupling λ_H becomes tiny at very high field values, and may turn negative, leading to vacuum instability.
M_t as low as 171 GeV leads to $\lambda_H \to 0$ at the Plank scale.

Top mass

- Top mass: fundamental parameter of the Standard Model.
- Ideal measurement: $t\bar{t}$ production at threshold at e^+e^-.
- LHC has the opportunity to measure it.

Lots of methods:





Several methods explored by CMS (see PAS TOP-15-012).

Notice: they do not increase precision with respect to PRD 93 (2016) 072004:

"The top quark mass is measured using the lepton+jets, all-jets and dilepton decay channels, giving values of

$172.35 \pm 0.16(\mathrm{st}) \pm 0.48(\mathrm{sy})$,

$172.32 \pm 0.25(\mathrm{st}) \pm 0.59(\mathrm{sy})$,

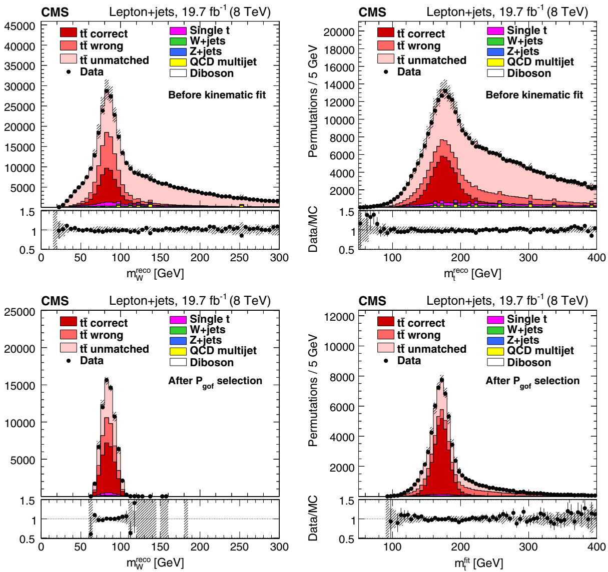
$172.82 \pm 0.19(\mathrm{st}) \pm 1.22(\mathrm{sy})$

GeV respectively.

Amazingly consistent determinations with different methods.
Most precise technique:

- Semileptonic decays: lepton + missing E_t + 4 jets, 2 b-tagged jets.
- Assuming on-shell W the neutrino kinematics can be further constrained (up to a two-fold ambiguity). Remaining two-fold ambiguity on b-jets assignment.
- Assuming on-shell W the jet energy scale can be fitted together with m_t.

RESOLUTION (Semileptonic, CMS)



Top resolution:

$\approx \pm 15$ GeV

Top mass at hadron colliders

Generator $m_{\rm top}$ parameter fitted to an experimentaly defined $m_{\rm top}^{\rm reco}$, essentially made up of a W and a b-jet.

1. The reconstructed mass must be closely related to the pole mass. There is an intrinsic uncertainty in relating the top pole mass to the top $\overline{\rm MS}$ mass, due to infrared renormalons, usually quoted to be few hundred MeV.
2. Doubts on the relation of this mass parameter, the so called "Monte Carlo" mass, to a theoretically well-defined mass

First objection: Mass renormalon

The relation of the Pole Mass to short distance parameters of the Standard Model Lagrangian (i.e. the $\overline{\mathrm{MS}}$ mass) is affected by an irreducible error of the order of typical hadronic scales.
Some authors have quoted an ambiguity of 1 GeV (Hoang, Dec. 11 2014).
Recent calculations give much smaller results:

- Beneke, Marquard, Steinhauser, P.N. 2016, v2, 9 Jun 2017: 110 MeV
- Hoang, Lepenik, Preisser, 26 Jun 2017: 250 MeV

No need to worry about it now.

Second objection: which mass

- The generator has a given accuracy: LO, NLO, etc. We should rephrase the problem: rather than "which mass" we should ask what is the theoretical error in the relation of the "theoretical" mass to the measured (mass sensitive) distribution.
- Certain features of $t\bar{t}$ events are experimentally measured. Should be use to constrain generator parameters.
- Theorists have tried to propose "golden" observables, for which such errors are small.

“golden” observables

- Butenschoen,Dehnadi,Hoang,Mateu,Preisser,Stewart,2016 Use boosted top jet mass + SCET.
- Agashe,Franceschini,Kim,Schulze,2016: peak of b-jet energy insensitive to production dynamics.
- Kawabata,Shimizu,Sumino,Yokoya,2014: shape of lepton spectrum. Insensitive to production dynamics and reduced sensitivity to strong interaction effects.
- Frixione, Mitov Use only lepton observables.
- Alioli, Fernandez, Fuster, Irles, Moch, Uwer, Vos ,2013; Bayu etal: M_t from $t\bar{t}j$ kinematics.

But how do we determine the error?

Experimentalists claim that several characteristics of the $t\bar{t}$ production process are actually measured, which should free us from generator dependence. However, the extracted mass is unavoidably a parameter in the generator.

We can formulate a general strategy for the determination of the theoretical errors as follows:

- Compute the observable in question using the most precise generators available.
- Study the theoretical errors in the traditional way: scale, coupling pdf uncertainties, comparison of generators that have formally the same accuracy for the observable at hand, non-perturbative parameters in MC's, alternative MC tunings.
- Restrict parameter variations requiring consistency with data.

Two simple examples:

- The b-jet energy is increased by UE activity captured in the jet cone.
- Radiation from b-quarks causes a reduction of the jet energy for out of cone losses.

Generators that differ in these two items will have different relations between the reconstructed top mass and the generator mass parameter. But the UE activity can be measured away from the jets; so either tune the generators to reproduce it (restricting the parameter variations with data), or subtract them directy from the jet energy (reducing sensitivity to the generator parameters).

Similarly, the generators can be tuned to reproduce the features of the b jet, or the mass measurement can be repeated with different r parameters, and the generators can be tuned to reproduce the observed r dependence of the reconstructed top mass.

Available Generators

- Modern generators for $t\bar{t}$ production have become available in recent times:
  - MC@NLO Frixione,Webber,P.N. and POWHEG Frixione,Ridolfi,P.N. hvq traditional NLO+PS $t\bar{t}$ generators. Do not include either exact spin correlations in decays or radiative corrections in decays. Routinely used by LHC experiments.
  - ttb_NLO_dec Campbell,Ellis,Re,P.N.. Includes exact spin correlations and NLO corrections in decay. Off shell effects included approximately (in such a way to be LO exact).
  - b_bbar_4l Ježo,Lindert,Nason,Oleari,Pozzorini,P.N. 2016 Includes exact NLO matrix element for $pp \to l\bar{\nu}_l \bar{\ell}\nu_\ell b\bar{b}$. It uses a recently introduced method for dealing with (coloured) narrow resonance in POWHEG.

Our task

We (Ferrario-Ravasio,Ježo,Oleari,P.N.) are tackling the following tasks:

- compare three NLO+PS generators:
  hvq, $t\bar{t}$_dec, $b\bar{b}$4l.
- studied the effect of scale variations in the $t\bar{t}$_dec and $b\bar{b}$4l generators.
- studied the α_s sensitivity of the results in the $b\bar{b}$4l generator.
- studied the PDF error in the $b\bar{b}$4l generators.
- performed an initial study on shower and hadronization uncertainties by comparing two shower generators: Pythia8 and Herwig7.

(As of now) most disturbing differences found in the last item.
This talk will focus upon Pythia8 and Herwig7 comparison.

Our task

- We focus upon the $pp \to l\bar{\nu}_l \bar{\ell}\nu_\ell b\bar{b}$ process. This is OK for lepton observables, but also for the b-jet energy peak.
  If we assume that the W can be fully reconstructed, our results will also imply a lower bound on the error in semileptonic and fully hadronic $t\bar{t}$ events, which is our main goal.
- Our most studied mass sensitive observable is the mass of the Wj_b system with matching signs.
- We look for parameter/setup variations that can lead to a displacement of the peak in m_{Wj_b} (this leads to an "irreducible" theoretical error on the top mass extraction).
- We also extract the mass after smearing the peak with a Gaussian, with half width equal to 15 GeV. This leads to an error that is related to the experimental resolution on our observable.
- "Irreducible errors" can actually be reduced. Some parameter/setup variations may be constrainable by data.

General approach

Assuming we have an observable O sensitive to the top mass, we will have in general

$$O = O_{\rm c} + B(m_t - m_{t,{\rm c}}) + \mathcal{O}((m_t - m_{t,{\rm c}})^2)$$

where $m_{t,{\rm c}} = 172.5$ GeV is our central value for the top mass. $O_{\rm c}$ and B differ for different generator setup. Given an experimental result for O, the extracted mass value is

$$m_t = m_{t,{\rm c}} + (O_{\rm exp} - O_{\rm c})/B$$

By changing the generator setup $O_{\rm c}, B \to O'_{\rm c}, B'$:

$$m_t - m'_t = -\frac{O_{\rm c} - O'_{\rm c}}{B} - (O_{\rm exp} - O'_{\rm c})(B - B')/(BB') \approx -\frac{O_{\rm c} - O'_{\rm c}}{B}.$$

Thus:

- Compute the B coefficient using a single setup for the generator.
- Compute the O_{c} coefficient (i.e. the value of the observable for $m_t = m_{t,\mathrm{c}}$) for all different setup we want to explore.
- Extract the difference in the extracted m_t between different setups, according to the equation

$$\Delta m_t = -\frac{\Delta O_{\mathrm{c}}}{B}.$$

ALL VERY PRELIMINARY!!!

NLO+PS generators

- `hvq`: (Frixione,Nason,Ridolfi, 2007), the first POWHEG implementation of $t\bar{t}$ production.
  NLO corrections only in production. Events with on-shell t and $\bar{t}$ are produced, and then "deformed" into off-shell events with decays, with a probability proportional to the corresponding tree level matrix element with off-shell effects and decays.
  Radiation in decays is only generated by the shower.
- `tt̄_dec`: (Campbell etal, 2014) Full spin correlations, exact NLO corrections in production and decay in the zero width approximation. Off shell effects implemented via a reweighting method, such that the LO cross section includes exactly all tree level off-shell effects.
- `bb̄4l`:(Ježo etal, 2016) Full NLO with off shell effects for $pp \to b\bar{b}e^+\nu_e\mu^-\bar{\nu}_\mu$,

Invariant mass of top decay products

m_{W-bj}

m_{W-bj}

We take m_{W-bj} as a proxy for all top-mass sensitive observables that rely upon the mass of the decay products.
Experimental effects are simply represented as a smearing of this distribution.
Here we will show results with no smearing, and with a Gaussian smearing with $\sigma = 15\,\mathrm{GeV}$.
We look for:

- Effects that displace the peak. These lead to an intrinsic error on the extraction of the mass.
- Effects that affect the shape of the peak in a wide region. These will affect the mass determination if the experimental smearing is included.

m_{W-bj}

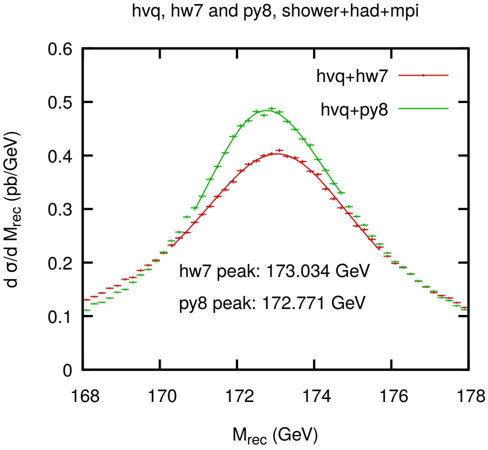
$W - bj$ is defined in the following way:

- Jets are defined using the anti-k_T algorithm with $R = 0.5$. The $b/\bar{b}$ jet is defined as the jet containing the hardest $b/\bar{b}$.
- $W^{\pm}$ is defined as the hardest $l^{\pm}$ paired with the hardest matching neutrino.
- The $W - bj$ system is obtained by matching a $W^{+/-}$ with a $b/\bar{b}$ jet (i.e. we assume we know the sign of the b).

In this case, $B \approx 1$!

Example: POWHEG-hvq, Pythia8 - Herwig7 comparison

The POWHEG-hvq generator interfaced to Pythia8 is widely used now by the experimental collaborations. We consider the differences we get when switching to Herwig7.

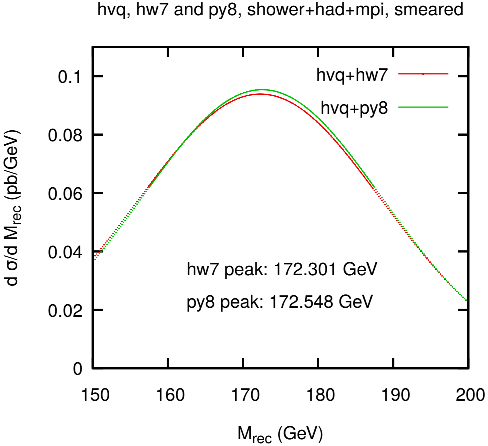


Peak position obtained by fitting peak distribution with a skewed Lorentian:

hw7 − py8: 263 MeV

Example: POWHEG-hvq, Pythia8 - Herwig7 comparison

Same, accounting for experimental errors by smearing the peak with a gaussian distribution with a width of 15 GeV.



$$f_{sm}(x) \propto \int \mathrm{d}y f(y) \times \exp\left[-\frac{(y-x)^2}{2\sigma^2}\right],$$

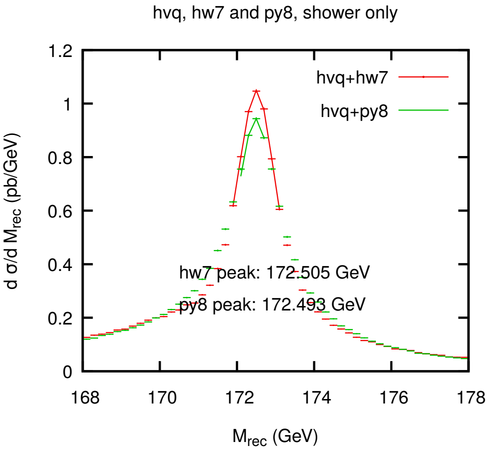
$\sigma = 15$ GeV,

Peak from a fit with a 4th degree polynomial.

hw7 − py8: −247 MeV

Example: POWHEG-hvq, Pythia8 - Herwig7 comparison

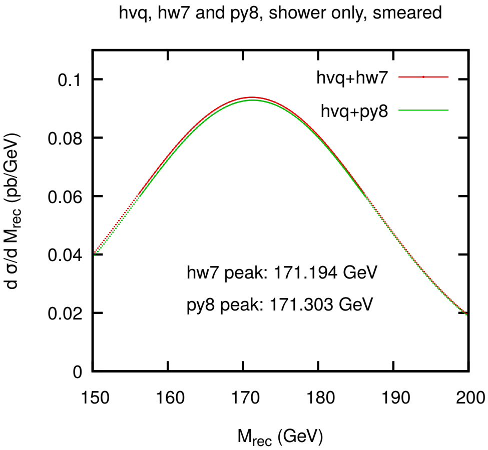
Same stuff, no hadronization and mpi;



hw7 − py8: 12 MeV

Example: POWHEG-hvq, Pythia8 - Herwig7 comparison

No hadronization and mpi, with smearing;



Smearing, $\sigma = 15$ GeV,

hw7 − py8: −109 MeV

Example: POWHEG-hvq, Pythia8 - Herwig7 comparison

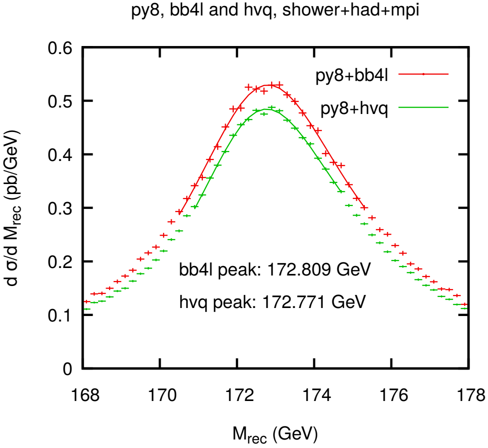
Summary of POWHEG-hvq hw7 - py8 comparison:

M_{rec} (GeV)						
	Full			Shower only		
	`Herwig7`	`Pythia8`	Δ	`Herwig7`	`Pythia8`	Δ
$\sigma = 0$	173.034	172.771	0.263	172.505	172.493	0.012
$\sigma = 15$	172.301	172.548	-0.247	171.194	171.303	-0.109

Sizeable difference, but well below the current $\pm 0.6 GeV$ experimental results.
The different shape around the peak region is worrysome.
Hadronization seems to be responsible for the discrepancy.

Pythia8, POWHEG-hvq - POWHEG-bb4l comparison

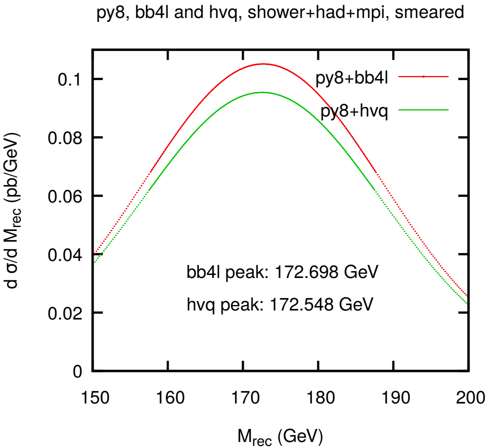
Now see what happens if we go from the old hvq to the new bb4l NLO+PS generator, using Pythia8 for the shower.



bb4l − hvq: 38 MeV

Pythia8, POWHEG-hvq - POWHEG-bb4l comparison

Same, accounting for experimental errors by smearing the peak with a gaussian distribution with a width of 15 GeV.

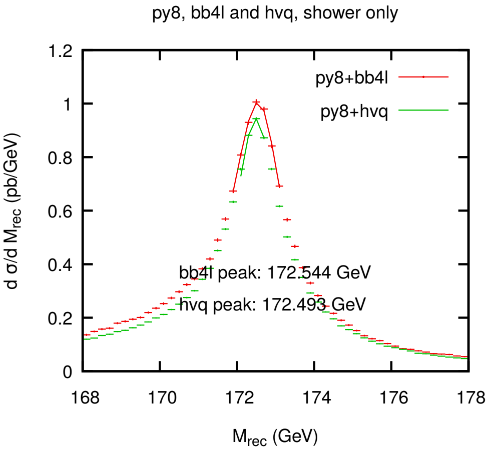


Smearing, $\sigma = 15$ GeV,

bb4l $-$ hvq: 150 MeV

Pythia8, POWHEG-hvq - POWHEG-bb4l comparison

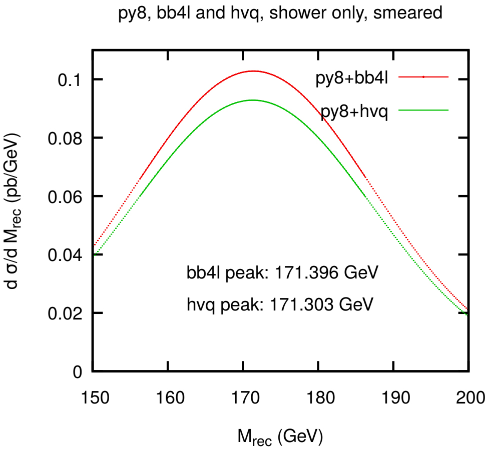
Same stuff, no hadronization and mpi;



bb4l − hvq: 51 MeV

Pythia8, POWHEG-hvq - POWHEG-bb4l comparison

No hadronization and mpi, with smearing;



Smearing, $\sigma = 15$ GeV,

bb4l $-$ hvq: 93 MeV

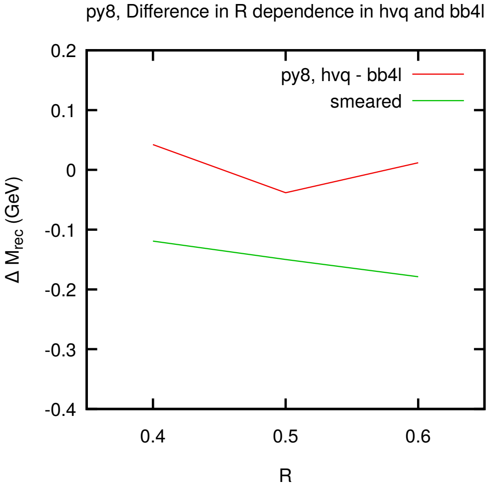
Pythia8, POWHEG-hvq - POWHEG-bb4l comparison

Summary of POWHEG-hvq hw7 - POWHEG-bb4l (with Pythia8) comparison:

$M_{\rm rec}$ (GeV)						
	Full			Shower only		
	bb4l	hvq	Δ	bb4l	hvq	Δ
$\sigma = 0$	172.809	172.771	0.038	172.544	172.493	0.051
$\sigma = 15$	172.698	172.548	0.150	171.396	171.303	0.093

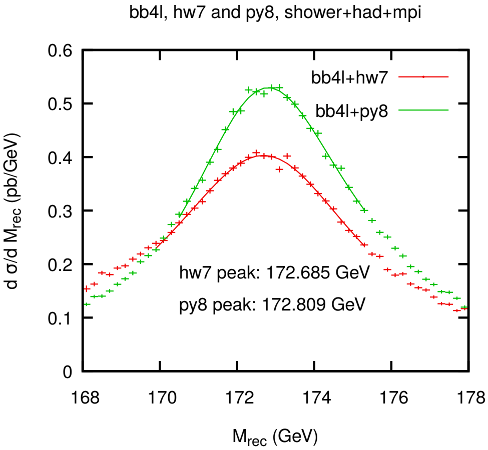
Very modest difference! Is it stable under change of the R parameter?

Pythia8, POWHEG-hvq - POWHEG-bb4l comparison



Fairly stable! From this, we are tempted to conclude that switching to the new generator makes no difference ...
But this is not the case: look at Herwig ...

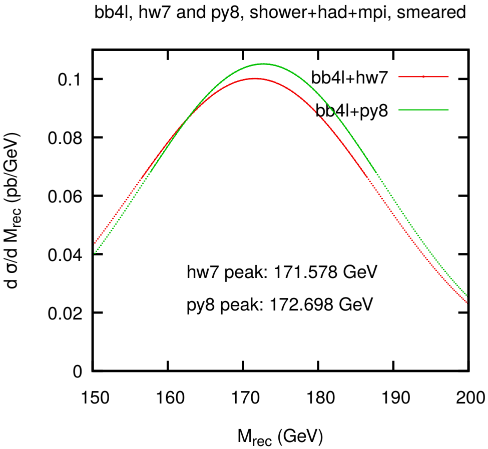
POWHEG-bb4l, Herwig7 - Pythia8 comparison



hw7 − py8: −0.124 MeV

POWHEG-bb4l, Herwig7 - Pythia8 comparison

Same, accounting for experimental errors by smearing the peak with a gaussian distribution with a width of 15 GeV.

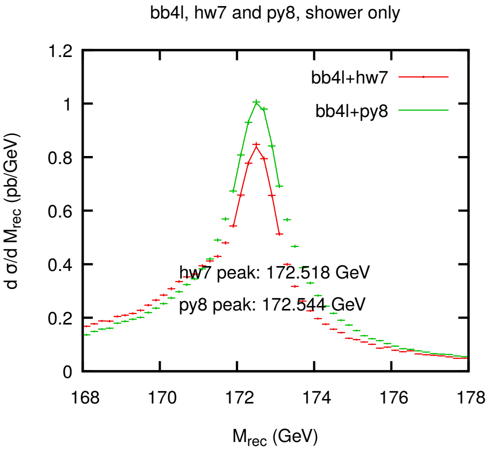


Smearing, $\sigma = 15$ GeV,

hw7 − py8: −1.12 MeV

POWHEG-bb4l, Herwig7 - Pythia8 comparison

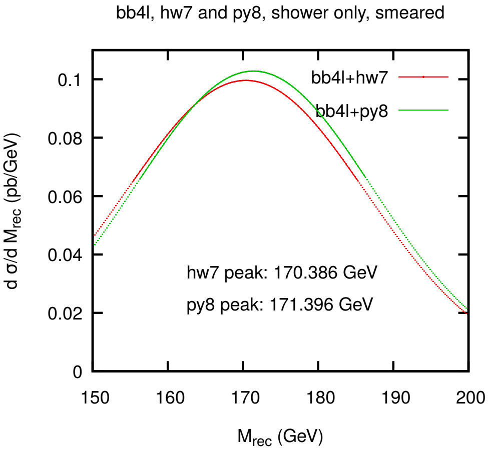
Same stuff, no hadronization and mpi;



hw7 − py8: −0.026 MeV

POWHEG-bb4l, Herwig7 - Pythia8 comparison

No hadronization and mpi, with smearing;



Smearing, $\sigma = 15$ GeV,

hw7 − py8: -1.01 MeV

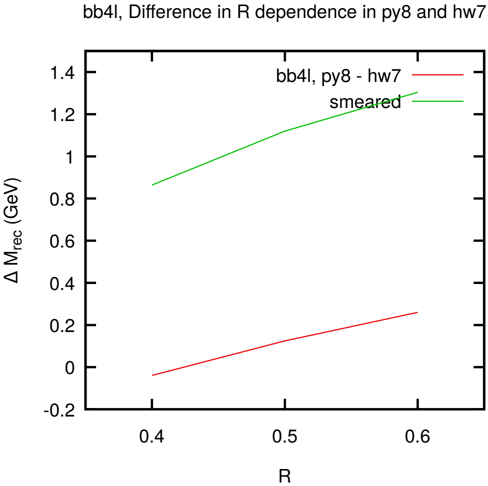
POWHEG-bb4l, Herwig7 - Pythia8 comparison

Summary of POWHEG-bb4l hw7 - py8 comparison:

$M_{\rm rec}$ (GeV)						
	Full			Shower only		
	hw7	py8	Δ	hw7	py8	Δ
$\sigma = 0$	172.685	172.809	0.124	172.518	172.544	0.026
$\sigma = 15$	171.578	172.698	1.12	170.386	171.396	1.01

Modest differences in the unsmeared case; but with smearing, we see very large differences.

POWHEG-bb4l, Herwig7 - Pythia8 comparison



Different slope in their R dependence: at least one of them needs tuning to fit it. This may reduce the differences in extracted mass.

Differences mainly caused by Shower/Matching effects.

Jet energy peak

Agashe,Franceschini,Kim,Schulze,2016

	$E_{b-\text{jet peak}}$ (GeV)	
	bb4l	hvq
hw7	68.88 ± 0.40	69.67 ± 0.26
py8	71.24 ± 0.40	70.77 ± 0.27
hw7, no had.	68.09 ± 0.45	68.30 ± 0.28
py8, no had	69.64 ± 0.44	69.04 ± 0.27

Here $B = 0.45$, so:

- bb4l, hw7 - py8: $\Delta m_t = 5$ GeV, (only shower: 3.4 GeV)
- hvq, hw7 - py8: $\Delta m_t = 2.4$ GeV (only shower: 0.74 GeV)

Frixione, Mitov, 2014

Deviations in top mass values:
comparison of bb4l and ttbNLOdec, both with Pythia8

	ΔM_{top} (GeV)		
	Mom 1	Mom 2	Mom 3
$p_t(l^+)$	-0.8 $\pm$ 0.4	-0.7 $\pm$ 0.3	-0.6 $\pm$ 0.5
$p_t(l^+l^-)$	1.1 $\pm$ 0.3	1.6 $\pm$ 0.2	2.6 $\pm$ 0.3
$m(l^+, l^-)$	-0.8 $\pm$ 0.6	-0.6 $\pm$ 0.4	-0.1 $\pm$ 0.7
$E(l^+l^-)$	-0.3 $\pm$ 0.5	-0.4 $\pm$ 0.4	-0.3 $\pm$ 0.5
$p_t(l^+) + p_t(l^-)$	-0.4 $\pm$ 0.4	-0.5 $\pm$ 0.3	-0.9 $\pm$ 0.4

Generally good agreement between the two;
the only (marginal) exception of $p_t(l^+l^-)$.

Lepton Observables

Deviations in top mass values:
comparison of bb4l and hvq, both with Pythia8

	ΔM_{top} (GeV)		
	Mom 1	Mom 2	Mom 3
$p_t(l^+)$	-0.1 $\pm$ 0.4	0.2 $\pm$ 0.3	0.6 $\pm$ 0.5
$p_t(l^+l^-)$	2.4 $\pm$ 0.3	2.8 $\pm$ 0.2	3.8 $\pm$ 0.3
$m(l^+,l^-)$	-1.8 $\pm$ 0.6	-1.2 $\pm$ 0.4	-0.4 $\pm$ 0.6
$E(l^+l^-)$	0.2 $\pm$ 0.5	0.4 $\pm$ 0.4	0.9 $\pm$ 0.5
$p_t(l^+)+p_t(l^-)$	-0.1 $\pm$ 0.4	-0.1 $\pm$ 0.3	-0.2 $\pm$ 0.4

Good agreement for 1st, 4th and 5th observable. These are the observables that were argued to be less sensitive to shower and spin correlation effects by Friione and Mitov.

Lepton Observables

Deviations in top mass values:
comparison of Pythia8 and Herwig7, both with bb4l

	ΔM_{top} (GeV)		
	Mom 1	Mom 2	Mom 3
$p_t(l^+)$	3.4 ± 0.4	4.0 ± 0.2	4.9 ± 0.4
$p_t(l^+l^-)$	4.6 ± 0.3	5.3 ± 0.2	6.5 ± 0.2
$m(l^+, l^-)$	0.7 ± 0.5	1.2 ± 0.3	1.8 ± 0.5
$E(l^+l^-)$	2.8 ± 0.4	3.0 ± 0.3	3.3 ± 0.4
$p_t(l^+) + p_t(l^-)$	3.2 ± 0.4	3.7 ± 0.2	4.2 ± 0.3

Bad agreement in general, also for 1st, 4th and 5th observable.

Lepton Observables

Deviations in top mass values:
comparison of Pythia8 and Herwig7, both with hvq

	ΔM_{top} (GeV)		
	Mom 1	Mom 2	Mom 3
$p_t(l^+)$	2.0 $\pm$ 0.4	2.6 $\pm$ 0.3	3.5 $\pm$ 0.5
$p_t(l^+l^-)$	2.7 $\pm$ 0.3	3.3 $\pm$ 0.2	4.2 $\pm$ 0.3
$m(l^+, l^-)$	0.6 $\pm$ 0.6	1.2 $\pm$ 0.4	2.0 $\pm$ 0.7
$E(l^+l^-)$	1.4 $\pm$ 0.5	1.6 $\pm$ 0.4	1.8 $\pm$ 0.5
$p_t(l^+) + p_t(l^-)$	2.0 $\pm$ 0.4	2.5 $\pm$ 0.3	3.2 $\pm$ 0.4

Still bad, although better than bb4l.

Lepton Observables

Deviations in top mass values:
comparison of bb4l and hvq, both with hw7

	ΔM_{top} (GeV)		
	Mom 1	Mom 2	Mom 3
$p_t(l^+)$	-1.5 $\pm$ 0.4	-1.2 $\pm$ 0.3	-0.8 $\pm$ 0.4
$p_t(l^+l^-)$	0.5 $\pm$ 0.3	0.8 $\pm$ 0.2	1.4 $\pm$ 0.2
$m(l^+, l^-)$	-1.9 $\pm$ 0.5	-1.2 $\pm$ 0.4	-0.2 $\pm$ 0.5
$E(l^+l^-)$	-1.2 $\pm$ 0.4	-1.1 $\pm$ 0.3	-0.7 $\pm$ 0.4
$p_t(l^+) + p_t(l^-)$	-1.3 $\pm$ 0.4	-1.3 $\pm$ 0.2	-1.2 $\pm$ 0.3

Still bad.

Checks and attempts to solve the issue

- B radiation in POWHEG: new impementation of B radiation Buonocore, Tramontano, P.N., from Buonocore master thesis. Irrelevant differences observed.
- 3 alternative (and orthogonal) implementation of NLO+PS shower matching in Herwig7 (also with the help of the authors). 2.5 alternative implementation of the interface with Pythia8. Found equivalent results.
- Herwig7 implements an angular ordered shower. There are issues related to the need of truncated-vetoed shower in the interface with POWHEG. There are, in Herwig7, variants in the implementation of the shower initial conditions that are equivalent to the inclusion of truncated shower. We have tried them, and found no important differences.

Conclusions

- Useful theoretical work can be done studying of oversimplified observables with state of the art generators.
- This work does not imply that the experimental results are flawed. It must be carried out to expose possible sources of error that might have been overlooked.
- Further work should be carried out to see if there are oversimplified observables that can mimic experimental constraints on the event structure that should be satisfied by generators.
- Surprising results for "golden" observables: also lepton observables influenced by the shower ...

Renormalon Issues

$$m_P = m + N\alpha_s \sum_{n=0}^{\infty} c_n(\mu, m)\alpha_s^n,$$

where m_p is the pole mass, m is the $\overline{\mathrm{MS}}$ mass, and $\alpha_s = \alpha_s(\mu)$. The asymptotic behaviour of the expansion is (in leading order)

$$\begin{aligned} \alpha_s^n c_n &\stackrel{n\to\infty}{\longrightarrow} \mu t_a^{(n)}, \\ t_a^{(n)} &\equiv (2b_0\alpha_s)^n n! \approx \sqrt{2\pi} e^{(n+1/2)\log n - n + n\log(2b_0\alpha_s)}, \end{aligned} \quad (1)$$

Minimum at $n_{\mathrm{m}} \approx 1/(2b_0\alpha_s)$. Using $\alpha_s = 1/(b_0 \log[\mu^2/\Lambda^2])$:

$$t_a^{(n_{\mathrm{m}})} = \sqrt{2\pi n_{\mathrm{m}}}\, e^{-n_{\mathrm{m}}} = \sqrt{2\pi n_{\mathrm{m}}}\, \frac{\Lambda}{\mu}$$

The ambiguity of the asymptotic formula should be μ independent. But the minimal term goes like

$$N\mu\alpha_s t_a^{(n_{\mathrm{m}})} = \alpha_s \sqrt{2\pi n_{\mathrm{m}}}\, \Lambda \quad (2)$$

Needs an extra factor of $\sqrt{n_{\mathrm{m}}}$ to be μ independent.

Around the minimum

$$t_a^{(n)} \approx t_a^{(n_{\rm m})} \left(1 + \frac{1}{2n}\left(n - n_{\rm m}\right)^2\right) \tag{3}$$

We can supplement the minimal term by a factor quantifying how many terms are close to the minimum

$$\frac{1}{2n}\left(n - n_{\rm m}\right)^2 < p \Longrightarrow \Delta_n = \sqrt{2pn_{\rm m}}$$

Δ_n times the minimal term is in fact μ independent, and equal to

$$N\frac{\sqrt{4\pi p}}{2b_0}\Lambda$$

Borel sum approach

We transform the series in the inverse Borel transform of a convergent series. Order by order in α_s we have the identity

$$N\alpha_s \sum_{n=0}^{a} c_n(\mu, m)\alpha_s^n = N \int_0^\infty \mathrm{d}r\, e^{-\frac{t}{\alpha_s}} \sum_{n=0}^{a} c_n(\mu, m)\, \frac{r^n}{n!}.$$

Plugging in the asymptotic value for the coefficients:

$$N\mu \int_0^\infty \mathrm{d}r\, e^{-\frac{r}{\alpha_s}} \sum_{n=0}^{a} (2b_0 r)^n = N\mu \int_0^\infty \mathrm{d}r\, \frac{e^{-\frac{r}{\alpha_s}}}{1 - 2b_0 r}$$

The singularity in $r = 1/(2b_0)$ is due to the renormalon. One can define the sum as the principal value for the integral, and the ambiguity as the imaginary part of the integral divided by π (Beneke, 1999)

$$N\mu \frac{1}{2b_0} e^{-\frac{1}{2b_0\alpha_s}} = \frac{N}{2b_0}\Lambda$$

Comparison of the two methods

The minimal term method and the Borel method agree in the estimate of the error provided that

- We extrapolate n to non integer values.
- $p = 1/(4\pi) = 0.08$.

Beneke etal vs. Hoang etal

- Beneke, Marquard, Steinhauser, P.N. 2016, v2, 9 Jun 2017
  Uses the Im/Pi prescription
- Hoang, Lepenik, Preisser, 26 Jun 2017:
  - Takes half the sum of all terms that are less than the minimal one multiplied by a factor f, where "f is a number larger but close to unity". Chooses $f = 5/4$,
  - Does not extrapolate to non integer n.
  - Does further scale variation on the terms they sum

It is clear that the "range" factor in H is larger than the one in B by

$$\sqrt{\frac{f-1}{\frac{1}{4\pi}}} = \sqrt{\frac{0.25}{0.080}} = 1.77\,.$$

Further enhacement arises from the scale variation procedure.

Beneke etal vs. Hoang etal

There is large arbitrarity in the determination of the error:

- In the Beneke etal calculation: the factor in front of the imaginary part of the Borel integral
- In the Hoang etal procedure: the choice of f.

As of B, the Im/Pi prescription has proven reliable in several phenomenological contexts where the Borel ambiguity could be related to some measurable quantities (Beneke 1999).