Resurgence and Nonperturbative Physics

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New Frontiers in Theoretical Physics, Cortona, May 28, 2014

GD & M. Ünsal, 1210.2423, 1210.3646, 1306.4405, 1401.5202

also with: G. Başar, A. Cherman, D. Dorigoni, R. Dabrowski: 1306.0921, 1308.0127, 1308.1108

Physical Motivation

- infrared renormalon puzzle in asymptotically free QFT
  (i) IR renormalons $\Rightarrow$ perturbation theory ill-defined
  (ii) $\mathcal{I}\bar{\mathcal{I}}$ interactions $\Rightarrow$ instanton-gas ill-defined
- non-perturbative physics without instantons

Bigger Picture

- non-perturbative definition of QCD in the continuum
- analytic continuation of path integrals
- dynamical and non-equilibrium physics from path integrals
- "exact" asymptotics in QFT and string theory: relation to localization in QFT

Mathematical Motivation

Resurgence: 'new' idea in mathematics (Écalle, 1980; Stokes, 1850)

- goal: explore implications for physics

<u>resurgence</u> = unification of perturbation theory and non-perturbative physics

- perturbation theory generally $\Rightarrow$ divergent series
- series expansion $\longrightarrow$ *trans-series* expansion
- trans-series well-defined under analytic continuation
- perturbation theory and non-perturbative physics are intricately entwined
- philosophical shift:

  view semiclassical expansions as potentially exact

- applications: ODEs, PDEs, QM, Matrix Models, QFT, String Theory, ...

Resurgent Trans-Series

- trans-series expansion:

$$f(g^2) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{k-1} a_{n,k,l}\, g^{2n} \left[\exp\left(-\frac{S}{g^2}\right)\right]^k \left[\log\left(-\frac{1}{g^2}\right)\right]^l$$

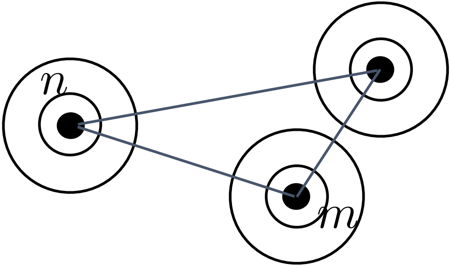
- J. Écalle (1980): set of functions with these trans-monomial elements is closed under:

(Borel transform)+(analytic continuation)+(Laplace transform)

- "any reasonable function" has a trans-series expansion
- trans-series expansion coefficients are highly correlated
- "resurgence" encodes inter-relations

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



recap: rough basics of Borel summation

(i) divergent, alternating:

$$\sum_{n=0}^{\infty}(-1)^n\, n!\, g^{2n} = \int_0^{\infty} dt\, e^{-t}\, \frac{1}{1+g^2\, t}$$

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(ii) divergent, non-alternating:

$$\sum_{n=0}^{\infty} n! \, g^{2n} = \int_0^{\infty} dt \, e^{-t} \, \frac{1}{1-g^2 \, t}$$

$\Rightarrow$ ambiguous imaginary non-pert. term: $\pm \frac{i\,\pi}{g^2} \, e^{-1/g^2}$

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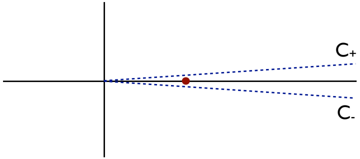
avoid singularities on $\mathbb{R}^+$: lateral Borel sums:



$\theta = 0^\pm \longrightarrow$ non-perturbative ambiguity: $\pm\mathrm{Im}[\mathcal{S}_0 f(g^2)]$

challenge: use physical input to resolve ambiguity

Borel summation in practice

direct quantitative correspondence between:

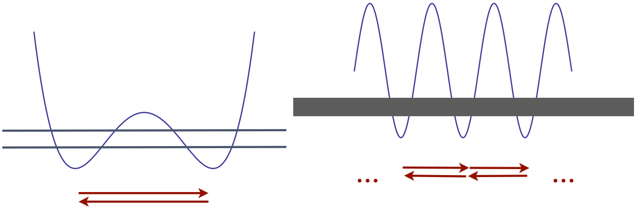
rate of growth $\leftrightarrow$ Borel poles $\leftrightarrow$ non-perturbative exponent

non-alternating factorial growth: $c_n \sim b^n \, n!$

positive Borel singularity: $t_c = \dfrac{1}{b\, g^2}$

non-perturbative exponent: $\pm i \, \dfrac{\pi}{b\, g^2} \, \exp\left[-\left(\dfrac{1}{b\, g^2}\right)\right]$

Analogue of IR Renormalon Problem in QM



- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

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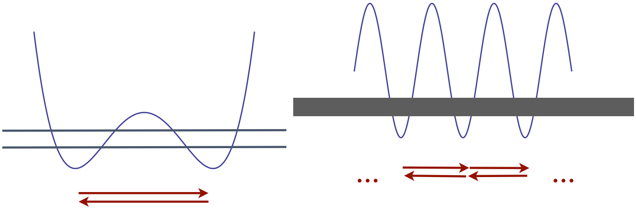


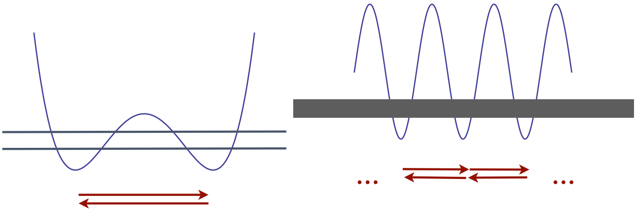
- degenerate vacua: double-well, Sine-Gordon, ...

splitting of levels: a real one-instanton effect: $\Delta E \sim e^{-\frac{S}{g^2}}$

surprise: pert. theory non-Borel summable: $c_n \sim \frac{n!}{(2S)^n}$

- stable systems
- ambiguous imaginary part
- $\pm i\, e^{-\frac{2S}{g^2}}$, a 2-instanton effect

Bogomolny/Zinn-Justin mechanism in QM



- degenerate vacua: double-well, Sine-Gordon, ...
  1. perturbation theory non-Borel summable: ill-defined/incomplete
  2. instanton gas picture ill-defined/incomplete: $\mathcal{I}$ and $\bar{\mathcal{I}}$ attract
- regularize both by analytic continuation of coupling

$\Rightarrow$ ambiguous, imaginary non-perturbative terms cancel !

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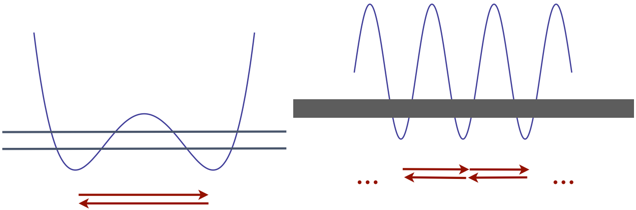

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"resurgence" $\Rightarrow$ cancellation to all orders

QFT: Renormalons

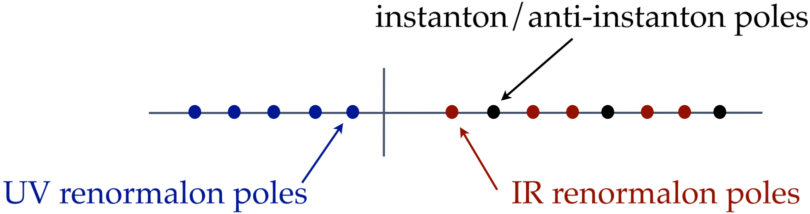
QM: divergence of perturbation theory due to factorial growth of number of Feynman diagrams

QFT: new physical effects occur, due to running of couplings with momentum

- faster source of divergence: "renormalons"
- both positive and negative Borel poles

perturbation theory: $\longrightarrow \quad \pm i\, e^{-\frac{2S}{\beta_0\, g^2}}$

instantons on $\mathbb{R}^2$ or $\mathbb{R}^4$: $\longrightarrow \quad \pm i\, e^{-\frac{2S}{g^2}}$



appears that BZJ cancellation cannot occur

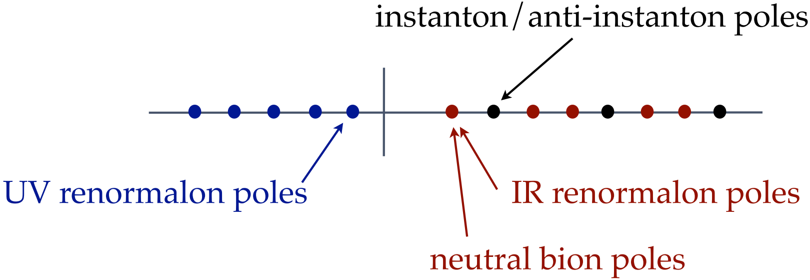
asymptotically free theories remain inconsistent

't Hooft, 1980; David, 1981

IR Renormalon Puzzle in Asymptotically Free QFT

resolution: there is another problem with the non-perturbative instanton gas analysis (Argyres, GD, Ünsal, 1206.1890 1210.2423)

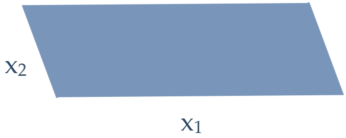
- scale modulus of instantons
- spatial compactification and principle of continuity

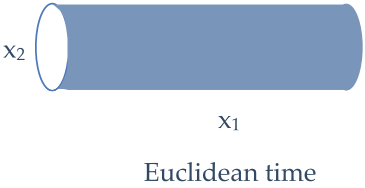


cancellation occurs !

$\mathbb{CP}^{N-1}$: regulate scale modulus problem with (spatial) compactification

$\mathbb{R}^2 \to S_L{}^1 \times \mathbb{R}^1$





instantons fractionalize   Kraan/van Baal; Lee/Yi; Bruckmann; Brendel et al, ...

temporal compactification: information about deconfined phase

spatial compactification: semi-classical small L regime continuously connected to large L:

principle of continuity

$\mathbb{CP}^{N-1}$ Model

$\mathbb{CP}^{N-1}$ model: 2d sigma model analogue of 4d Yang-Mills

- asymptotically free: $\beta_0 = N$ (independent of N_f)
- instantons, theta vacua, fermion zero modes, ...
- divergent perturbation theory (non-Borel summable)
- renormalons (both UV and IR)
- large-N analysis
- non-perturbative mass gap: $m_g = \Lambda = \mu\, e^{-4\pi/(g^2 N)}$
- couple to fermions, SUSY, ...
- analogue of center symmetry (GD, Ünsal, 1210.2423)

Fractionalized Instantons in $\mathbb{CP}^{N-1}$ on $S^1 \times \mathbb{R}^1$

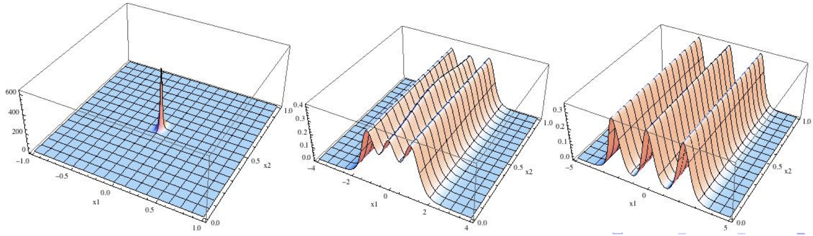
$\mathbb{Z}_N$ twisted instantons fractionalize Bruckmann, 2007; Brendel et al, 2009

- *spatial* compactification $\Rightarrow$ $\mathbb{Z}_N$ twist:

$$v_{\text{twisted}} = \begin{pmatrix} 1 \\ \left(\lambda_1 + \lambda_2\, e^{-\frac{2\pi}{L} z}\right)\, e^{\frac{2\pi}{L}\,\mu_2\, z} \end{pmatrix}$$

(twist in x_2)+(holomorphicity)$\Rightarrow$ fractionalization along x_1

$$\Rightarrow \qquad S_{\text{inst}} \longrightarrow \frac{S_{\text{inst}}}{N} = \frac{S_{\text{inst}}}{\beta_0}$$



Fractionalized Bions in $\mathbb{CP}^{N-1}$ GD/Ünsal, 1210.2423

bions: topological molecules of instantons/anti-instantons

- characterized by (extended) Cartan matrix (as in YM)
- "orientation" dependence of $\mathcal{I}\bar{\mathcal{I}}$ interaction:
- charged bions: $\hat{A}_{ij} < 0$; repulsive bosonic interaction

$$\mathcal{B}_{ij} = [\mathcal{K}_i \bar{\mathcal{K}}_j] \sim e^{-S_i(\varphi)-S_j(\varphi)} e^{i\theta(\alpha_i-\alpha_j)}$$

- neutral bions: $\hat{A}_{ii} > 0$; attractive bosonic interaction

$$\Re\mathcal{B}_{ii} = \Re[\mathcal{K}_i \bar{\mathcal{K}}_i] \sim e^{-2S_i(\varphi)}$$

- kink-anti-kink amplitude is two-fold ambiguous:

$$\left[\mathcal{K}_i \bar{\mathcal{K}}_i\right]_\pm = \left(\ln\left(\frac{g^2\,N}{8\pi}\right) - \gamma\right) \frac{16}{g^2\,N}\, e^{-\frac{8\pi}{g^2\,N}} \pm i\pi\, \frac{16}{g^2\,N}\, e^{-\frac{8\pi}{g^2\,N}}$$

Perturbation Theory in Twisted $\mathbb{CP}^{N-1}$

- small radius limit $\longrightarrow$ effective QM Hamiltonian

$$H^{\rm zero} = \frac{g^2}{2} P_\theta^2 + \frac{\xi^2}{2g^2} \sin^2\theta + \frac{g^2}{2\sin^2\theta} P_\phi^2 \quad , \qquad \xi = \frac{2\pi}{N}$$

- Born-Oppenheimer approximation: drop high ϕ-sector modes

effective Mathieu equation:

$$-\frac{1}{2}\psi'' + + \frac{\xi^2}{2g^2}\sin^2(g\theta)\psi = E\,\psi$$

- Stone-Reeve (Bender-Wu methods):

$$\mathcal{E}(g^2) \equiv E_0 \xi^{-1} = \sum_{n=0}^{\infty} a_n (g^2)^n, \quad a_n \sim -\frac{2}{\pi}\left(\frac{N}{8\pi}\right)^n n! \left(1 - \frac{5}{2n} + \dots\right)$$

- non-Borel summable!

- perturbative sector: lateral Borel-Écalle summation

$$B_{\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_{\pm}} dt\, B\mathcal{E}(t)\, e^{-t/g^2} = \mathrm{Re}\, B\mathcal{E}(g^2) \mp i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

BZJ cancellation in Twisted $\mathbb{CP}^{N-1}$ (GD, Ünsal, 1210.2423)

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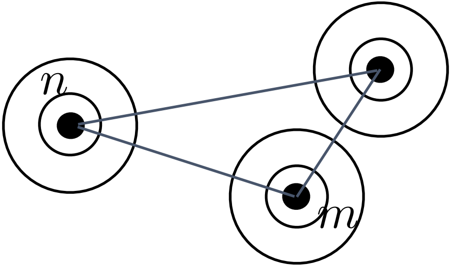
- non-perturbative sector: bion-bion amplitudes

$$\left[\mathcal{K}_i\bar{\mathcal{K}}_i\right]_\pm = \left(\ln\left(\frac{g^2\, N}{8\pi}\right) - \gamma\right)\frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}} \pm i\pi\, \frac{16}{g^2\, N}\, e^{-\frac{8\pi}{g^2\, N}}$$

exact cancellation !

application of resurgence to nontrivial QFT *resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities*

J. Écalle, 1980



The Bigger Picture

Q: should we expect resurgent behavior in QM and QFT?

• QM with degenerate vacua: trans-series arise naturally from uniform WKB, and perturbation theory generates everything: all multi-instanton effects are encoded in perturbation theory!

(GD, Ünsal, 1306.4405, 1401.5202)

Q: what is behind this resurgent structure?

• basic property of all-orders steepest descents integrals: could this extend to (path) functional integrals ?

• resurgence 'enforces' proper analytic continuation properties

All-Orders Steepest Descents: Darboux Theorem

- all-orders steepest descents for contour integrals (Berry/Howls 1991: *hyperasymptotics*)

$$I^{(n)}(k) = \int_{C_n} dz\, e^{-k\, f(z)} = \frac{1}{\sqrt{k}}\, e^{-k\, f_n}\, T^{(n)}(k)$$

- $T^{(n)}(k)$: beyond the Gaussian approximation
- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(k) \sim \sum_{r=0}^{\infty} \frac{T_r^{(n)}}{k^r}$$

All-Orders Steepest Descents: Darboux Theorem

- universal resurgent relation between different saddles:

$$T^{(n)}(k) = \frac{1}{2\pi\, i} \sum_m (-1)^{\gamma_{nm}} \int_0^\infty \frac{dv}{v} \frac{e^{-v}}{1 - v/(k\, F_{nm})} T^{(m)} \left(\frac{v}{F_{nm}}\right)$$

- exact resurgent relation between fluctuations about n^{th} saddle and about neighboring saddles m

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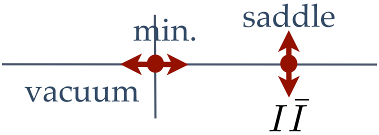
$$T_r^{(n)} = \frac{(r-1)!}{2\pi i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)} T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)} T_2^{(m)} + \dots \right.$$

- universal factorial divergence of fluctuations (Darboux)
- fluctuations about different saddles explicitly related !

$d = 0$ partition function for periodic potential $V(z) = \sin^2(z)$

$$I(k) = \int_0^{\pi} dz \, e^{-k \, \sin^2(z)}$$

two saddle points: $z_0 = 0$ and $z_1 = \frac{\pi}{2}$.



All-Orders Steepest Descents: Darboux Theorem

- large order behavior about saddle z_0:

$$
\begin{aligned}
T_r^{(0)} &= \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)} \\
&\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1-\frac{\frac{1}{4}}{(r-1)}+\frac{\frac{9}{32}}{(r-1)(r-2)}-\frac{\frac{75}{128}}{(r-1)(r-2)(r-3)}+
\end{aligned}
$$

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$$

- low order coefficients about saddle z_1:

$$
T^{(1)}(k) \sim i\sqrt{\pi}\left(1-\frac{1}{4}\cdot\frac{1}{k}+\frac{9}{32}\cdot\frac{1}{k^2}-\frac{75}{128}\cdot\frac{1}{k^3}+\dots\right)
$$

- fluctuations about the two saddles are explicitly related
- resurgence at work!

could something like this work for path integrals?

"functional Darboux theorem" ?

Resurgence in Path Integrals: "Functional Darboux Theorem"

- periodic potential: $V(x) = \frac{1}{g^2}\,\sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n!\left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\,e^{-2\frac{1}{2g^2}}\left(1 - \frac{5}{2}\cdot g^2 - \frac{13}{8}\cdot g^4 - \dots\right)$$

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- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n! \left(1 - \frac{53}{6}\cdot\frac{1}{3}\cdot\frac{1}{n} - \frac{1277}{72}\cdot\frac{1}{3^2}\cdot\frac{1}{n(n-1)} - \ldots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\,e^{-2\frac{1}{6g^2}} \left(1 - \frac{53}{6}\cdot g^2 - \frac{1277}{72}\cdot g^4 - \ldots\right)$$

another perspective on resurgence:

resurgence can be viewed as a method for making asymptotic expansions consistent with global analytic continuation properties

e.g. asymptotics of special functions: Stokes phenomenon and saddle point structure

analytic continuation of path integrals is a major problem in physics and mathematics

Analytic Continuation of Path Integrals: Lefschetz Thimbles

functional version: path integral

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}\left(S_{\rm real}[A]+i\, S_{\rm imag}[A]\right)} \sim \sum_{\text{thimbles}\, k} e^{-\frac{i}{g^2}\, S_{\rm imag}[A]} \int_{\Gamma_k} \mathcal{D}A\, e^{-\frac{1}{g^2} S_{\rm real}[A]}$$

thimble = "functional steepest descents contour"

remaining path integral has real measure: amenable to
(i) Monte Carlo
(ii) semiclassical expansion
(iii) exact results?

resurgence: asymptotic expansions about different saddles are closely related

requires a deeper understanding of complex configurations and analytic continuation of path integrals ...

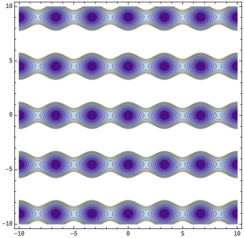
"functional Darboux" suggests possibilities ...

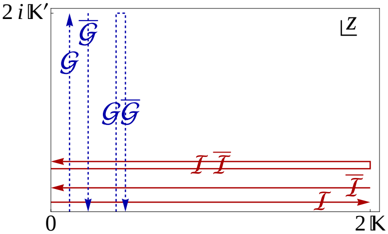
Ghost Instantons: Quantum Mechanical Path Integrals

(Başar, GD, Ünsal, arXiv:1308.1108)

$$\mathcal{Z}(g^2|m) = \int \mathcal{D}\phi\, e^{-S[\phi]} = \int \mathcal{D}\phi\, e^{-\int d\tau \left(\frac{1}{4}\dot{\phi}^2 + \frac{1}{g^2}\,\mathrm{sd}^2(g\,\phi|m)\right)}$$

- doubly periodic potential: *real* & complex instantons



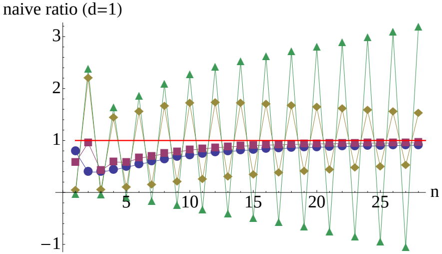


- actions:

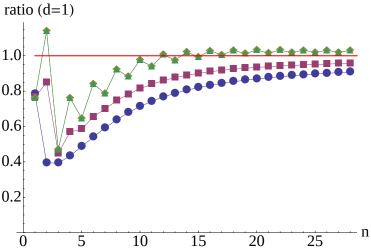
$$\frac{S_{\mathcal{I}}(m)}{g^2} = \frac{2\arcsin(\sqrt{m})}{g^2\sqrt{m(1-m)}} \geq \frac{2}{g^2} \quad , \quad \frac{S_{\mathcal{G}}(m)}{g^2} = \frac{-2\arcsin(\sqrt{1-m})}{g^2\sqrt{m(1-m)}} \leq -\frac{2}{g^2}$$

Ghost Instantons: Quantum Mechanical Path Integrals

- large order growth of QM perturbation theory



without ghost instantons



with ghost instantons

$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$

Ghost Instantons: Quantum Mechanical Path Integrals

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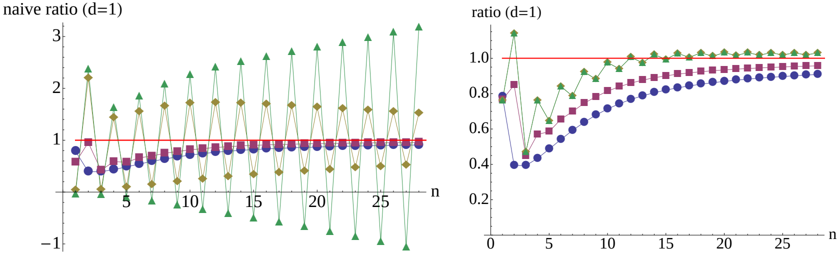

without ghost instantons *with ghost instantons*

$$a_n(m) \sim -\frac{16}{\pi} n! \left(\frac{1}{(S_{\mathcal{I}\bar{\mathcal{I}}}(m))^{n+1}} - \frac{(-1)^{n+1}}{|S_{\mathcal{G}\bar{\mathcal{G}}}(m)|^{n+1}} \right)$$

- complex instantons directly affect perturbation theory, even though they are not in original path integral measure !

Yang-Mills, $\mathbb{CP}^{N-1}$, $O(N)$, PCM, ... all have non-BPS solutions with finite action

- "unstable": negative modes of fluctuation operator
- what do these mean ?

resurgence: ambiguous imaginary non-perturbative terms should cancel ambiguous imaginary terms coming from lateral Borel sums of perturbation theory

$$\int \mathcal{D}A\, e^{-\frac{1}{g^2}S[A]} = \sum_{\text{all saddles}} e^{-\frac{1}{g^2}S[A_{\text{saddle}}]} \times (\text{fluctuations}) \times (\text{qzm})$$

- 2d Principal Chiral Model: (Cherman, Dorigoni, GD, Ünsal, 1308.0127)

$$S_b = \frac{N}{2\lambda}\int d^2x \,\mathrm{tr}\, \partial_\mu U \partial^\mu U^\dagger, \quad U \in SU(N),$$

Conclusions

- Resurgence systematically unifies perturbative and non-perturbative world
- series $\longrightarrow$ trans-series: sectors are inter-related
- there is extra 'magic' in perturbation theory
- IR renormalon puzzle in asymptotically free QFT
- multi-instanton physics from perturbation theory
- basic property of steepest descents expansions
- resurgence required for analytic continuation
- moral: consider all saddles, including non-BPS

Open Problems

- natural path integral construction
- analytic continuation of path integrals
- relation to localization
- relating strong- and weak-coupling expansions: dualities
- relation to OPE
- relation to SUSY and extended SUSY
- ODE/Integrable Model correspondence
- ...