Order - disorder operators in planar and almost planar graphs

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joint works with:
Hugo Duminil-Copin, Vincent Tassion, and Simone Warzel

Mathematics and Physics at the Crossroads
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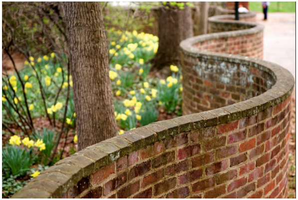
Rome: exhilarating, surprising, humbling

A suitable background for discussions of integrable systems

A case in point: Jeffersonian serpentine walls [U. Virginia, 1820's]:



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one of many innovations by which Jefferson attempted to combine aesthetics with utility."

Also on Wikipedia:

• "Many crinkle crankle walls are found in the East Anglia area of England where the marshes of the fen country were drained by Dutch engineers starting in the mid-1600s."

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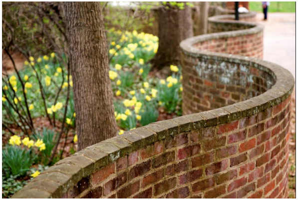

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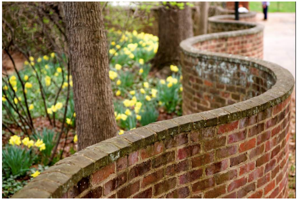

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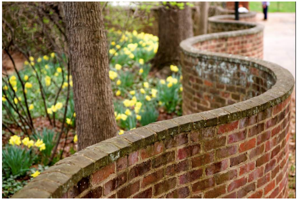

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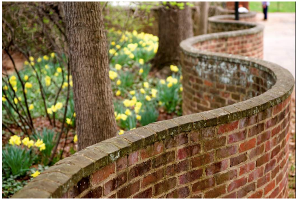

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Stochastic geometric perspective on Ising spin systems and related models, through tools which apply quite generally, but link with integrability in the planar case.

Among the concepts to be discussed:

Pfaffian correlation functions

Ising model's order - disorder operators (Kaufmann / Kadanoff spinors)

and a generalization of this notion

dimer cover models, and their disorder operators

planar and "almost planar" models ...

emergent planarity

The talk is based on the joint works:

• M. Aizenman H. Duminil-Copin, V. Tassion, S. Warzel: "Fermionic correlation functions and emergent planarity in 2D Ising models " (tentative title)

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Talk's main themes

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1. A feature of planar Ising models

A planar Ising model: $\mathcal{G} = V(\mathcal{G}), \mathcal{E}(\mathcal{G})$ a planar graph, and

$$H(\sigma) = - \sum_{\{x,y\} \in \mathcal{E}(\mathcal{G})} J_{x,y} \sigma_x \sigma_y - h \sum_{x \in V(\mathcal{G})}$$

Theorem (Groeneveld-Boel-Kasteleyn '78, $^{(*)}$)
Let $\langle - - \rangle$ be an equilibrium state of a ferromagnetic Ising model on a planar graph with a connected boundary segment Γ.

Then, for any collection of boundary sites $\{x_1, ..., x_{2n}\} \subset \Gamma$, ordered cyclicly along Γ:

$$\langle \prod_{j=1}^{2n} \sigma_{x_j} \rangle = \sum_{\text{pairings } \pi} \varepsilon(\pi) \prod_{j=1}^{n} \langle \sigma_{x_{\pi(2j-1)}} \sigma_{x_{\pi(2j)}} \rangle \equiv Pf\left(S_2(x_j, x_k)\right)$$

where $\varepsilon(\pi) = \pm 1$ is the pairing's parity.

- *Realized in increasing generality, starting with Schultz-Mattis-Lieb '65 for graphs with a regular transf. matrix.*
- *Not true for the correlation functions in the bulk, nor for non-planar models (BK '78)*
- *The statement is valid for amorphous graphs and arbitrary sets of couplings.*
- *Our proof & explanation (ADTW) utilizes the random current representation.*

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1 b. Remarks on the Pfaffian structure (to be clarified in the talk)

$$\boxed{\langle \prod_{j=1}^{2n} \sigma_{x_j} \rangle \;=\; \sum_{\text{pairings } \pi} \varepsilon(\pi) \prod_{j=1}^{n} \langle \sigma_{x_{\pi(2j-1)}} \sigma_{x_{\pi(2j)}} \rangle \;\equiv\; \mathrm{Pf}\left(S_2(x_j, x_k)\right)}$$

1) A Pfaffian structure for the scalar correlation function in the bulk would not be consistent with conformal invariance. (A simple argument.)

2) A relevant topological distinction between the two cases.

3) Reason why it does not hold for finite range (and thus non-planar) models.

3) An extension of the correlation functions (into the correlators of order-disorder pairs) for which the Pfaffian structure does extend to the bulk.

4) A generalization of the principle by a construction which applies also to the (related) dimer cover model.

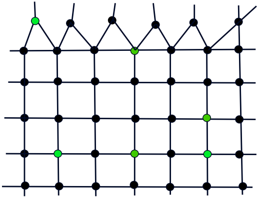
5) Emergent planarity – at the critical point from a stochastic geometric perspective.

6) The latter statement is linked with universality results of Spencer-Pinson, and Giuliani-Greenblatt-Mastropietro '12.

Ising spins on a general graph $\mathcal{G}$:

$$\sigma : \mathcal{G} \to \{-1, +1\}$$

$$H(\sigma) = - \sum_{(x,y)\in\mathcal{E}} J_{x,y}\sigma_x\sigma_y - h\sum_{x\in\mathcal{G}} \sigma_x$$



Gibb's equil. measure

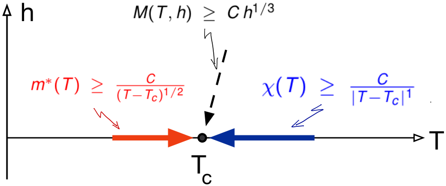
$$\boxed{\Pr_\Lambda(\sigma) = e^{-\beta H_\Lambda(\sigma)}/Z_\Lambda}$$

$$Z_\Lambda = \sum_{\sigma\in\{-1,1\}} e^{-\beta H_\Lambda(\sigma)}$$

The spontaneous magnetization:

$$m^*(T) \equiv M(T, h = 0+) := \langle\sigma_x\rangle_{T,h=0+}$$

is $\begin{cases} 0 & T > T_c \\ > 0 & T < T_c \end{cases}$



Phase diagram for $\boxed{\langle\cdot\rangle = \lim_{\Lambda\nearrow\mathcal{G}} \mathbb{E}_\Lambda[\cdot]}$

[On transitive graphs the corresponding critical exponents are bounded by their mean field values (ABF'87):
$\gamma \geq 1, \quad \beta \leq 1/2, \quad \delta \geq 3$.]

The (ferr.) Ising spin system on a graph $\mathcal{G}$ of edge set $\mathcal{E}$ (finite subsets $\Lambda \subset \mathcal{G}$) is:

$$\sigma : \mathcal{G} \mapsto \{-1, 1\}, \quad \Pr_\Lambda(\sigma) = \frac{e^{-\beta H_\Lambda(\sigma)}}{Z_\Lambda}$$

with $H(\sigma) = -\sum_{(x,y)\in\mathcal{E}} J_{x,y}\,\sigma_x\sigma_y \; - \; h\sum_{x\in\mathcal{G}}\sigma_x$; $\quad J_{x,y} \geq 0$ (ferromag. interaction)

The Random Current representation (starting from the *high temp. exp., as GHS did*)

$\mathbf{n} : \mathcal{E} \mapsto \{0, 1, 2, ...\}$ $\quad \boxed{\partial\mathbf{n} := \{x \in \mathcal{G} \; : \; (-1)^{\sum_y n_{x,y}} = -1\}}$ - *the set of sources*

weights: $\boxed{w(n) := \prod_{b\in\mathcal{E}} (\beta J_b)^{n_b} / n_b!}$ *with "b" an alternative symbol for* $(x, y) \in \mathcal{E}$

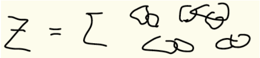
Basics relations (for $h = 0$):

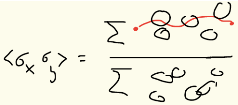
$$\boxed{Z \; := \sum_\sigma e^{-\beta H(\sigma)} \; = \sum_{\mathbf{n}:\, \partial\mathbf{n}=\emptyset} w(\mathbf{n})}$$

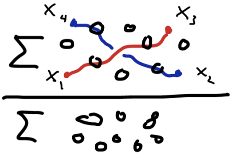
and for any $A \subset \mathcal{G}$:

$$\boxed{\langle \prod_{x\in A} \sigma_x \rangle = \sum_{\mathbf{n}:\, \partial\mathbf{n}=A} w(\mathbf{n})/Z}$$

pictorially:





$\langle \sigma_{x_1} \ldots \sigma_{x_4} \rangle =$ 

This yields a suggestive explanation of the phenomenon of upper critical dimension: in high dimensions (as it turns out $d > 4$), at large separations:

$$\langle \sigma_{x_1} \ldots \sigma_{x_4} \rangle \approx \left[\langle \sigma_{x_1} \sigma_{x_2} \rangle \langle \sigma_{x_3} \sigma_{x_4} \rangle + \langle \sigma_{x_1} \sigma_{x_3} \rangle \langle \sigma_{x_2} \sigma_{x_4} \rangle + \langle \sigma_{x_1} \sigma_{x_4} \rangle \langle \sigma_{x_2} \sigma_{x_3} \rangle \right] [1 + o(1)]$$

Theorem (A 81) *For the n.n. Ising models on $\mathbb{Z}^d$ in $d > 4$, if for some $\kappa(\delta) \to \infty$ the scaled correlation functions converge (pointwise for $x_1, \ldots, x_{2n} \in \mathbb{R}^d$)*

$$S_{2n}(x_1, \ldots, x_{2n}) \; = \; \lim_{\delta \to 0} \kappa(\delta)^{2n} \langle \prod_{j=1}^{2n} \sigma_{[x_j/\delta]} \rangle_{T_c}$$

then the limiting functions satisfy $\boxed{S_{2n}(x_1, \ldots, x_{2n}) \; = \; \sum_{\text{pairings}\,\pi} \prod_{j=1}^{n} S_2(x_{\pi(2j-1)}, x_{\pi(2j)})}$

Under the above conditions also (A-Barsky-Fernandez '87) :

$$\boxed{\gamma = 1, \qquad \beta = 1/2, \qquad \delta = 3 .}$$

4. An instructive stochastic geometric expression

Defining $u_4(x_1, ..., x_4)$ so that:

$$\langle \sigma_{x_1}\sigma_{x_2}\sigma_{x_3}\sigma_{x_4}\rangle = \left[\langle \sigma_{x_1}\sigma_{x_2}\rangle \langle \sigma_{x_3}\sigma_{x_4}\rangle + \langle \sigma_{x_1}\sigma_{x_3}\rangle \langle \sigma_{x_2}\sigma_{x_4}\rangle + \langle \sigma_{x_1}\sigma_{x_4}\rangle \langle \sigma_{x_2}\sigma_{x_3}\rangle\right] + u_4(x_1, ..., x_4)$$

we have:

Lemma: *For any Ising model on a finite graph:*

$$\boxed{u_4(x_1, ..., x_4) \;=\; -2\,\langle \sigma_{x_1}\sigma_{x_2}\rangle \langle \sigma_{x_3}\sigma_{x_4}\rangle \;\mathit{Prob}\left(C_{\mathbf{n}_1+\mathbf{n}_2}(x_1) \ni x_2, x_3, x_4 \;\middle|\; {\partial \mathbf{n}_1 = \{x_1, x_2\} \atop \partial \mathbf{n}_2 = \{x_3, x_4\}}\right)}$$

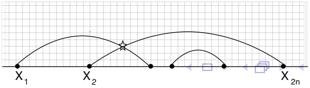
Note:

i) In situations where $|u_4(x_1, ..., x_4)|/\langle \sigma_{x_1}\sigma_{x_2}\sigma_{x_3}\sigma_{x_4}\rangle \to 0$ one gets Gaussian limits

ii) If for intertwined pairs: Prob(...) $\to 1$, then one gets a fermonic expression.

ii) The argument has a simple extension to to all even-n boundary correlation functions (ADTW).

iv) Important here are not just the statistics, but the apparent “free-ness” (or integrability) of the model.



An interesting contrast

For $d > 4$ the critical correlations $S_{2n}(x_1, ..., x_{2n}) = \kappa(\delta)^{2n} \langle \prod_{j=1}^{2n} \sigma_{[x_j/\delta]} \rangle_{T_c}$

satisfy $$\boxed{S_{2n}(x_1, ..., x_{2n}) \approx \sum_{\pi} \prod_{j=1}^{n} S_2(x_{\pi(2j-1)}, x_{\pi(2j)})}$$ Gauss-Wick rule (Aiz 81)

with equality in the scaling limit ($\delta \to 0$, $\kappa(\delta) \to \infty$ adjusted so the limit exists),

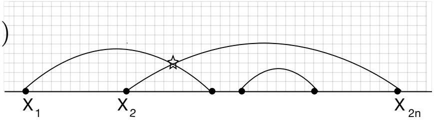
In $d = 2$ dimensions for any ferromag. Ising model on a planar graph, with a connected boundary segment, the boundary fields satisfy (SML 65, McCoy-Wu'73, GBK 78):

$$\boxed{S_{2n}(x_1, ..., x_{2n}) = \sum_{\pi} \varepsilon(\pi) \prod_{j=1}^{n} S_2(x_{\pi(2j-1)}, x_{\pi(2j)})}$$

Fermi-Wick rule

$x_j \in [0, \infty) \times \mathbb{R}^{d-1}$

$$= \text{Pf}\left(S_2(x_j, x_k)\right)$$



Curiously, both relation have a relatively simple explanation through the "random current representation". Using it, the Pfaffian structure of correlations (on which more can be read in Chelkak-Cimasoni-Kassel '15) appears as a consequence of elementary topological arguments (ADTW).

"Almost planar" – finite-range models on planar graphs.

For a class of such models we have the following statement of emergent planarity.

Theorem (ADTW '16) *In any finite range ferromagnetic Ising model in* $\mathcal{G} = \mathbb{Z} \times \mathbb{Z}_+$ *whose couplings J are: i) translation invariant, ii) acyclic, iii) invariant under reflections: for any cyclicly ordered* $(x_1, \ldots, x_{2n}) \in \partial\mathcal{G} := \mathbb{Z} \times \{0\}$

$$\boxed{\langle \sigma_{x_1} \cdots \sigma_{x_{2n}} \rangle_{\beta_c} \;=\; Pf_n\left(\left[\langle \sigma_{x_i} \sigma_{x_j} \rangle_{\beta_c} \right]_{1 \le i,j \le 2n} \right) (1 + o(1))}$$

where $o(1)$ *is a quantity tending to zero with the smallest distance in* $\mathcal{G}$ *between any two* x_i*.*

In the stochastic geometric argument the effective planarity emerges due to the critical random currents' fractal nature (at β_c).

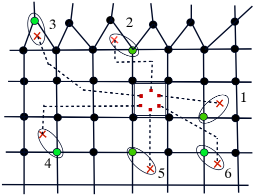
Related universality results -stability of the law under weak perturbations- were previously derived using rigorous (perturbative) renormalization arguments by Pinson-Spencer and Giuliani-Greenblatt-Mastropietro '12.

6. 'Order - disorder' variables for 2D models

A natural question: Does the fermionic structure extend to variables in the bulk?

Our answer: "Yes / but": a natural extension is found in the order-disorder operators.

ℓ_j: dual lines linking sites of $\{x_j'\}$ with $x_0^* \in \mathcal{G}^*$ (the grand central).



Coupling-reversing transform's:

$$(R_\ell J)_{x,y} = -J_{x,y}$$

for edges $\{x, y\}$ crossed by ℓ.

The "order - disorder" variables $\boxed{\tau_{\hat{x}} = \sigma_x \mu_{x'}}$ are defined by:

$$\boxed{\langle \prod_{j=1}^{2n} \tau_{\hat{x}_j} \rangle := \sum_\sigma \left(\prod_{j=1}^{2n} \sigma_{x_j} \right) e^{-\beta R_{\ell_1} \dots R_{\ell_j} \dots H(\sigma)} / Z}$$

Of particular interest: τ_j for neighboring pairs $\hat{x}_j = (x_j, x'_j) \in \mathcal{G} \times \mathcal{G}^*$.

Theorem 4 In planar Ising models, of pair interaction $\mathcal{J}$ with $Z_{\mathcal{G}}(\mathcal{J}) \neq 0$, for any collection of "order - disorder" variables labeled cyclicly in terms of the disorder lines

$$\boxed{\langle \prod_{j=1}^{2n} \tau_{\hat{x}_j} \rangle = \sum_{\text{pairings}\,\pi} \varepsilon(\pi) \prod_{j=1}^{n} \langle \tau_{\hat{x}_{\pi(2j-1)}} \tau_{\hat{x}_{\pi(2j)}} \rangle \equiv \mathrm{Pf}\left(\langle \tau_{\hat{x}_j} \tau_{\hat{x}_k} \rangle \right)}\,.$$

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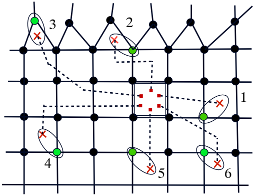

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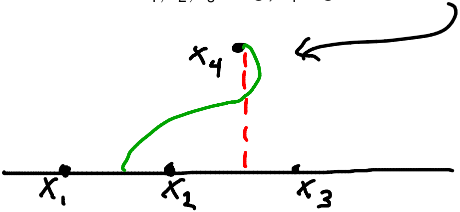
6 b. ‘Order-disorder’ operators’ stochastic geometric interpretation

i) In terms of random currents:

$$\langle \tau_{\hat{x}_1} \tau_{\hat{x}_2} \rangle \;=\; \sum_{\substack{\partial n_1 = \{x_1, x_2\} \\ n_2 = \emptyset}} \frac{w(n_1)}{Z} \frac{w(n_2)}{Z} (-1)^{(n_1, \gamma_{1,2})} \; \mathbb{1}[\;\;]$$

$n_1 = n_2 = 0$

which in the case of $x_1, x_2, x_3 \in \partial \mathcal{G}$, $x_4 \in \mathcal{G}$ reduces to:



ii) In terms of the Kac-Ward (“parafermionic”) amplitudes

$$\begin{aligned} \langle \tau_{\hat{x}_1} \tau_{\hat{x}_2} \rangle &= e^{i\angle(\tilde{x}_1, \tilde{x}_2)} \langle \bar{e}_2 | \, (\mathbb{1} - KW)^{-1} \, | e_1 \rangle \\ &= e^{i\angle(\tilde{x}_1, \tilde{x}_2)} \sum_{\gamma : e_1 \to \bar{e}_2} \chi_{KW}(\gamma) \; e^{i \int_\gamma dArg(e)/2} \end{aligned}$$

The disorder operators represent partial (incomplete) gauge transformations.

IN $D = 2$ dimensions a disorder operator, μ_ℓ, act on the Hamiltonian,
changing it across a line ℓ (and more generally across a $D-1$ dimensional surface).

The underlying gauge symmetry $\Longrightarrow$

> μ_ℓ's effect on the Hamiltonian's Gibbs states is homologous,
> i.e., a homotopy-invariant function of ℓ, up to a simple gauge transformation.

Another example: disorder operators for the dimer cover model

Given a finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of vertex set $\mathcal{V}$, a perfect matching or dimer cover is a subset of the edge set, $\omega \subset \mathcal{E}$, such that every vertex is covered by exactly one edge.

The set of perfect matchings is denoted $\Omega_{\mathcal{G}}$. The dimer-cover partition function counts the number of the graph's perfect matchings.
Perfect matchings can also be weighted through a complex-valued edge function $K : \mathcal{E} \mapsto \mathbb{C}$. Given an edge weight, the weighted dimer-cover partition function is

$$Z_{\mathcal{G},K} := \sum_{\omega \in \Omega_{\mathcal{G}}} \chi(\omega)\,, \qquad \text{with} \quad \chi(\omega) := \prod_{b \in \omega} K_b\,.$$

Of particular interest is the effect on the dimer-cover partition function of the removal of a collection of sites, $M \subset \mathcal{V}$, which are regarded as covered by separate monomers.
The collection of perfect matchings of the remaining vertices is denoted by $\Omega_{\mathcal{G}}(M)$ and

$$\boxed{Z_{\mathcal{G},K}(M) := \sum_{\omega \in \Omega_{\mathcal{G}}(M)} \chi(\omega)}$$

stands for the weighted partition function of the monomer-depleted graph.

Not all graphs admit a perfect matching (recall unbalanced bipartite graphs).

The dimer model's correlation function and its gauge symmetry

The monomer correlation function for an even collection of disjoint sites $\{x_1, ..., x_{2n}\} \subset \mathcal{V}$ is

$$\boxed{S_{2n}(x_1, ..., x_{2n}) := \langle \prod_{j=1}^{n} \eta_{x_j} \rangle_{\mathcal{G},K} \;:=\; \frac{Z_{\mathcal{G},K}(\{x_1, ..., x_{2n}\})}{Z_{\mathcal{G},K}}}$$

As was noted already in the early work of P. W. Kasteleyn, (Kas'63), in the dependence of $Z_{\mathcal{G},K}$ on the kernel K the dimer model has a Z_2 gauge symmetry:

For subsets $B \subset \mathcal{V}$ the (edge) boundary is denoted

$$\partial B = \{[x, y] \in \mathcal{E} \mid \text{if exactly one of the two points is in } B\} \,.$$

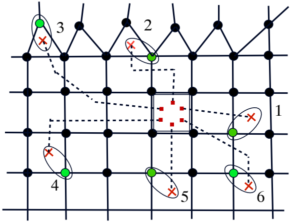
For each such set, let $T_{\partial B} : \mathbb{C}^{\mathcal{E}} \to \mathbb{C}^{\mathcal{E}}$ be the map

$$(T_{\partial B} K)_b \;=\; \begin{cases} -K_b & \text{if } b \in \partial B \\ K_b & \text{otherwise.} \end{cases}$$

Under its action:

$$\boxed{Z_{\Lambda, T_{\partial B} K} \;=\; (-1)^{|B|}\, Z_{\Lambda,K}}\,.$$

As for the Ising model, we define the *expectation values* of products of order-disorder operators $\mu_j := \eta_{x_j}\tau_{\ell_j}$ as:



$$\langle\prod_{j=1}^{2n}\mu_j\rangle_{\mathcal{G},K} \;:=\; \frac{Z_{\mathcal{G},T_{\ell_1}\circ\cdots\circ T_{\ell_{2n}}K}\left(\{x_1,\ldots,x_{2n}\}\right)}{Z_{\mathcal{G},K}}\,,$$

with $p_j := (x_j,\ell_j)$ denoting an order-disorder pair.

We then have

Theorem (ALW '16) *For any finite planar graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ with edge weights $K:\mathcal{E}\mapsto\mathbb{C}$, and any collection of canonical pairs of order-disorder variables $p_j=(x_j,\ell_j)$, $j\in\{1,\ldots,2n\}$, which are cyclicly ordered*

$$\langle\prod_{j=1}^{2n}\mu_j\rangle_{\mathcal{G},K} \;=\; \sum_{\pi\in\Pi_n}\mathrm{sign}(\pi)\prod_{j=1}^{n}\langle\mu_{\pi(2j-1)}\,\mu_{\pi(2j)}\rangle_{\mathcal{G},K} \;\equiv\; \mathrm{Pf}_n\left(\langle\mu_j\mu_k\rangle_{\mathcal{G},K}\right)\,.$$

As before, for collections of boundary sites this reduces to Pfaffian relation for the regular monomer correlation functions.

The order-disorder variables form the Kaufman spinors, and are operators of interest in the Kadanoff - Ceva list.

In addition, their product also yields the energy density operator. Through this relation, the above fermonic rule yields yet another intuitive explanation, a-la Kadanoff, of some of the (already well known) critical exponents, e.g.:

- the energy- energy correlations decay in 2D as $1/r^2$
  (and hence the energy density has, in 2D, a logarithmic cusp at T_c).
- boundary spin correlators decay as $1/r$, etc.

Emergent planarity: There is still room for a more complete mathematical grasp of the stochastic geometry of the critical models. As we saw, this may adds some robust insight on the emergent planarity at criticality in two dimensional models with non-planar interactions / weights, thus supplementing the rigorous (perturbative) renormalization group analysis.

Thank you for your attention.