Janus-Facedness of the Pion: Analytic Instantaneous Bethe–Salpeter Models

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Goldstonic quark–antiquark bound states

Within quantum chromodynamics, the pions or, as a matter of fact, all light pseudoscalar mesons must be interpretable as both quark–antiquark bound states and almost massless (pseudo) Goldstone bosons of the spontaneously —and to a minor degree even explicitly—broken chiral symmetries of QCD.

Relativistic quantum field theory describes bound states by Bethe–Salpeter amplitudes $\Phi(p)$ controlled by some homogeneous Bethe–Salpeter equation defined, for two bound particles of individual and relative momenta $p_{1,2}$ and p, by their propagators $S(p_{1,2})$ and an integral kernel $K(p,q)$ encompassing their interactions, suppressing dependences on the total momentum p_1+p_2:

$$\Phi(p) = \frac{\mathrm{i}}{(2\pi)^4}\, S_1(p_1) \int \mathrm{d}^4 q\, K(p,q)\, \Phi(q)\, S_2(-p_2)\ .$$

Suitably adapted inversion techniques [1] allow us to retrieve the underlying interactions analytically in form of a (configuration-space) central potential $V(r)$, $r \equiv |\boldsymbol{x}|$, from presumed solutions to the Bethe–Salpeter equation [2].

By that, we are put in a position to construct exact analytic Bethe–Salpeter solutions for all massless pseudoscalar mesons [3] in the sense of establishing rigorous analytic relationships between interactions and resulting solutions: all analytic findings [4] may then be confronted with numerical outcomes [5].

Crucial simplifying-assumptions sequence

1. Assuming, for any involved quark, both instantaneous interactions and free propagation with a mass dubbed constituent, simplifies the Bethe–Salpeter equation to a bound-state equation for the Salpeter amplitude
$$\phi(\boldsymbol{p}) \propto \int \mathrm{d}p_0\, \Phi(p)\ .$$
For a spin-$\frac{1}{2}$ fermion and a spin-$\frac{1}{2}$ antifermion of equal masses m bound to a spin-singlet state (which is the case for, e.g., pseudoscalar mesons), this wave function involves only two independent components, $\varphi_{1,2}(\boldsymbol{p})$:
$$\phi(\boldsymbol{p}) = \left[\varphi_1(\boldsymbol{p})\,\frac{\gamma_0\,(\boldsymbol{\gamma}\cdot\boldsymbol{p}+m)}{E(p)} + \varphi_2(\boldsymbol{p})\right]\gamma_5\ ,$$
$$E(p) \equiv \sqrt{\boldsymbol{p}^2+m^2}\ , \qquad p \equiv |\boldsymbol{p}|\ .$$

2. Upon assuming the quark interactions in the kernel to respect spherical and Fierz symmetries, the bound-state equation for $\phi(\boldsymbol{p})$ becomes a set of two coupled radial eigenvalue equations for the bound-state mass M:
$$2\,E(p)\,\varphi_2(p) + 2\int\limits_0^\infty \frac{\mathrm{d}q\,q^2}{(2\pi)^2}\,V(p,q)\,\varphi_2(q) = M\,\varphi_1(p)\ ,$$
$$2\,E(p)\,\varphi_1(p) = M\,\varphi_2(p)\ ,$$
$$V(p,q) \equiv \frac{8\pi}{p\,q}\int\limits_0^\infty \mathrm{d}r\,\sin(p\,r)\sin(q\,r)\,V(r)\ , \qquad q \equiv |\boldsymbol{q}|\ .$$

3. In the truly massless Goldstone case $M=0$, the system decouples, one component vanishes $[\varphi_1(p)\equiv 0]$, and the surviving component satisfies
$$E(p)\,\varphi_2(p) + \int\limits_0^\infty \frac{\mathrm{d}q\,q^2}{(2\pi)^2}\,V(p,q)\,\varphi_2(q) = 0\ .$$
Denoting by $T(r)$ the Fourier transform of the kinetic term $E(p)\,\varphi_2(p)$, $V(r)$ can be found from the latter's configuration-space representation:
$$V(r) = -\frac{T(r)}{\varphi_2(r)}\ .$$

Constraints on Bethe–Salpeter amplitude

Information on $\varphi_2(p)$ can be extracted from the full quark propagator $S(p)$, determined by its mass function $M(p^2)$ and a renormalization factor $Z(p^2)$:

$$S(p) = \frac{\mathrm{i}\, Z(p^2)}{\not p - M(p^2) + \mathrm{i}\,\varepsilon}\ , \qquad \not p \equiv p^\mu\, \gamma_\mu\ , \qquad \varepsilon \downarrow 0\ .$$

Studies of $S(p)$ within the Dyson–Schwinger framework, preferably done in Euclidean space indicated by underlined variables, entail crucial insights. In the chiral limit, a Ward–Takahashi identity relates [6] this quark propagator to the flavour-nonsinglet pseudoscalar-meson Bethe–Salpeter amplitude [3]:

$$\Phi(\underline{k}) \approx \frac{M(\underline{k}^2)}{\underline{k}^2 + M^2(\underline{k}^2)}\, \gamma_5 + \text{subleading contributions}\ .$$

In order to devise an analytic scenario, we exploit two pieces of information:

1. Phenomenologically sound Dyson–Schwinger models [7] get for $M(\underline{k}^2)$, in the chiral limit, at large $\underline{k}^2$ a decrease basically proportional to $1/\underline{k}^2$.
2. From axiomatic QFT, we may infer [8] that the presence in $M(\underline{k}^2)$ of an inflexion point at spacelike momenta $\underline{k}^2 > 0$ entails quark confinement.

Of course, such requirements on $M(\underline{k}^2)$ are reflected by $\Phi(\underline{k})$. A compatible ansatz for $\Phi(\underline{k})$, involving a mass parameter μ and a mixing parameter η, is

$$\Phi(\underline{k}) = \left[\frac{1}{(\underline{k}^2 + \mu^2)^2} + \frac{\eta\, \underline{k}^2}{(\underline{k}^2 + \mu^2)^3}\right] \underline{\gamma}_5\ , \qquad \mu > 0\ , \qquad \eta \in \mathbb{R}\ .$$

An integration w.r.t. the Euclidean momentum's time component results in

$$\varphi_2(p) \propto \frac{1}{(p^2 + \mu^2)^{3/2}} + \eta\, \frac{p^2 + \mu^2/4}{(p^2 + \mu^2)^{5/2}}\ , \qquad p \equiv |\boldsymbol{p}|\ ,$$

in configuration space expressible in terms of modified Bessel functions K_n:

$$\varphi_2(r) \propto 4\,(1 + \eta)\, K_0(\mu\, r) - \eta\, \mu\, r\, K_1(\mu\, r)\ .$$

If $\eta < -1$ or $\eta > 0$, $\varphi_2(r)$ has one zero, which induces a singularity in $V(r)$.

For special values of m/μ, $V(r)$ can be given by an analytic expression [3,4]. Henceforth, any quantity is understood in units of the adequate power of μ.

Confining potentials: Analytic results [3,4]

As consequence of our particular ansatz for $\varphi_2(r)$, for $\eta \neq -1$ all $V(r)$ must develop, at spatial origin, a logarithmically softened Coulombic singularity:

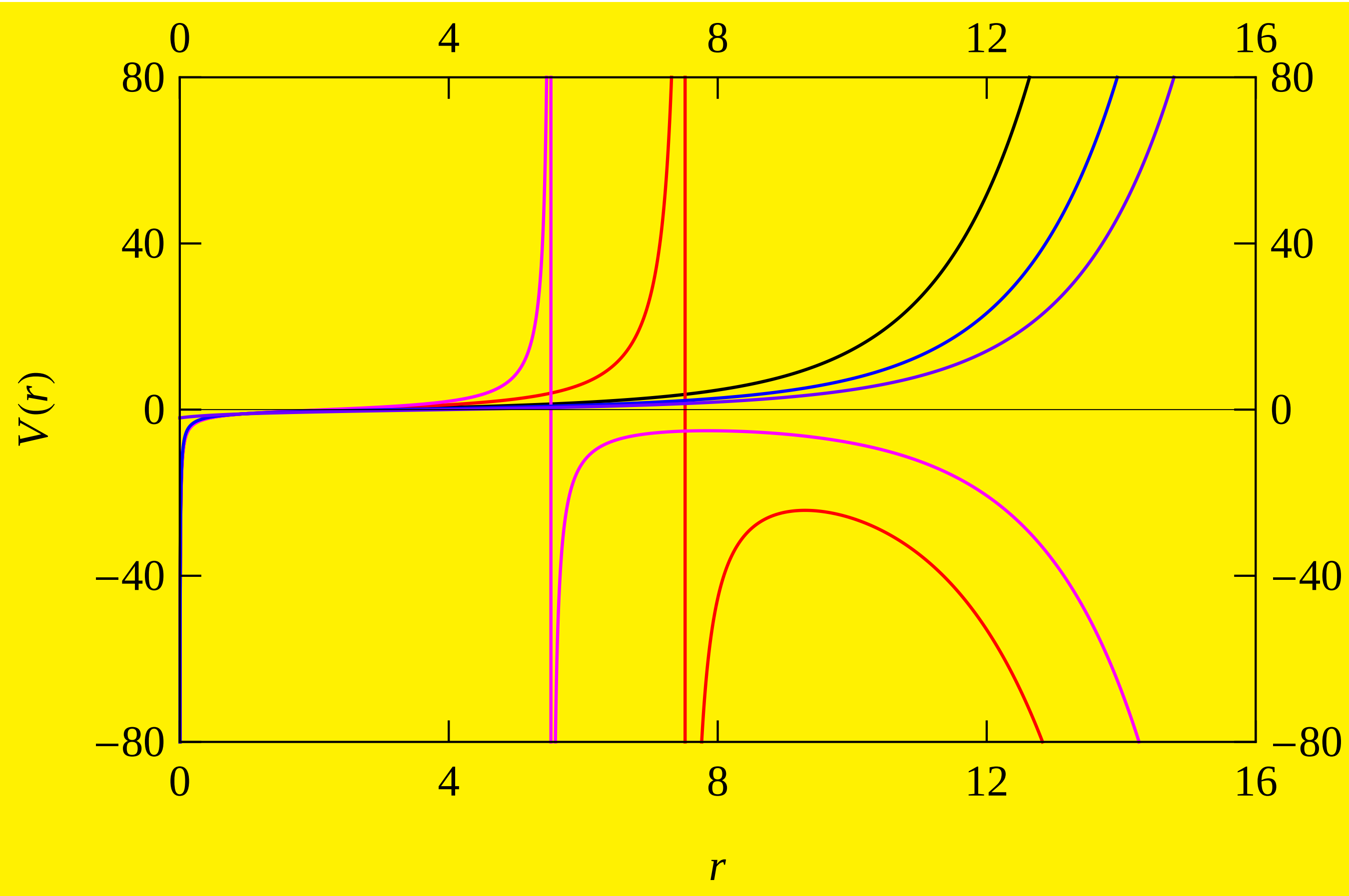
$$V(r) \xrightarrow[r\to 0]{} \frac{\text{const}}{r \ln r} \xrightarrow[r\to 0]{} -\infty \qquad (\text{const} > 0) \qquad \text{for } \eta \neq -1 \ .$$

Analytically manageable scenario of massless quarks ($m = 0$)

For our $\varphi_2(r)$, $V(r)$ involves modified Bessel (I_n) and Struve ($\mathbf{L}_n$) functions and rises—confiningly—to infinity either at the zero of $\varphi_2(r)$ or for $r \to \infty$:

$$V(r) = \frac{N(r)}{D(r)} \ , \qquad \begin{aligned} N(r) &\equiv \pi \left[4 + \eta\,(4 + r^2)\right] \left[\mathbf{L}_0(r) - I_0(r)\right] \\ &\quad + \pi\,(4 + 5\,\eta)\,r \left[\mathbf{L}_1(r) - I_1(r)\right] + 4\,(2 + 3\,\eta)\,r \ , \\ D(r) &\equiv 2\,r \left[4\,(1 + \eta)\,K_0(r) - \eta\,r\,K_1(r)\right] \ . \end{aligned}$$

$V(r)$ of the Fierz-symmetric kernel $K(p,q)$ for $m = 0$ and mixture $\eta = 0$[3] (black), $\eta = 1$ (red), $\eta = 2$ (magenta), $\eta = -0.5$ (blue), or $\eta = -1$ (violet):

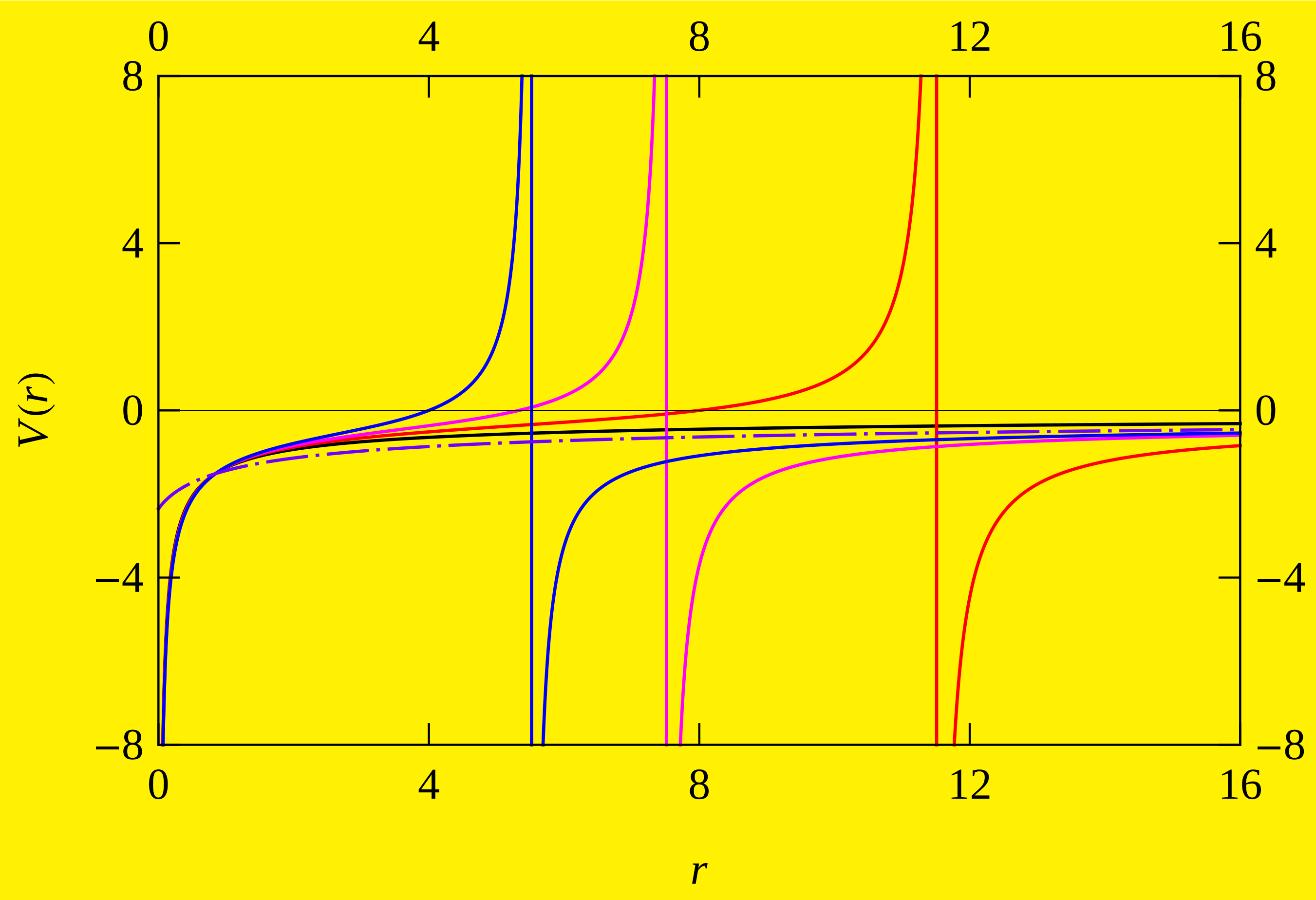


Analytically expressible case: quarks of common mass $m = \mu$

Here, $T(r)$ exhibits a mixture of Yukawa and exponential behaviour. Thus,

$$V(r) = -\frac{\pi\,[8+\eta\,(8-3\,r)]\exp(-r)}{4\,r\,[4\,(1+\eta)\,K_0(r)-\eta\,r\,K_1(r)]} \xrightarrow[r\to\infty]{} -\frac{\text{const}}{\sqrt{r}} \xrightarrow[r\to\infty]{} 0\ .$$

$V(r)$ of the Fierz-symmetric kernel $K(p,q)$ for $m=1$ and mixture $\eta=0$ [3] (black), $\eta=0.5$ (red), $\eta=1$ (magenta), $\eta=2$ (blue), and $\eta=-1$ (violet):



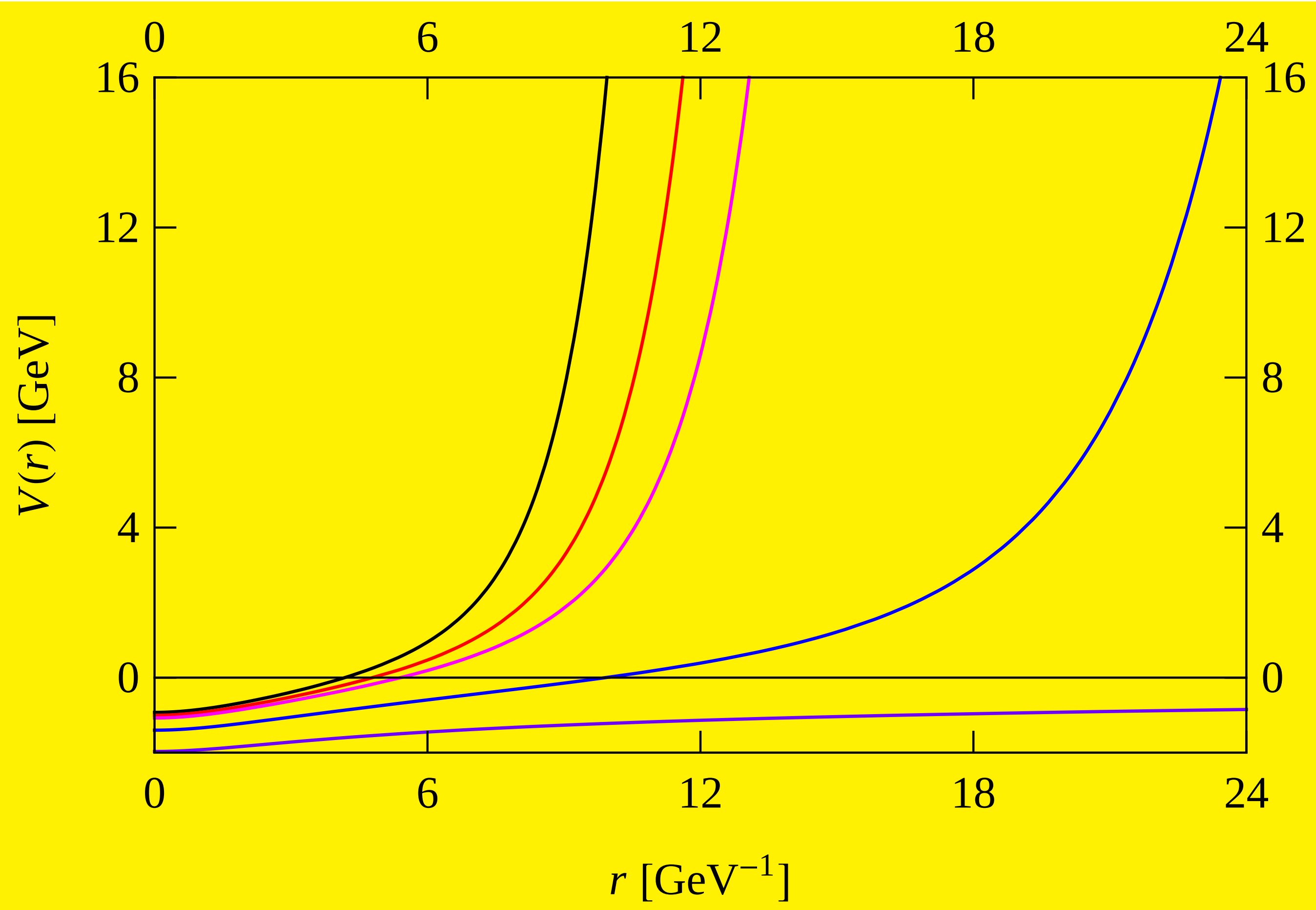
Test of reliability: Numerical derivation [5]

We check our results using the pointwise form of the chiral-limit quark mass function $M(\underline{k}^2)$, provided graphically in Ref. [7]: We parametrize $M(\underline{k}^2)$ by

$$M(\underline{k}^2) = 0.708\ \text{GeV}\exp\left(-\frac{\underline{k}^2}{0.655\ \text{GeV}^2}\right) + \frac{0.0706\ \text{GeV}}{\left[1+\left(\frac{\underline{k}^2}{0.487\ \text{GeV}^2}\right)^{1.48}\right]^{0.752}}\ .$$

N.B.: $1.48 \times 0.752 = 1.1$, pretty close to unity. Feeding this parametrization into our inversion procedure, we get potentials which are finite at $r = 0$ and rise, with r, to infinity for sufficiently small m but stay negative for large m.

$V(r)$ arising from $M(\underline{k}^2)$ of Ref. [7] for $m = 0$ (black), $m = 0.35$ GeV (red), $m = 0.50$ GeV (magenta), $m = 1.0$ GeV (blue), $m = 1.69$ GeV (violet) [5]:



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