Fractal Structure of Hadrons and Non-Extensive Statistics

EUGENIO MEGÍAS*,1
Department of Atomic, Molecular and Nuclear Physics, and Carlos I Institute of Theoretical and Computational Physics, University of Granada, Spain

AIRTON DEPPMAN
Instituto de Física, Universidade de São Paulo, Brazil

TOBIAS FREDERICO
Instituto Tecnológico da Aeronáutica, São Paulo, Brazil

DÉBORA P. MENEZES
Departamento de Física, CFM, Universidade Federal de Santa Catarina, Florianópolis - SC - Brazil

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Abstract

The role played by non-extensive thermodynamics [1] in physical systems has been under intense debate for the last decades. Some possible mechanisms that could give rise to non extensive statistics have been formulated along the last few years, in particular the existence of a fractal structure in thermodynamic functions for hadronic systems [2]. We investigate the properties of such fractal thermodynamical systems and propose a diagrammatic method for calculations of relevant quantities. Finally, the fractal scale invariance is discussed in terms of the Callan-Symanzik equation.

1. Tsallis Statistics and QCD Thermodynamics

- Tsallis statistics constitutes a generalization of Boltzmann-Gibbs (BG) statistics, under the assumption that the entropy of the system is non-additive. For two independent systems A and B

$$S_{A+B} = S_A + S_B + (1-q) S_A S_B \,, \tag{1}$$

where the entropic index q measures the degree of non-extensivity [1]. Let us define the q-exponential $e_q^{(\pm)}(x) = [1 \pm (q-1)x]^{\pm 1/(q-1)}$, with $e_q^{(+)}(x)$ defined for $x \geq 0$ and $e_q^{(-)}(x)$ for $x < 0$, and the q-log function $\log_q^{(\pm)}(x) = \pm(x^{\pm(q-1)} - 1)/(q-1)$. Then the grand-canonical partition function for a non-extensive ideal quantum gas is [3]

$$\log \Xi_q(V,T,\mu) = -\xi V \int \frac{d^3p}{(2\pi)^3} \sum_{r=\pm} \Theta(rx) \log_q^{(-r)} \left(\frac{e_q^{(r)}(x) - \xi}{e_q^{(r)}(x)} \right), \tag{2}$$

where $x = \beta(E_p - \mu)$, the particle energy is $E_p = \sqrt{p^2 + m^2}$, with m being the mass and μ the chemical potential, $\xi = \pm 1$ for bosons and fermions respectively, and Θ is the step function. Eq. (2) reduces to the Bose-Einstein and Fermi-Dirac partition functions in the limit $q \to 1$.

- The thermodynamics of Quantum Chromodynamics (QCD) in the confined phase can be studied within the HRG approach, which is based on the assumption that physical observables in this phase admit a representation in terms of hadronic states which are treated as non-interacting and point-like particles [4]. These states are taken as the conventional hadrons listed in the review by the Particle Data Group. Within this approach the partition function is then given by [3, 5]

$$\log \Xi_q(V,T,\{\mu\}) = \sum_i \log \Xi_q(V,T,\mu_i) \,, \tag{3}$$

where μ_i refers to the chemical potential for the *i-th* hadron. We summarize our results in Fig. 1.

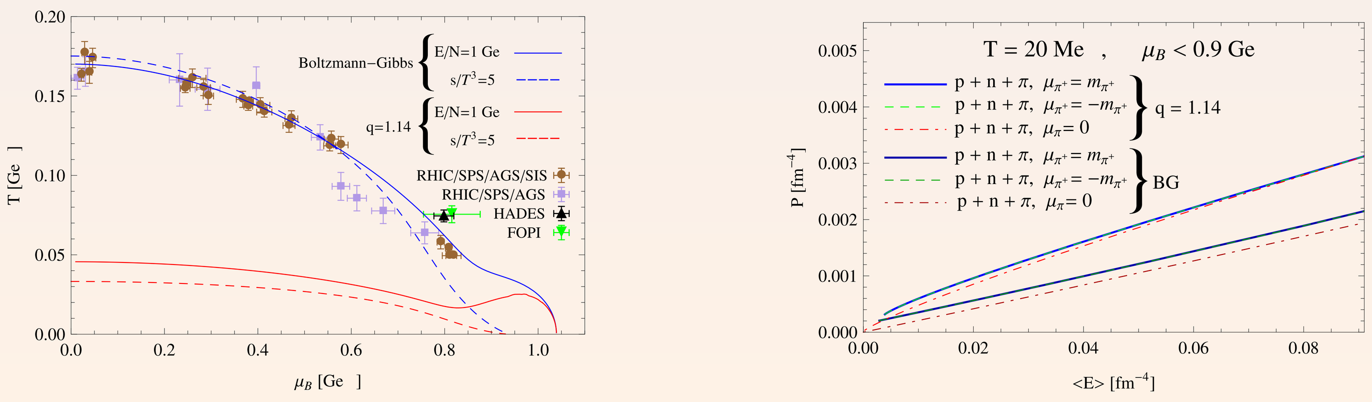

Fig. 1: *Left: Chemical freeze-out line $T = T(\mu_B)$. Right: Equation of State (EoS).*

2. Tsallis Statistics and Thermofractals

- The emergence of the non-extensive behavior has been attributed to different causes: long-range interactions, correlations and memory effects [6]; temperature fluctuations; and finite size of the system. We will study a natural derivation of non-extensive statistics in terms of Thermofractals. These are systems in thermodynamical equilibrium presenting the following properties [2, 7]:

1. Total energy is given by:

$$U = F + E \,, \tag{4}$$

   where $F \equiv$ kinetic energy, and $E \equiv$ internal energy of N' constituent subsystems.
2. Constituent particles are thermofractals: the distribution $P_{\rm TF}(E)$ is self-similar or self-affine ➞ at some level of the subsystem hierarchy $P_{\rm TF}(E)$ is equal to those in the other levels.
3. At level n the phase space is so narrow that one can consider $P_{\rm TF}(E_n) dE_n = \rho dE_n$.

- The energy distribution of a thermodynamical system is given, according to BG statistics, by

$$P(U) dU = A \exp(-U/kT) dU \,, \tag{5}$$

where A is a normalization constant. The phase space, in the case of thermofractals, must include momentum degrees of freedom of free particles as well as the internal degrees of freedom. According to property 2 of self-similar thermofractals [2], the internal energy is given by

$$dE = \frac{F}{kT} [P_{\rm TF}(\varepsilon)]^\nu d\varepsilon \,, \qquad \frac{\varepsilon}{kT} = \frac{E}{F} \,, \tag{6}$$

where ν is an exponent to be determined. Then, the total energy distribution is given by

$$P(U) dU = A' F^{\frac{3N}{2}-1} \exp\left(-\frac{\alpha F}{kT}\right) dF \, [P_{\rm TF}(\varepsilon)]^\nu d\varepsilon \,, \qquad \alpha = 1 + \frac{\varepsilon}{kT} \,, \tag{7}$$

with $N' = N + \frac{2}{3}$ an effective number of particles taking into account the internal degrees of freedom. After integration in F, the thermodynamical potential is given by

$$\Omega = \int dU P(U) = \int_0^\infty A \left[1 + \frac{\varepsilon}{kT}\right]^{-3N/2} \left[\tilde{P}(\varepsilon)\right]^\nu d\varepsilon \,, \qquad A = \Gamma\left[\frac{3}{2}N\right] (kT)^{\frac{3}{2}N} A' . \tag{8}$$

It is possible to impose the identity

$$P(U) \propto P_{\rm TF}(\varepsilon) \,, \tag{9}$$

corresponding to a self-similar solution for the thermofractal probability distribution. Then, the simultaneous solution for Eqs. (8) and (9) is obtained with [7, 8]

$$P_{\rm TF}(\varepsilon) = A \left[1 + \frac{\varepsilon}{kT}\right]^{-\frac{3N}{2}\frac{1}{1-\nu}} \quad \longrightarrow \quad P_{{\rm TF},(1)}(\varepsilon) = A_{(1)} e_q \left[-\frac{\varepsilon}{k\tau}\right] . \tag{10}$$

Of course one has $\varepsilon = \sum_{i=1}^{N'} \varepsilon_i^{(1)}$, so that at the first level of the thermofractal hierarchy one finds subsystems that are thermofractals with effective energies $\varepsilon^{(1)} \sim \varepsilon/N$. The distribution of thermofractals then obeys Tsallis statistics with $\tau = \frac{2(1-\nu)}{3} T$ and $q - 1 = \frac{2}{3N}(1-\nu)$.

3. Diagrammatic Representation and Callan-Symanzik Equation

- Thermofractals are scale invariant, and this should be accomplished with the scale invariance of the distribution of kinetic and internal energy. Then

$$\frac{F^{(0)}}{T^{(0)}} = \frac{F^{(n)}}{T^{(n)}} \quad \Longrightarrow \quad \lambda_n := \frac{E^{(n)}}{E^{(0)}} \left(\frac{1}{N}\right)^{\frac{n}{1-D}} , \tag{11}$$

where D is the fractal dimension. From the thermofractal structure one can obtain the fractal dimension of hadrons, resulting in $D = 0.69$ [2], a value close to that resulting from intermittence analysis [9]. It is possible to have a diagrammatic representation of the probability densities of thermofractals that can facilitate calculations of Ω and other relevant quantities [8]. The basic diagrams are summarized in Fig. 2.

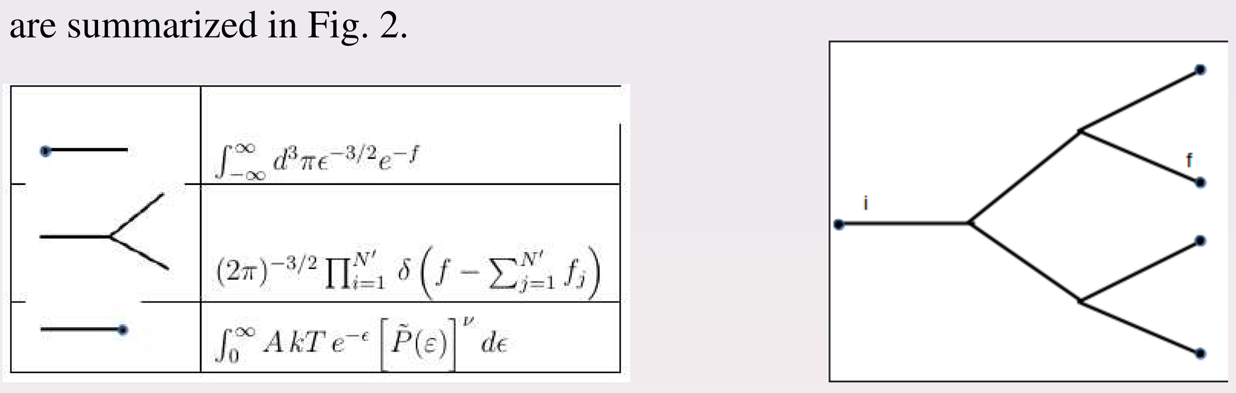

Fig. 2: *Left: Basic diagrams for thermofractals and their mathematical expressions. Right: Example of a tree graph representing different levels of a thermofractal.*

- On the other hand, the vertex function of thermofractals can be written in the form

$$\Gamma(E,\varepsilon,T) \propto (kT)^{-(1-D)} g \left[\prod_{i=1}^{N'} \left(2\pi \frac{E_i}{kT_i}\right)^{-3/2}\right] [P_{\rm TF}(\varepsilon_i)]^\nu . \tag{12}$$

Then one can derive the Callan-Symanzik equation for thermofractals, which writes

$$\left[M\frac{\partial}{\partial M} + \sum_{i=1}^{N'} \beta_i \frac{\partial}{\partial m_i} + \beta_{\bar g} \frac{\partial}{\partial \bar g} + \gamma\right] \Gamma = 0 \,, \tag{13}$$

where $m_i \equiv E_i$ is the thermofractal mass, which is identified with the thermofractal internal energy,

$$\beta_i = M \frac{\partial m_i}{\partial M} \,, \qquad \beta_{\bar g} = M \frac{\partial \bar g}{\partial M} \,, \tag{14}$$

and we have defined the effective coupling

$$\bar g(m,\varepsilon,t) = g \prod_{i=1}^{N'} \left[P_{\rm TF}\left(\frac{m(p_i) e^{t/d}}{M_0}\right)\right]^{\nu/2} , \qquad t := -d \log(M^2/M_0^2) . \tag{15}$$

4. Conclusions

- We have reviewed the non-extensive statistics in the form of Tsallis statistics of a quantum gas at finite T and μ, and applied it to study the EoS and phase diagram of QCD.
- We have reviewed the structure of a thermodynamical system presenting fractal properties, and shown that it naturally leads to non-extensive statistics.
- A diagrammatic formulation for practical calculations with the fractal structure was introduced.
- Based on the scale invariance of thermofractals, the Callan-Symanzik equation was obtained. This opens the opportunity to develop a *'field theoretical approach'* for thermofractals, leading to a possible theoretical understanding of the non-extensive properties of hadronic systems [10].

[1] C. Tsallis, *J. Stat. Phys.* **52** (1988) 479.
[2] A. Deppman, *Phys. Rev. D* **93**, 054001 (2016).
[3] E. Megías, D.P. Menezes and A. Deppman, *Physica A* **421**, 15 (2015).
[4] R. Hagedorn, *Lect. Notes Phys.* **221** (1985) 53.
[5] D.P. Menezes, A. Deppman, E.Megías and L.B. Castro, *Eur. Phys. J. A* **51**, no. 12, 155 (2015).
[6] L. Borland, *Phys. Lett. A* **245** (1998) 67.
[7] A. Deppman and E. Megías, *EPJ Web Conf.* **141**, 01011 (2017).
[8] A. Deppman, E. Megías, D.P. Menezes and T. Frederico, eprint arXiv:1801.01160.
[9] D. Ghosh *et al.*, *Phys. Rev. C* **58**, 3553 (1998).
[10] A. Deppman, E. Megías, D.P. Menezes and T. Frederico, in progress (2018).