Conformal Ward Identities and Couplings of QED and QCD to Gravity

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Abstract

We present a general study of 3-point functions of conformal field theory in momentum space, following a reconstruction method for tensor correlators, based on the solution of the conformal Ward identities (CWI's), introduced in recent works by Bzowski, McFadden and Skenderis (BMS). We investigate and detail the structure of the CWI's and their non-perturbative solutions, and compare them to perturbation theory, taking QED and QCD as examples.

Introduction

Exact results in conformal field theories (CFT's) have gathered a lot of attention along the years, mostly because the symmetry of such theories has been essential for determining the structure of the correlators, especially for 2- and 3-point functions. Such analysis have traditionally been performed in coordinate space, by imposing on correlation functions the corresponding conformal Ward identities (CWI's) that are solved far more easily in this case [1]. The solutions of the conformal constraints are determined up to few constants, which characterize the conformal class of a specific CFT. For instance the $\langle TJJ\rangle$ correlator can be expressed, in general d dimensions, as

$$\langle T^{\mu_1\nu_1}(x_1)J^{\mu_2\alpha_2}(x_2)J^{\mu_3\alpha_3}(x_3)\rangle=\frac{\delta^{\alpha_2\alpha_3}}{x_{12}^d x_{13}^d x_{23}^{d-2}}I_\alpha^{\mu_2}(x_{21};a)\,I_\beta^{\mu_3}(x_{31};a)\,t^{\mu_1\nu_1\alpha\beta}(X_{23};a)$$

showing that only one independent constant is left free to parametrize the correlator in d dimensions. For the $\langle TTT\rangle$ case

$$\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\nu_3}(x_3)\rangle=\frac{1}{x_{12}^d x_{13}^d x_{23}^d}\mathcal{I}^{\mu_1\nu_1,\alpha_1\beta_1}(x_{13})\,\mathcal{I}^{\mu_2\nu_2,\alpha_2\beta_2}(x_{23})\,t_{\alpha_1\beta_1\alpha_2\beta_2}{}^{\mu_3\nu_3}(X_{12}).$$

In the case of a Lagrangian realization, the constants are determined by the field content, i.e. the number of scalars, fermions, vectors etc. appearing in the Lagrangian, and for a sufficient number of independent family sectors and particle multiplicities, they are expected to saturate the exact solutions. The latter, obviously, are valid beyond perturbation theory and are parametrised by the same number of independent constants.

Reconstruction in Momentum Space

In momentum space this analysis becomes quite involved. The reconstruction method introduces a minimal set of form factors that can reconstruct the entire correlator. Starting from their transverse traceless parts it is possible to build up the entire correlator using the canonical Ward identities. For the $\langle TJJ\rangle$ for example

$$\begin{aligned}\langle\langle T^{\mu_1\nu_1}J^{\mu_2}J^{\mu_3}\rangle\rangle=&\langle\langle t^{\mu_1\nu_1}j^{\mu_2}j^{\mu_3}\rangle\rangle+\langle\langle T^{\mu_1\nu_1}j^{\mu_2}_{loc}J^{\mu_3}\rangle\rangle+\langle\langle T^{\mu_1\nu_1}J^{\mu_2}j^{\mu_3}_{loc}\rangle\rangle+\langle\langle t^{\mu_1\nu_1}_{loc}J^{\mu_2}J^{\mu_3}\rangle\rangle\\&-\langle\langle T^{\mu_1\nu_1}j^{\mu_2}_{loc}j^{\mu_3}_{loc}\rangle\rangle-\langle\langle t^{\mu_1\nu_1}_{loc}j^{\mu_2}_{loc}J^{\mu_3}\rangle\rangle-\langle\langle t^{\mu_1\nu_1}_{loc}J^{\mu_2}j^{\mu_3}_{loc}\rangle\rangle+\langle\langle t^{\mu_1\nu_1}_{loc}j^{\mu_2}_{loc}j^{\mu_3}_{loc}\rangle\rangle.\end{aligned}$$

where

$$j^\mu(p)=\pi^\mu_\alpha(p)J^\alpha(p)\qquad j^\mu_{loc}(p)=J^\mu(p)-j^\mu(p)\qquad \pi^\mu_\alpha=\delta^\mu_\alpha-\frac{p^\mu p_\alpha}{p^2}$$

$$t^{\mu\nu}(p)=\Pi^{\mu\nu}_{\alpha\beta}(p)T^{\alpha\beta}(p)\qquad t^{\mu\nu}_{loc}(p)=T^{\mu\nu}(p)-t^{\mu\nu}(p)\qquad \Pi^{\mu\nu}_{\alpha\beta}=\pi^{(\mu}_\alpha\pi^{\nu)}_\beta-\frac{1}{d-1}\pi^{\mu\nu}\pi_{\alpha\beta}.$$

The transverse and traceless part can be decomposed into simple tensors using a particular prescription of the momenta as

$$\begin{aligned}\langle t^{\mu_1\nu_1}(p_1)j^{\mu_2\alpha_2}(p_2)j^{\mu_3\alpha_3}(p_3)\rangle=\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)\pi^{\mu_2}_{\alpha_2}(p_2)\pi^{\mu_3}_{\alpha_3}(p_3)\big(&A_1^{a_2a_3}\,p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_1^{\alpha_3}+A_2^{a_2a_3}\,\delta^{\alpha_2\alpha_3}p_2^{\alpha_1}p_2^{\beta_1}\\&+A_3^{a_2a_3}\,\delta^{\alpha_1\alpha_2}p_2^{\beta_1}p_1^{\alpha_3}+A_3^{a_2a_3}(p_2\leftrightarrow p_3)\delta^{\alpha_1\alpha_3}p_2^{\beta_1}p_3^{\alpha_2}+A_4^{a_2a_3}\,\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\beta_1}\big),\end{aligned}$$

where the form factors depend on the momenta. It is worth mentioning that the construction of the transverse and traceless part is symmetric in all the momenta and indices. In the same way one can obtain the analogous decomposition for the $\langle TTT\rangle$ case. At this stage the main aim is to find the form factors A_i and for this reason one can exploit the action of the special conformal operator K^κ on the transverse and traceless part, obtaining a set of scalar equations of the form

$$\begin{array}{llll}K_{31}A_1=0 & K_{31}A_2=2A_1 & K_{31}A_3=-4A_1 & K_{31}A_4=-2A_3(p_2\leftrightarrow p_3)\\ K_{21}A_1=0 & K_{21}A_2=2A_1 & K_{21}A_3=0 & K_{21}A_4=-2A_3.\end{array}$$

These equations, called primary CWI's, are solved in terms of 3K integrals (i.e. integrals of three Bessel functions). Then, another set of conformal equations, called secondary CWI's, will fix the solutions modulo only one arbitrary constant for the $\langle TJJ\rangle$ case and two for $\langle TTT\rangle$ case. For instance

$$A_1=\alpha_1 J_{4\{000\}}=\alpha_1\int_0^\infty dx\,x^{\frac{d}{2}+3}\,p_1^{\frac{d}{2}}\,p_2^{\frac{d}{2}-1}\,p_3^{\frac{d}{2}-1}\,K_{\frac{d}{2}}(p_1x)\,K_{\frac{d}{2}-1}(p_2x)\,K_{\frac{d}{2}-1}(p_3x),$$

and so on for the other form factors. Notice that for odd spacetime dimensions d the Bessel functions K can be written as

$$K_\nu(x)=\sqrt{\frac{\pi}{2x}}\,e^{-x}\sum_{j=0}^{\lfloor|\nu|-1/2\rfloor}\frac{(|\nu|-1/2+j)!}{j!\,(|\nu|-1/2-j)!\,(2x)^j},\qquad \nu+1/2\in\mathbb{Z}.$$

We have showed in [2] how to solve this conformal equations in terms of generalized hypergeometric functions. For instance, in the case of A_1 form factor we obtain

$$A_1(p_1,p_2,p_3)=p_1^{\Delta-2d-4}\sum_{a,b}c^{(1)}(a,b)\,x^a y^b\,F_4(\alpha(a,b)+2,\,\beta(a,b)+2;\,\gamma(a),\,\gamma'(b);\,x,y).$$

The general solution of the CWI's depends on two independent constants for $\langle TTT\rangle$ and only one arbitrary constant for the $\langle TJJ\rangle$. The study of the correspondence between perturbative and non-perturbative solutions is performed, in perturbation theory, by the inclusion 2 sectors (fermion and scalar) in general d dimensions, with the addition of a gauge sector in $d=4$, depending on the specific correlator

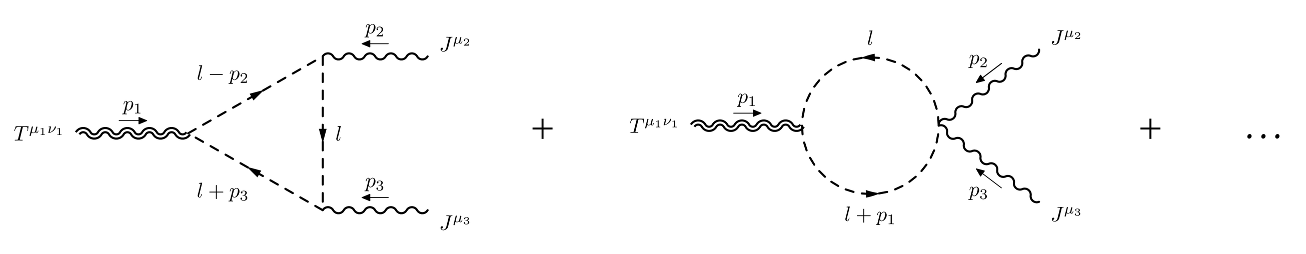
Perturbative solutions

The general solution, derived from the conformal constraints, gets very involved in the presence of divergences and requires an entirely new regularization procedure for such 3K integrals. Notice that such integrals are not the master integrals of perturbation theory, and cannot be handled by the ordinary reduction procedures which are typical of analysis in perturbative QCD at higher orders. This motivate us to reconsider the BMS reconstruction [4] by checking its equivalence to the perturbative results in order to proceed with their simplification. The matching between the non-perturbative and the perturbative solution, and the check of their equivalence, has been done by working in $d=3$ and $d=5$ dimensions. For instance, the conformal perturbative realizations for the Abelian $\langle TJJ\rangle$ can be done using the perturbative actions

$$S_{\text{Abelian}}=\int d^dx\,e\left[\frac{i}{2}e^\mu_a\bar\psi\gamma^a\overleftrightarrow{D}_\mu\psi-\frac14F_{\mu\nu}F^{\mu\nu}\right],\qquad D_\mu\psi=\left(\partial_\mu-i\bar eA_\mu+\frac18\left[\gamma^b,\gamma^c\right]e^\alpha_b\nabla_\mu e_{c\alpha}\right)\psi$$

$$S=\int d^dx\sqrt{-g}\left(|D_\mu\phi|^2+\frac{d-2}{8(d-1)}R|\phi|^2\right),\qquad D_\mu\phi=\left(\partial_\mu-i\bar eA_\mu\right)\phi$$

which allow to calculate the diagrams

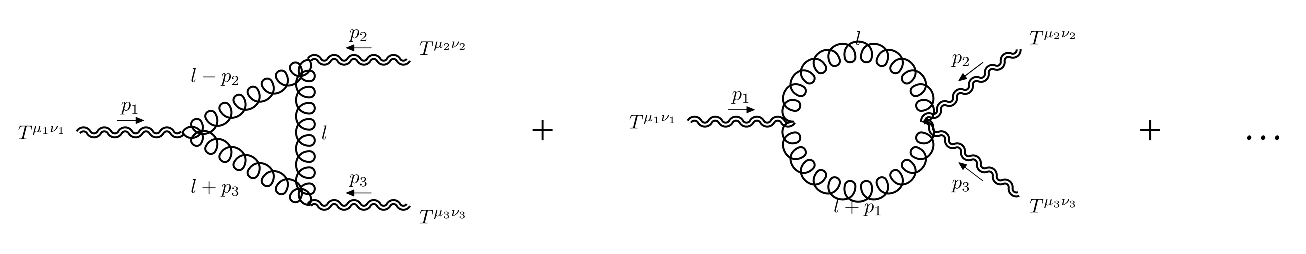


whose topologies appear both in the Abelian and in the non Abelian cases. The conformal perturbative realizations for the $\langle TTT\rangle$ in the Abelian and non Abelian correlator can be obtained from the actions

$$S_{\text{Abelian}}=\int d^dx\sqrt{-g}\left\{-\frac14F^{\mu\nu}F_{\mu\nu}-\frac{1}{2\xi}\left(\nabla_\mu A^\mu\right)^2+\partial^\mu\bar c\,\partial_\mu c+\frac12\left(g^{\mu\nu}\nabla_\mu\phi\nabla^\nu\phi-\chi R\phi^2\right)+\frac{i}{2}e^\mu_a\bar\psi\gamma^a\overleftrightarrow{D}_\mu\psi\right\}$$

$$\begin{aligned}S_{\text{Non-Abelian}}=\int d^dx\sqrt{-g}\Big\{&-\frac14F^{a\mu\nu}F^a_{\mu\nu}-\frac{1}{2\xi}\left(\nabla_\mu A^{a\mu}\right)^2\\&+i\nabla^\mu\bar c^aD^{ab}_\mu c^b+\left(|D_\mu\phi|^2+\frac{d-2}{4(d-1)}R|\phi|^2\right)+\frac{i}{2}e^\mu_a\bar\psi^i\gamma^a\overleftrightarrow{D}^{ij}_\mu\psi^j\Big\}\end{aligned}$$

by computing the diagrams



Notice that the gauge fields contribute only in the case of $d=4$. The explicit expressions of the form factors in $d=3$ and $d=5$ for the $\langle TJJ\rangle$ and $\langle TTT\rangle$ are calculated using the star-triangle relations in order to express the scalar integrals B_0 and C_0. For instance in $d=3$ we obtain

$$A_{1,TJJ}=\frac{\pi^3e^2(8n_F+n_S)\big(4p_1+p_2+p_3\big)}{3(p_1+p_2+p_3)^4},\qquad A_{1,CWI's}=\frac{2\alpha_1\big(4p_1+p_2+p_3\big)}{(p_1+p_2+p_3)^4}$$

where the expression on the left refers to the perturbative result and the one on the right is the non-perturbative one. Similar results hold for the $\langle TTT\rangle$ (three-graviton vertex).

Conclusions

The form factors computed in perturbation theory satisfy the same anomalous conformal Ward identities as the non-perturbative ones, in $d=4$. They both satisfy CWI's in general (d) dimensions. In $d=3$ and $d=5$ the two solutions completely match. We conclude that, at least for these correlation functions, **free field theory in momentum space at one loop provide the same information derived from the non-perturbative solutions, and the two can be freely interchanged, being equivalent.** This implies that there should be significant cancellations among the contributions of the 3K integrals or among those given by us in the form of hypergeometric functions, in such a way that they can be expressed in terms of the elementary master integrals B_0 and C_0, directly derived from perturbation theory.

References

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