Anomalous dimensions without Feynman diagrams

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July, 20 2017

The Renormalization Group approach to Wilson Fisher fixed points

- Two small parameters: the coupling constant g which turns on the interaction in the Lagrangian and $\epsilon = d_u - d$
- $d_u :=$ upper critical dimension,where the perturbation is marginal
- At $d < d_u$ the perturbation becomes slightly relevant at the Gaussian UV fixed point and the system flows to the infrared WF fixed point
- The anomalous dimensions of local operators are expressed in terms of a (scheme-dependent) loop expansion
- The vanishing of the Callan-Zymanzik $\beta(g)$ fixes the relation between g and ϵ and gives scheme-independent ϵ-expansions

The CFT approach to Wilson Fisher fixed points

- No Lagrangians
- One small parameter: $\epsilon = d_u - d$
- $d_u :=$ upper critical dimension, where the free field theory has a scalar primary with scaling dimension $\Delta = \frac{d+2}{2}$
- In $d = d_u - \epsilon$, and only there, there is a smooth conformal deformation of the free theory that can be identified with the WF fixed point

➡ This smoothness fixes uniquely, to the first non-trivial order in the ϵ-expansion, the anomalous dimensions and the OPE coefficients of infinite classes of scalar local operators

- A CFT in d dimensions is defined by a set of local operators $\{\mathcal{O}_k(x)\}$ $x \in \mathcal{R}^d$ and their correlation functions

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$$

- Local operators can be multiplied. Operator Product Expansion:

$$\mathcal{O}_1(x)\mathcal{O}_2(y) = \sum_k c_{12k}(x-y)\mathcal{O}_k(\frac{x+y}{2})$$

- $\mathcal{O}_{\Delta,\ell,f}(x)$ are labelled by a scaling dimension Δ

$$\mathcal{O}_{\Delta,\ell,f}(\lambda x) = \lambda^{-\Delta}\mathcal{O}_{\Delta,\ell,f}(x)$$

  an $SO(d)$ representation ℓ (spin), and possibly a flavor index f
- among the local operators there are the identity and a (unique) energy -momentum tensor $T_{\mu\nu}(x) = \mathcal{O}_{d,2}(x)$
- ⇨ a CFT has no much to do with Lagrangians, coupling constants and equations of motion, even if one often uses these terms for simplicity

- ❈ Acting with the Lie algebra of $SO(d+1,1)$ on a local operator generates a whole representation of the conformal group. The local operator of minimal Δ is said a primary, the others are descendants
- ❈ Not all the primaries define irreducible representations:
- ❈ There are primaries admitting an invariant subspace: there is a descendant which is also primary. It corresponds to a null state
- ➯ Denoting with $[\Delta, \ell]$ a null state and with $[\Delta', \ell']$ its parent primary, in view of the fact that they belong to the same representation, they must share the eigenvalues $c_2, c_4, \ldots$ of all the Casimir operators $C_2, C_4, \ldots$
  $c_2(\Delta, \ell) = c_2(\Delta', \ell')\ ;\ c_4(\Delta, \ell) = c_4(\Delta', \ell')\ ;\ \ldots$
- ❈ since $[\Delta, \ell]$ and $[\Delta', \ell']$ belong to the same rep.➯ $\Delta = \Delta' + n$ and the first two eq.s fix uniquely the possible pairs

❄ Eigenvalues of the Casimir operators

$$c_2(\Delta,\ell)=\frac{1}{2}\Delta(\Delta-d)+\ell(\ell+d-2)$$

$$c_4(\Delta,\ell)=\Delta^2(\Delta-d)^2+\tfrac{1}{2}d(d-1)\Delta(\Delta-d)+\ell^2(\ell+d-2)^2$$
$$+\tfrac{1}{2}(d-1)(d-4)\ell(\ell+d-2)$$

➩ There are three families of null states:

Parent primary	Descendant primary		
Δ'_k	Δ_k	ℓ	
$1-\ell'-k$	$1-\ell+k$	$\ell'+k$	$k=1,2,\dots$
$\frac{d}{2}-k$	$\frac{d}{2}+k$	ℓ'	$k=1,2,\dots$
$d+\ell'-k-1$	$d+\ell+k-1$	$\ell'-k$	$k=1,2,\dots,\ell$

➩ The canonical scalar field ϕ_f of scaling dimension $\Delta_{\phi_f}=\frac{d}{2}-1$ could have a primary descendant of dimension $\Delta=\frac{d}{2}+1$

CFT, useful formulae

- The four-point function of local scalar operators $\mathcal{O}_i(x)$ in a d-dimensional CFT can parametrised as

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{\Delta^+_{12}}|x_{34}|^{\Delta^+_{34}}} \left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta^-_{12}} \left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta^-_{34}}$$

$\Delta^{\pm}_{ij} = \Delta_i \pm \Delta_j$ and Δ_i scaling dimension of $\mathcal{O}_i$, $g(u,v)$ is a function of the cross-ratios $u = \frac{x^2_{12}x^2_{34}}{x^2_{13}x^2_{24}}$ and $v = \frac{x^2_{14}x^2_{23}}{x^2_{13}x^2_{24}}$

- $g(u,v)$ can be expanded in terms of *conformal blocks* $G^{a,b}_{\Delta,\ell}(u,v)$ (eigenfunctions of the Casimir operators $C_2, C_4, \ldots$ of $SO(d+1,1)$) :

$$g(u,v) = \sum_{\Delta,\ell} \mathrm{p}_{\Delta,\ell} G^{a,b}_{\Delta,\ell}(u,v).$$

$a = -\frac{\Delta^-_{12}}{2};\ b = \frac{\Delta^-_{34}}{2}$

WF fixed points

A CFT in $d-\epsilon$, defined by a set of local operators $\mathcal{O}_i$, is a *smooth deformation* of the free field theory in d dimensions if

1. $\exists\, \mathcal{O}_i \leftrightarrow \mathcal{O}^f_i \,:\, \Delta_{\mathcal{O}_i} = \Delta_{\mathcal{O}^f_i} + \gamma_i^{(1)}\epsilon + \gamma_i^{(2)}\epsilon^2 + \dots$
2. $\mathcal{O}^f_i \times \mathcal{O}^f_j = \sum_k \lambda^f_{ijk}\mathcal{O}^f_k,\;\; \mathcal{O}_i \times \mathcal{O}_j = \sum_k(\lambda^f_{ijk} + O(\epsilon))\mathcal{O}_k$

- This definition does not imply that primary operators of free theory are also primary in the deformed CFT.
- For general space dimensions d the deformations $\Delta_{\mathcal{O}} \to \Delta_{\mathcal{O}_f} + O(\epsilon)$ do not define a one-to-one correspondence with the spectrum of the free theory hence they are not smooth deformations
- Consistent smooth deformations exist only at special values of d (upper critical dimension), reduce the number of primaries for $\epsilon \neq 0$ and define WF fixed points
- They are encoded in the analytic properties of conformal blocks $G^{a,b}_{\Delta,\ell}$ as functions of Δ

Poles and null states

- The conformal blocks can be written as a sum of poles (+ an entire function in the whole Δ complex plane)
- Poles only occur at the special $[\Delta'_k, \ell']$ primaries admitting a primary descendant, i.e. a null state.
- The residue of the pole is proportional to a conformal block:

$$G^{a,b}_{\Delta',\ell'} \sim r(\Delta'_k) \frac{G^{a,b}_{\Delta_k,\ell}}{\Delta' - \Delta'_k}$$

- The complete list of the null states for general d coincides with the three families listed in the previous table.

The scalar null state at $\Delta_1 = \frac{d}{2} + 1 = \Delta_{\phi_f} + 2$

$$G^{a,b}_{\Delta'} = r(\Delta_{\phi_f})\frac{G^{a,b}_{(\Delta_{\phi_f}+2)}}{\Delta'-\Delta_{\phi_f}} + \text{rest}, \quad r(\Delta_{\phi_f}) = \frac{(\Delta^2_{\phi_f}-\Delta^2_{12})(\Delta^2_{\phi_f}-\Delta^2_{34})}{4d(d-2)}$$

In a free field theory this primary descendant has always a vanishing residue in all the possible OPEs that generate ϕ_f:

$[\phi_f^p] \times [\phi_f^{p\pm1}] = \sqrt{p\pm 1}[\phi_f] + \dots$ $\qquad$ $([\phi_f]^p = \phi_f^p/\sqrt{p!})$

$\Rightarrow \; \Delta^-_{12} = \pm\Delta_{\phi_f} \quad \Rightarrow r(\Delta_{\phi_f}) = 0$

Turning on the interaction in $d-\epsilon$, i.e. putting $\phi_f^n \to \phi^n$ with

$\Delta_{\phi^n} = \Delta_{\phi_f^n} + \gamma_n^{(1)}\epsilon + \gamma_n^{(2)}\epsilon^2 + \dots \Rightarrow r(\Delta') \neq 0 \Rightarrow$

$$G^{a,b}_{\Delta_\phi} = \frac{(d-2)(\gamma_p^{(1)}-\gamma_{p\pm1}^{(1)})^2\epsilon^2}{4d(\gamma_\phi^{(1)}\epsilon+\gamma_\phi^{(2)}\epsilon^2+\dots)}G^{a_f,b_f}_{\Delta_{\phi_f+2}} + \dots$$

For general d this is not a smooth deformation since the the free theory does not have a local operator of dimension Δ_{ϕ_f+2} unless there is a primary ϕ_f^n with that dimension, i.e.

$$n\,\Delta_{\phi_f} = \Delta_{\phi_f} + 2\,, \Rightarrow \quad d = 2\frac{n+1}{n-1} \;:\; \text{only 3 solutions with integer d}$$

$$(d=3, n=5), \;\; (d=4, n=3), \;\; (d=6, n=2)$$

❋ Matching the coefficient of $G^{a_f,b_f}_{\Delta_{\phi_f+2}}$ with that of $G^{a_f,b_f}_{\Delta_{\phi_f^n}}$ we obtain constraints among anomalous dimensions $\gamma^{(i)}_n = \gamma^{(i)}_{\phi^n}$

$d = 4 \quad :$

$$[\phi_f] \times [\phi_f^2] = \sqrt{2}[\phi_f] + \sqrt{3}[\phi_f^3] + \text{spinning operators}$$

$$\langle \phi_f \phi_f^2 \phi_f \phi_f^2 \rangle \Rightarrow g_f(u,v) = 2G^{a_f,b_f}_{\Delta_{\phi_f}} + 3G^{a_f,b_f}_{\Delta_{\phi_f^3}} + \text{spinning conf. blocks}$$

$$\langle \phi\, \phi^2 \phi\, \phi^2 \rangle \Rightarrow g(u,v) = (2 + O(\epsilon))G^{a,b}_{\Delta_\phi} + \dots$$

$$\lim_{\epsilon \to 0} g(u,v) = 2\left(G^{a_f,b_f}_{\Delta_{\phi_f}} + \frac{\epsilon(\gamma^{(1)}_{\phi^2})^2}{8(\gamma^{(1)}_\phi + \epsilon\gamma^{(2)}_\phi)} G^{a_f,b_f}_{\Delta_{\phi_f+2}=\Delta_{\phi_f^3}} \right) + \text{spinning conf. b.}$$

$$\gamma^{(1)}_\phi = 0\,, \qquad \frac{(\gamma^{(1)}_{\phi^2})^2}{\gamma^{(2)}_\phi} = 12.$$

In the general case from the fusion rule

$$[\phi^p] \times [\phi^{p+1}] = \sqrt{p+1}\,\left([\phi] + \sqrt{\frac{3}{2}}p\,[\phi^3] + \sqrt{\frac{5}{6}}p(p-1)\,[\phi^5]\right) + \dots$$

we get in the $d = 4$ case the recursion relation

$$\frac{(\gamma^{(1)}_{p+1} - \gamma^{(1)}_p)^2}{\gamma^{(2)}_\phi} = 12\,p^2$$

$\gamma^{(1)}_0 = \gamma^{(1)}_1 = 0 \quad \Rightarrow$

$$\gamma^{(1)}_{\phi^p} \equiv \gamma^{(1)}_p = \frac{\kappa_4}{2}\,p(p-1), \;\; \kappa_4 = \pm\sqrt{12\gamma^{(2)}_\phi}$$

and similarly in $d = 3$

$$\gamma^{(1)}_{\phi^p} \equiv \gamma^{(1)}_p = \frac{\kappa_3}{3}\,p(p-1)(p-2), \;\; \kappa_3 = \pm\sqrt{10\gamma^{(2)}_\phi}$$

In $d = 4$ there is another way to calculate $\gamma^{(1)}_{\phi^3} = 3\kappa_4$

- ❋ The scaling dimensions of the null states are universal and depend only on d
- ❋ in $d = 4 - \epsilon$ the primary descendant of ϕ_f has scaling dimensions $\Delta_{\phi_f} + 2 = 3 - \epsilon/2$ which should coincide with the scaling dimensions of ϕ^3
- ❋ the smooth deformation requires $\Delta_{\phi^3} = 3\Delta_{\phi_f} + \gamma^{(1)}_{\phi^3}\epsilon = 3 + (\gamma^{(1)}_{\phi^3} - \frac{3}{2})\epsilon + O(\epsilon^2)$

➾ $\gamma^{(1)}_{\phi^3} = 1$, then $\kappa_4 = \frac{1}{3}, \quad \gamma^{(2)}_{\phi} = \frac{1}{108}$

Similarly in $d = 3$ ➾ $\gamma^{(1)}_{\phi^5} = 20\kappa_3$, matching with the primary descendant of ϕ yields $\gamma^{(1)}_{\phi^5} = 2$, thus

$$\kappa_3 = \frac{1}{10}, \quad \gamma^{(2)}_{\phi} = \frac{1}{1000}$$

- ❈ All these results in $d = 4$ and $d = 3$ coincide with those obtained with Feynman diagrams in quantum field theory

OPE coefficients in $d = 4$

Other results can be obtained by considering deformations of OPE free theories in which a ϕ_f^3 contribution on the RHS appears

$$[\phi_f^2] \times [\phi_f^5] = \sqrt{10}[\phi_f^3] + 5\sqrt{2}[\phi_f^5] + \sqrt{21}[\phi_f^7] + \text{spinning op.}$$

or

$$[\phi_f] \times [\phi_f^4] = 2[\phi_f^3] + \sqrt{5}[\phi_f^5] + \text{spinning op.}$$

the ϕ_f^3 contribution should be replaced by the conformal block of ϕ in the deformed theory.

$$\lambda^2_{\phi^2\phi^5\phi} = 5\gamma_\phi^{(2)}\epsilon^2 + O(\epsilon^3) = \frac{5}{108}\epsilon^2 + O(\epsilon^3);$$
$$\lambda^2_{\phi\phi^4\phi} = 2\gamma_\phi^{(2)}\epsilon^2 + O(\epsilon^3) = \frac{1}{54}\epsilon^2 + O(\epsilon^3)$$

Generalizations

* For any generalized free field of dimension $\Delta_\phi = \frac{d}{2} - k$ and any integer m one can define an upper critical dimension $d_u = 2k\, m/(m-1)$ (in general a fractional number) in which
  - ϕ^{2m} is a marginal perturbation
  - in $d_u - \epsilon$ there is a (generalized) WF critical point characterized by the following spectrum of anomalous dimensions

$$\gamma_p^{(1)} = \frac{m-1}{(m)_m}(p-m+1)_m\,, \quad (p>1)$$

$$\gamma_\phi^{(2)} = (-1)^{k+1} 2 \frac{m\left(\frac{k}{m-1}\right)_k}{k\left(\frac{m\,k}{m-1}\right)_k}(m-1)^2\left[\frac{(m!)^2}{(2m)!}\right]^3 \qquad (1)$$

$O(N)$- invariant models

- generalized free theories with scalar fields $\phi_i, i = 1, 2, \ldots, N$ transforming as vectors under $O(N)$
- $\gamma^{(i)}_{p,s} \equiv$ anomalous dimensions of symmetric traceless rank-s tensors $\phi^{2p}\,\phi_{i_1}\phi_{i_2}\ldots\phi_{i_s} - \mathrm{traces}$

➯ for $d_u = 4k$ $\gamma^{(1)}_{p,s} = \frac{s(s-1)+p(N+6(p+s)-4)}{N+8}$, $\gamma^{(2)}_{\phi} = \frac{(-1)^{k+1}(k)_k(N+2)}{2k(2k)_k(N+8)^2}$

➯ for $d_u = 3k$

$$\gamma^{(1)}_{p,s} = \frac{(2p+s-2)(s(s-1)+p(3N+10(p+s)-8))}{3(3N+22)}$$

$$\gamma^{(2)}_{\phi} = \frac{(-1)^{k+1}(k/2)_k(N+2)(N+4)}{8k(3k/2)_k(3N+22)^2}$$

Conclusions

1. It is possible to define smooth deformations and Wilson Fisher fixed points in $d-\epsilon$ only using CFT notions, with no reference to Lagrangians, coupling constants or equations of motion
2. $O(N)$ symmetric models and generalized free fields allow to define a more general class of WF fixed points
3. Simple constraints on anomalous dimensions and OPE coefficients up to $O(\epsilon^2)$ are easily obtained. Higher order calculations require more constraints from conformal bootstrap equations.

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