Some exact Bradlow vortex solutions

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Based on arXiv:1701.04356

IMP, CAS

2017年7月20日

The Zoo

soliton	codimension	homotopy group
domain wall	1	$\pi_0(\mathcal{M}_{\text{vacuum}})$
vortex	2	$\pi_1(G) \simeq \mathbb{Z}$
monopole	3	$\pi_2(G/H) \simeq \mathbb{Z}$
instanton	4	$\pi_3(G) \simeq \mathbb{Z}$
Skyrmion	3	$\pi_3\left(\frac{SU(2)\times SU(2)/\mathbb{Z}_2}{SU(2)}\right) \simeq \mathbb{Z}$
baby-Skyrmion	2	$\pi_2(S^2) \simeq \mathbb{Z}$
Hopfions	3	$\pi_3(S^2) \simeq \mathbb{Z}$
$\vdots$	$\vdots$	$\vdots$

Plan of the talk

- five vortex equations
- integrability on constant-curvature manifolds
- Witten's solution
- Baptista metric
- Bradlow bound(s)
- Bradlow vortices

Warm-up: the Abelian Higgs model

The Abelian Higgs model

Consider 2+1 dimensions and

$$-\mathcal{L} = \frac{1}{4e^2}F_{\mu\nu}^2 + |D_\mu\phi|^2 + \frac{e^2}{2}(|\phi|^2 - v)^2, \tag{1}$$

with

$$D_\mu\phi = \partial_\mu + iA_\mu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$

Bogomol'nyi trick for the (static) energy density:

$$\mathcal{E} = \frac{1}{2e^2}\left(F_{12}^2 - e^2(|\phi|^2 - v^2)\right)^2 + |D_1\phi + iD_2\phi|^2 - v^2F_{12} - i\epsilon^{ij}\partial_i(\phi^\dagger D_j\phi) \tag{3}$$

BPS-equations:

$$F_{12} = e^2(|\phi|^2 - v^2), \qquad D_1\phi + iD_2\phi = 0. \tag{4}$$

BPS-bound:

$$E \geq -v^2\int d^2x\, F_{12}. \tag{5}$$

Axial vortices (building intuition)

Ansatz:

$$\phi = vh(r)e^{iN\phi}, \qquad A_i = \epsilon_{ij}\frac{Nx^j}{r^2}a(r). \tag{6}$$

Boundary conditions:

$$h(0) = 0, \quad h(\infty) = 1, \quad a(0) = 0, \quad a(\infty) = 1, \tag{7}$$

BPS-equations:

$$h' = \frac{N}{r}(1-a)h, \qquad \frac{Na'}{r} = e^2v^2(1-h^2). \tag{8}$$

Polar coordinates: $x^1 + ix^2 = re^{i\theta}$;
Field strength:

$$F_{12} = -\frac{Na'}{r}. \tag{9}$$

Master equation: (axially symmetric) Taubes equation

$$-u'' - \frac{1}{r}u' = m^2(e^{2u} - 1). \tag{10}$$

with $u \equiv \log h$ and $m \equiv ev$.

General solution to 1st BPS equation:

$$\bar{D}\phi = \bar{\partial}\phi + i\bar{A}\phi = (-s^{-1}\bar{\partial}s + i\bar{A})vs^{-1}\phi_0(z) = 0, \tag{11}$$

where ϕ_0 is holomorphic (and ϕ is "covariantly" holomorphic):

$$\phi = vs^{-1}(z, \bar{z})\phi_0(z). \tag{12}$$

$\bar{A}$ is given by the Maurer-Cartan form

$$\bar{A} = -i\bar{\partial}\log s = i\bar{\partial}\psi. \tag{13}$$

Field strenth:

$$F_{12} = 2iF_{\bar{z}z} = 4\bar{\partial}\partial\psi. \tag{14}$$

Master eqaution:

$$4\bar{\partial}\partial\psi = m^2(e^{2\psi}|\phi_0(z)|^2 - 1), \tag{15}$$

Setting $e^{2\psi}|\phi_0(z)|^2 = e^{2u}$ we get

$$4\bar{\partial}\partial u = m^2(e^{2u} - 1) + 2\pi\sum_{i=1}^{N}\delta^{(2)}(z - z_i). \tag{16}$$

Vortex equation on curved surface

Assuming a compatible Riemannian metric of the form

$$ds^2 = -dt^2 + \Omega_0(z,\bar{z})dzd\bar{z} \tag{17}$$

The energy changes as

$$E = \int d^2x\, \Omega_0 \left[\frac{1}{4e^2\Omega_0^2}F_{\mu\nu}^2 + \Omega_0^{-1}|D_\mu\phi|^2 + \frac{e^2}{2}(|\phi|^2 - v^2)\right]. \tag{18}$$

Thus the static energy:

$$\begin{aligned} E^{\text{static}} = \int d^2x\, \Omega_0 \Bigg[&\frac{1}{2e^2}\left(\Omega_0^{-1}F_{12} - e^2(|\phi|^2 - v^2)\right)^2 \\ &+ \Omega_0^{-1}|D_1\phi + iD_2\phi|^2 - v^2\Omega_0^{-1}F_{12} \\ &- i\Omega_0^{-1}\partial_i(\phi^\dagger D_j\phi)\Bigg]. \end{aligned} \tag{19}$$

Giving the BPS equations on curved background:

$$\Omega_0^{-1}F_{12} - e^2(|\phi|^2 - v^2) = 0, \qquad D_1\phi + iD_2 = 0, \tag{20}$$

Master equation on curved background

Taubes equation on curved background:

$$\frac{4}{\Omega_0}\bar{\partial}\partial u = m^2(e^{2u} - 1) + \frac{2\pi}{\Omega_0}\sum_{i=1}^{N}\delta^{(2)}(z - z_i). \tag{21}$$

Witten's solution:

$$2u = v - \log\Omega_0 \tag{22}$$

yielding

$$2\bar{\partial}\partial(v - \log\Omega_0) = m^2\Omega_0(\Omega_0^{-1}e^v - 1) + 2\pi\sum_{i=1}^{N}\delta^{(2)}(z - z_i) \tag{23}$$

which we can write as

$$2\bar{\partial}\partial\log\Omega_0 = m^2\Omega_0, \qquad 2\bar{\partial}\partial v = m^2 e^v + 2\pi\sum_{i=1}^{N}\delta^{(2)}(z - z_i). \tag{24}$$

Both geometry and the vortices are determined by Liouville's equation.

Solving Liouville's equation yields:

$$\Omega_0 = \frac{4}{m^2(1-|z|^2)^2}, \tag{25}$$

for the background geometry and

$$e^v = \frac{4}{m^2(1-|g(z)|^2)^2}\left|\frac{dg}{dz}\right|^2, \tag{26}$$

vortex positions $\Leftrightarrow$ ramification points of g.
Vortex condensate:

$$|\phi|^2 = \Omega_0^{-1}e^v = \frac{(1-|z|^2)^2}{(1-|g(z)|^2)^2}\left|\frac{dg}{dz}\right|^2, \tag{27}$$

In the Poincare disc model, solutions are given by g being the Blaschke rational function

$$g(z) = \prod_{i=1}^{N+1}\frac{z-a_i}{1-\bar{a}_i z}. \tag{28}$$

Generalizing the Taubes equation

$$-\frac{4}{\Omega_0}\bar{\partial}\partial u = m^2 - m^2 e^{2u} - \frac{2\pi}{\Omega_0}\sum_{i=1}^{N}\delta^{(2)}(z - z_i). \tag{29}$$

$\Rightarrow$ Manton's five vortex equations:

$$-\frac{4}{\Omega_0}\bar{\partial}\partial u = -C_0 + Ce^{2u} + \frac{2\pi}{\Omega_0}\sum_{i=1}^{N}\delta^{(2)}(z - z_i). \tag{30}$$

By rescaling Ω_0 and shifting u, we can reduce the possibilities to

$$\{C_0, C\} = \{-1, 0, 1\}, \tag{31}$$

four of which

$$(C_0, C) = (1, -1),\ (0, 0),\ (1, 0),\ (0, -1), \tag{32}$$

cannot give a positive magnetic flux $F_{12} > 0$.

Manton's five vortex equations

Table: Vortex equation constants C_0 and C for five different theories.

C_0	C	name	analytic solutions on
-1	-1	Taubes	$\mathbb{H}^2$
0	1	Jackiw-Pi	$\mathbb{R}^2$, T^2
1	1	Popov	S^2
-1	0	Bradlow	$\mathbb{H}^2$
-1	1	Ambjørn-Olesen-Manton	$\mathbb{H}^2$

Taubes: Instantons on $\mathbb{H}^2 \times S^2 \Rightarrow$ vortex on $\mathbb{H}^2$.

Popov: Instantons on $\mathbb{H}^2 \times S^2 \Rightarrow$ vortex on S^2 **and** change the overall sign of RHS.

Ambjørn-Olesen-Manton equation

- exact analytic solution on $\mathbb{H}^2$
- period solution on $\mathbb{R}^2$
- describes W condensation in the electroweak theory
- can play a role in non-Abelian vector bootstrap mechanism generating a primordial magnetic field

Finding analytic (integrable) vortex solutions

Consider the background Gaussian curvature:

$$K_0 = -\frac{2}{\Omega_0}\bar{\partial}\partial \log \Omega_0. \tag{33}$$

Now define the (singular) **Baptista** metric

$$\Omega \equiv \Omega_0 e^{2u}, \tag{34}$$

corresponding curvature:

$$K = -\frac{2}{\Omega}\bar{\partial}\partial \log \Omega = -\frac{2}{e^{2u}\Omega_0}\bar{\partial}\partial \left(\log \Omega_0 + 2u\right), \tag{35}$$

multiply by e^{2u} to arrive at:

$$e^{2u}K = K_0 - \frac{4}{\Omega_0}\bar{\partial}\partial u. \tag{36}$$

Use the vortex equation (ignoring delta functions from now on):

$$e^{2u}K = K_0 - C_0 + Ce^{2u}. \tag{37}$$

Finding analytic (integrable) vortex solutions

Rearranging:

$$(K - C)e^{2u} = K_0 - C_0, \tag{38}$$

multiply by Ω_0 to finally arrive at:

$$(K - C)\Omega = (K_0 - C_0)\Omega_0. \tag{39}$$

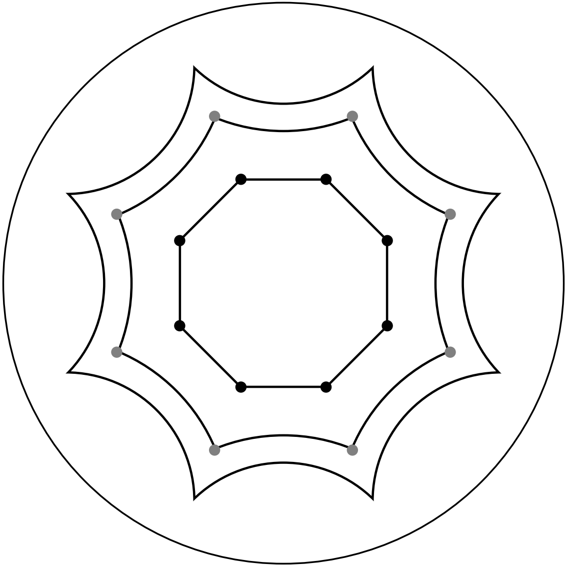
(Known) Integrability $\Leftrightarrow$

$$K = C, \qquad K_0 = C_0, \tag{40}$$

Baptista curvature $= C$; and background curvature C_0.

General solution:

$$e^{2u} = \frac{(1 + C_0|z|^2)^2}{(1 + C|g(z)|^2)^2} \left| \frac{dg}{dz} \right|^2 . \qquad (41)$$



Energy giving rise to the general equation

Energy:

$$E = \int d^2x\, \Omega_0 \left[\frac{1}{4e^2\Omega_0^2} F_{\mu\nu}^2 - C\Omega_0^{-1}|D_\mu\phi|^2 + \frac{e^2}{2}\left(C|\phi|^2 - C_0 v^2\right) \right]. \quad (42)$$

Static energy:

$$E^{\text{static}} = \int d^2x\, \Omega_0 \Bigg[\frac{1}{2e^2} \left(\Omega_0^{-1} F_{12} + e^2 (C|\phi|^2 - C_0 v^2) \right)^2 - C\Omega_0^{-1}|D_1\phi + iD_2\phi|^2 + C_0\, v^2 \Omega_0^{-1} F_{12} + Ci\Omega_0^{-1}\partial_i(\phi^\dagger D_j \phi) \Bigg]. \quad (43)$$

Giving the BPS equations on curved background:

$$\Omega_0^{-1} F_{12} + C\, e^2 (|\phi|^2 - v^2) = 0, \qquad D_1\phi + iD_2 = 0, \quad (44)$$

Notice: the second BPS equation does not exist for $C = 0$.

Integrating the general vortex equation:

$$2\pi N = -C_0 A_0 + CA, \tag{45}$$

where

$$A = \int d^2x\ \Omega = \int d^2x\ \Omega_0 e^{2u}, \tag{46}$$

is the Baptista area.
Taubes equation $\Rightarrow$

$$A_0 \leq 2\pi N \tag{47}$$

called the **Bradlow** bound.
Bradlow equation $\Rightarrow$

$$A_0 = 2\pi N. \tag{48}$$

Amjørn-Olesen-Manton equation $\Rightarrow$

$$A_0 \geq 2\pi N \tag{49}$$

Jackiw-Pi equation $\Rightarrow$

$$A = 2\pi N, \tag{50}$$

i.e. the Baptista area is equal to 2π times the vortex number.

So far **all** exact solutions have been found are for constant background curvature: $K_0 = \text{const}$.

But... the Bradlow equation is very simple...

The Bradlow equation reads:

$$-4\bar{\partial}\partial u = \Omega_0 - 2\pi \sum_{i=1}^{N} \delta^{(2)}(z - z_i), \tag{51}$$

Notice that the field is directly related to the background geometry.

Axially symmetric Bradlow vortex on $\mathbb{D}^2$

Let's consider axial symmetry and a disc:

$$u = -\frac{r^2}{4} + u_0 + N \log r, \tag{52}$$

Fixing the boundary condition $u = 0$:

$$u = -\frac{r^2 - R^2}{4} + N \log \frac{r}{R}. \tag{53}$$

Generalized Bradlow bound yields:

$$N = \frac{1}{2} R^2. \tag{54}$$

which fixed the radius in terms of number of vortices N.

The general solution can readily be found:

$$u = -\frac{|z|^2}{4} + u_0 + \frac{1}{2}\sum_{i=1}^{N} \log|z - z_i|^2 + g(z) + \overline{g(z)}, \qquad (55)$$

However, it is impossible to impose $u = 0$ at the boundary of a circular disc.
Attempting yields

$$-u_0 = \frac{1}{2}\sum_{i=1}^{N} \log|Re^{i\theta} - z_i|^2 = \frac{1}{2}\sum_{i=1}^{N} \log\left|\frac{Rz}{|z|} - z_i\right|^2, \qquad (56)$$

and $g = \bar{g} = 0$.
But, still BC are not satisfied..

Conjecture

The only solution satisfying the Bradlow equation on the flat disc, $\mathbb{D}^2$*, with a finite radius* $R < \infty$ *and the boundary condition* $u(R) = 0$*, is the axially symmetric solution (see above) where all* $z_i = 0$*,* $\forall i$*.*

Assume a large disc and consider:

$$u = \frac{R^2 - |z|^2}{4} + \frac{1}{2}\sum_{i=1}^{N} \log \frac{|z - z_i|^2}{R^2}, \tag{57}$$

then u at the boundary reads

$$u(|z| = R) = \frac{1}{2}\sum_{i=1}^{N} \log \left| \frac{z}{|z|} - \frac{z_i}{R} \right|^2 \simeq -\frac{1}{2}\sum_{i=1}^{N} \left[\frac{z_i|z|}{Rz} + \frac{\bar{z}_i|z|}{R\bar{z}} + \mathcal{O}\left(\frac{|z_i|^2}{R^2}\right) \right], \tag{58}$$

which becomes small for $R \gg |z_i|, \forall i$.

$$\begin{aligned} E &= \int_{M_0} d^2x\, \Omega_0 \left\{ \frac{1}{2e^2\Omega_0^2} F_{12}^2 + \Omega_0^{-1}|D_a\phi|^2 + \Omega_0^{-1}|\phi|^2 F_{12} + \frac{1}{2}e^2v^4 \right\} \\ &= \int_{M_0} d^2x\, \Omega_0 \Bigg\{ \frac{1}{2e^2}\left(\Omega_0^{-1}F_{12} + e^2v^2\right)^2 + \Omega_0^{-1}|D_1\phi + iD_2\phi|^2 \\ &\qquad - i\Omega_0^{-1}\epsilon^{ab}\partial_a(\bar{\phi}D_b\phi) \Bigg\} \\ &\quad - v^2 \int_{M_0} d^2x\, F_{12}, \end{aligned} \tag{59}$$

BPS-equation:

$$D_1\phi + iD_2\phi \equiv 2D_{\bar{z}}\phi = 0, \tag{60}$$

$$-\frac{1}{\Omega_0}F_{12} = m^2, \tag{61}$$

Total energy:

$$E = v^2m^2A_0 = e^2v^4A_0 = 2\Lambda A_0, \tag{62}$$

Boundary term for the disc solution

$$
\begin{aligned}
&- i \int_D d^2x\, \epsilon^{ab} \partial_a(\bar{\phi} D_b \phi) = 2 \int_D d^2x\, \partial_{\bar{z}}(e^{2u} \partial_z u) \\
&= -i \oint_{\partial D} dz\, e^{\frac{R^2-|z|^2}{2}} \prod_{j=1}^{N} \frac{|z-z_j|^2}{R^2} \left(-\frac{\bar{z}}{4} + \sum_{i=1}^{N} \frac{1}{2(z-z_i)} \right) . \qquad (63)
\end{aligned}
$$

is in general complicated;
for $N = 1$, it simplifies:

$$
2\pi \left[\left(1 + \frac{|z_1|^2}{R^2} \right) \left(-\frac{R^2}{4} + \frac{1}{2} \right) - \frac{|z_1|^2}{2R^2} \right] = -\frac{\pi |z_1|^2}{2}, \qquad (64)
$$

which can be seen to vanish for $z_i = 0\ \forall i$.
The negative sign can be interpreted as the boundary pulling in the vortices and the symmetric configuration being marginally *unstable*.
For axial symmetry and general N:

$$
-i \oint_{\partial D} dz \left(-\frac{R^2}{4z} + \frac{N}{2z} \right) = 2\pi \left(-\frac{R^2}{4} + \frac{N}{2} \right) = 0, \qquad (65)
$$

by the generalized Bradlow bound.

So for nonconstant background curvature..

Let us consider metrics of the form (for simplicity):

$$ds^2 = dt^2 - \Omega_0(|z|^2)dzd\bar{z}, \tag{66}$$

Formal solution:

$$u = u_0 - F(|z|^2) + \frac{1}{2}\sum_{i=1}^{N} \log|z - z_i|^2 + g(z) + \overline{g(z)}, \tag{67}$$

with

$$\nabla^2 F = \Omega_0. \tag{68}$$

Axial symmetry and $u(R) = 0$ yields

$$u = F(R^2) - F(|z|^2) + \frac{N}{2}\log\frac{|z|^2}{R^2}. \tag{69}$$

Consider:

$$ds^2 = dt^2 - \kappa^{-1}(1 \pm |z|^{2k})^{\ell} dz d\bar{z}, \tag{70}$$

with $k \in \mathbb{Z}_{>0}$, $\ell \in \mathbb{Z}$, $\kappa \in \mathbb{R}_{>0}$.
Gaussian background curvature:

$$K_0 = -\frac{1}{2\Omega_0}\nabla^2 \log \Omega_0 = \mp \frac{2\kappa \ell k^2 |z|^{2k-2}}{(1 \pm |z|^{2k})^{\ell+2}}. \tag{71}$$

is indeed in general nonvanishing.

Special cases: constant curvature cases: $k = 1$ and $\ell = -2$: upper sign is S^2 and lower sign is $\mathbb{H}^2$.
Analytic solution:

$$F^{(\ell,k)} = \frac{|z|^2}{4\kappa} {}_3F_2\left[k^{-1}, -\ell, k^{-1}; 1 + k^{-1}, 1 + k^{-1}; \mp |z|^{2k}\right], \tag{72}$$

where ${}_3F_2$ is a hypergeometric function.

As a good check, let us first consider $\ell = -2$ and $k = 1$ for which the Gaussian curvature is constant:

$$F^{(-2,1)} = \pm \frac{1}{4\kappa} \log(1 \pm |z|^2), \tag{73}$$

Another check is $\ell = 0$

$$F^{(0,k)} = \frac{|z|^2}{4\kappa}, \tag{74}$$

i.e. flat disc

Other families of solutions that can be written as fractions are, for $\ell = 1$:

$$\kappa \boldsymbol{F}^{(1,k)} = \frac{|z|^2}{4} \pm \frac{|z|^{2k+2}}{4(1+k)^2}, \tag{75}$$

and for $\ell = 2$:

$$\kappa \boldsymbol{F}^{(2,k)} = \frac{|z|^2}{4} \pm \frac{|z|^{2k+2}}{2(1+k)^2} + \frac{|z|^{4k+2}}{4(1+2k)^2}, \tag{76}$$

and for generic $\ell \geq 1$:

$$\boldsymbol{F}^{(\ell\geq 1,k)} = \frac{|z|^2}{4\kappa} \sum_{p=0}^{\ell} \begin{pmatrix} \ell \\ p \end{pmatrix} \frac{(\pm 1)^p |z|^{2pk}}{(1+pk)^2}. \tag{77}$$

Finally, for $\ell = -1$ we can write the solution as

$$\boldsymbol{F}^{(-1,k)} = \frac{|z|^2}{4\kappa k^2} \Phi\left[\mp |z|^{2k}, 2, k^{-1}\right] = \frac{|z|^2}{2\kappa} \sum_{p=0}^{\infty} \frac{(\pm 1)^p |z|^{2pk}}{(1+pk)^2}, \tag{78}$$

where Φ is the Hurwitz-Lerch transcendent.

Using the generalized Bradlow bound:

$$2\pi N = \int_{M_0} d^2x\, \Omega_0 = A_0, \tag{79}$$

For the metrics:

$$A_0^{(\ell,k)} = \frac{2\pi}{\kappa}\int_0^R dr\, r(1 \pm r^{2k})^\ell = \frac{\pi R^2}{\kappa}\, {}_2F_1\left[k^{-1}, -\ell; 1 + k^{-1}; \mp R^{2k}\right], \tag{80}$$

As a consistency check, we can set $\ell = 0$

$$A_0^{(0,k)} = \frac{\pi R^2}{\kappa}, \tag{81}$$

for flat disc, $\mathbb{D}^2$.

We can again simplify the hypergeometric function in cases of positive ℓ; in particular for $\ell = 1$:

$$A_0^{(1,k)} = \frac{\pi R^2}{\kappa}\left(1 \pm \frac{R^{2k}}{1+k}\right), \tag{82}$$

and for $\ell = 2$:

$$A_0^{(2,k)} = \frac{\pi R^2}{\kappa}\left(1 \pm \frac{2R^{2k}}{1+k} + \frac{R^{4k}}{1+2k}\right), \tag{83}$$

and for generic $\ell \geq 1$:

$$A_0^{(\ell,k)} = \frac{\pi R^2}{\kappa}\sum_{p=0}^{\ell}\binom{\ell}{p}\frac{(\pm 1)^p R^{2pk}}{1+pk}. \tag{84}$$

Let us consider the lower sign with the radius $R = 1 - \epsilon$, where ϵ is an infinitesimal real number. In this case, we can expand the Gaussian hypergeometric function to get

$$A_0^{(\ell,k)} = -\frac{\pi^2 R^2 \csc(\pi\ell)\Gamma\left(1+k^{-1}\right)}{\kappa\Gamma(-\ell)\Gamma\left(1+k^{-1}+\ell\right)}(1+2\epsilon)+\epsilon^{\ell}\left(-\frac{2^{1+\ell}k^{\ell}\pi r^2}{\kappa(1+\ell)}\epsilon+\mathcal{O}(\epsilon^2)\right). \tag{85}$$

However, for $\ell \geq 0$, the area renders finite and as a few examples we get for $R = 1$

$$A_0^{(0,k)} < \frac{\pi}{\kappa}, \tag{86}$$

$$A_0^{(1,k)} < \frac{\pi}{\kappa}\frac{\Gamma(1+k^{-1})}{\Gamma(2+k^{-1})}, \tag{87}$$

$$A_0^{(2,k)} < \frac{2\pi}{\kappa}\frac{\Gamma(1+k^{-1})}{\Gamma(3+k^{-1})}, \tag{88}$$

and for general $\ell \geq 0$:

$$A_0^{(\ell\geq 0,k)} < \frac{\ell!\pi}{\kappa}\frac{\Gamma(1+k^{-1})}{\Gamma(\ell+1+k^{-1})}. \tag{89}$$

Let us consider $\ell = -2$, for which we get

$$A_0^{(-2,k)} = \left(1 - \frac{1}{k}\right) \frac{\pi}{\kappa} \Gamma\left(1 + \frac{1}{k}\right) \Gamma\left(1 - \frac{1}{k}\right) = \frac{k-1}{k^2} \pi^2 \csc\left(\frac{\pi}{k}\right). \tag{90}$$

This area is maximal for the two limits: $k = 1$ and $k \to \infty$: both yielding

$$A_0^{(-2,1)} = A^{(-2,\infty)} = \frac{\pi}{\kappa}, \tag{91}$$

Thanks! ありがとう！ 谢谢！ Merci! Tak! Danke! Tack! Grazie!

Happy birthday Ken!!
誕生日おめでとう小西さん！！