IS QUANTUM THEORY EXACT?

FQT 2015 (FRASCATI - LNF)

Virtually abelian Quantum Walks and their Symmetries

Marco Erba

(Università degli Studi di Pavia)

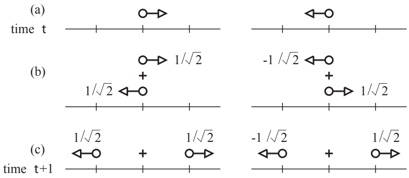
Contents

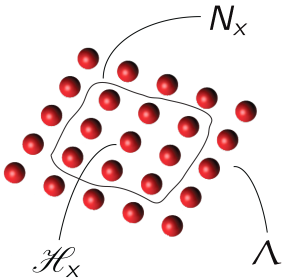
When random walks go *quantum*

[Y. Aharonov, L. Davidovich, and N. Zagury.

Quantum random walks.

Physical Review A, **48**:1687–1690, 1993]





- $\mathscr{H}_x \cong \mathbb{C}^s \to$ *quantum coin system*
- QW on a lattice Λ with *transition rule* N_x:

$$A : \mathscr{H}_x \longrightarrow \bigoplus_{i \in N_x} \mathscr{H}_i$$
$$|\psi\rangle_x \longmapsto A\,|\psi\rangle_x$$

- Total space: $\mathscr{H} = \bigoplus_{x \in \Lambda} \mathscr{H}_x \cong \ell^2(\Lambda) \otimes \mathbb{C}^s$

Discrete approximations of QFT

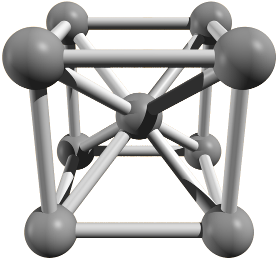
- Lattice gauge theories:
  non-perturbative approach

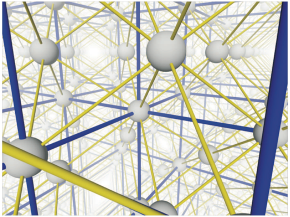
- Generalised Schrödinger equations
  from Markov processes

- Bialynicki-Birula, 1994:
  lattice geometry and evolution
  strictly related

- Dispersive Schrödinger equation:

$$i\partial_t \psi(\mathbf{x}, t) = \left[\mathbf{v} \cdot \nabla + \tfrac{1}{2}\mathbf{D}\left(\nabla^T \nabla\right)\right] \psi(\mathbf{x}, t)$$





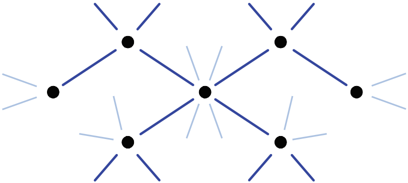
- Operationalism
  [P.W. Bridgman,
  The logic of modern physics]

- Informational Paradigm
  [*Church-Turing-Deutsch Principle*]

"It from Bit"
[R.P. Feynman]

G. Chiribella, G.M. D'Ariano, and P. Perinotti.
Informational derivation of quantum theory.
Physical Review A **84**, 012311 (2011)

Principles

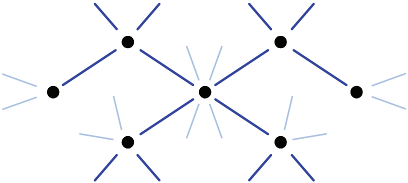


Denumerable set of quantum systems

in mutual interaction

(neither continuum nor metrics assumed)

Principles



Denumerable set of quantum systems

in mutual interaction

(neither continuum nor metrics assumed)

(1) *Locality*: interaction with a finite number of neighbours;
(2) *Homogeneity*: universality of the physical law;
(3) *Isotropy*: no preferential direction of interactions;
(4) *Unitarity*: A is a unitary operator.

G.M. D'Ariano and P. Perinotti.
Derivation of the Dirac Equation from Principles of Information Processing.
Physical Review A **90**, 062106 (2014)

Ambient space: Cayley graph

Principles lead to identifying **sites** with **elements of a group** G

S: generating set for G (*links*)
R: words defined on S (*relators*) $\longrightarrow$ **Presentation** of G:
$F_S/N_R = G \sim \langle\ S^+ \mid R\ \rangle$

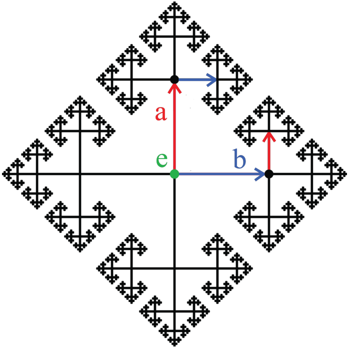
Ambient space: Cayley graph

Principles lead to identifying **sites** with **elements of a group** G

S: generating set for G (*links*)
R: words defined on S (*relators*)

$\longrightarrow$

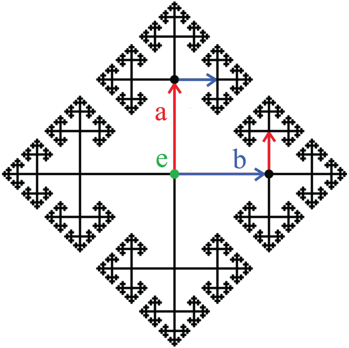
Presentation of G:
$F_S/N_R = G \sim \langle\ S^+ \mid R\ \rangle$



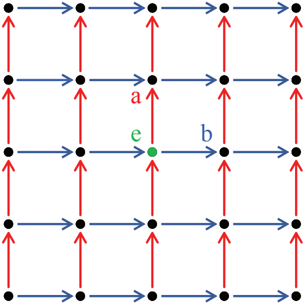
Ambient space: Cayley graph

Principles lead to identifying **sites** with **elements of a group** G

S: generating set for G (*links*)
R: words defined on S (*relators*) $\longrightarrow$ **Presentation** of G:
$F_S/N_R = G \sim \langle\, S^+ \mid R \,\rangle$



$\overset{ab=ba}{\Longrightarrow}$

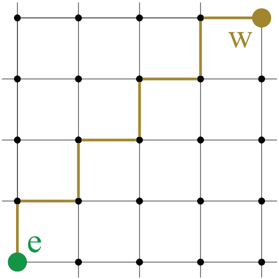


Let continuum emerge from discrete

No canonical presentation of G, but...

WORD METRIC & LENGTH
(and Weyl's tile argument...)

Quasi-isometric embedding
of the Cayley graph $\Gamma(G)$ in $\mathbb{R}^d$

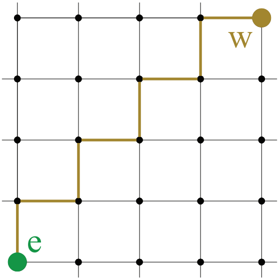


Let continuum emerge from discrete

No canonical presentation of G, but...

WORD METRIC & LENGTH
(and Weyl's tile argument...)

Quasi-isometric embedding
of the Cayley graph $\Gamma(G)$ in $\mathbb{R}^d$



Quasi-isometry between (G, d_G) and (M, d_M)

Function $\mathscr{E} : G \to M$ satisfying, for some fixed $a \geqslant 1$ and $b, c \geqslant 0$:

1. $\forall g, g' \in G\colon \frac{1}{a} d_G(g, g') - b \leqslant d_M(\mathscr{E}(g), \mathscr{E}(g')) \leqslant a d_G(g, g') + b$;
2. $\forall m \in M \; \exists \; g \in G\colon d_M(m, \mathscr{E}(g)) \leqslant c$.

Diagonalization of the walk operator

$$A = \sum_{f \in S} T_f \otimes A_f \in \mathrm{Aut}\left(\ell^2(G) \otimes \mathbb{C}^s\right)$$

Diagonalization of the walk operator

$$A = \sum_{f \in S} T_f \otimes A_f \in \mathrm{Aut}\left(\ell^2(G) \otimes \mathbb{C}^s\right)$$

(1) *Locality* $\Longrightarrow |S|$ finite

(2) *Homogeneity* $\Longrightarrow |S|$ independent of the site

(3) *Isotropy* $\Longrightarrow$ Automorphism group for $\Gamma(G)$

(4) *Unitarity* $\Longrightarrow A^\dagger A = AA^\dagger = T_e \otimes \mathbb{1}_s$

Diagonalization of the walk operator

$$A = \sum_{f \in S} T_f \otimes A_f \in \mathrm{Aut}\left(\ell^2(G) \otimes \mathbb{C}^s\right)$$

(1) *Locality* $\Longrightarrow |S|$ finite

(2) *Homogeneity* $\Longrightarrow |S|$ independent of the site

(3) *Isotropy* $\Longrightarrow$ Automorphism group for $\Gamma(G)$

(4) *Unitarity* $\Longrightarrow A^\dagger A = AA^\dagger = T_e \otimes \mathbb{1}_s$

If $G \cong \mathbb{Z}^d$, then $T_f |\mathbf{k}\rangle = e^{i\mathbf{k}\cdot\mathbf{f}} |\mathbf{k}\rangle$

$$\Downarrow$$

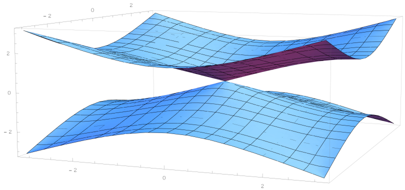
$$A = \int_{\mathscr{B}} \mathrm{d}\mathbf{k} \; |\mathbf{k}\rangle\langle\mathbf{k}| \otimes \left(\sum_{\mathbf{f}} e^{i\mathbf{k}\cdot\mathbf{f}} A_{\mathbf{f}}\right)$$

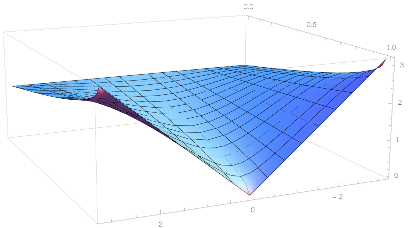
Relativistic limit

$$\sum_{\mathbf{f}} e^{i\mathbf{k}\cdot\mathbf{f}} A_{\mathbf{f}} = \sum_{j=1}^{s} \exp\{i\omega_j(\mathbf{k})\} |j\rangle\langle j| \longrightarrow \text{dispersion relation}$$

Relativistic limit

$$\sum_{\mathbf{f}} e^{i\mathbf{k}\cdot\mathbf{f}} A_{\mathbf{f}} = \sum_{j=1}^{s} \exp\{i\omega_j(\mathbf{k})\} |j\rangle\langle j| \longrightarrow \text{dispersion relation}$$





$$\omega_{\pm}(\mathbf{k}) = \pm \arccos\left\{\frac{\sqrt{1-m^2}}{2}(\cos k_x + \cos k_y)\right\} \xrightarrow[|\mathbf{k}|\ll 1]{} \pm\sqrt{m^2 + {k_x}^2 + {k_y}^2}$$

Also Weyl, Dirac and Maxwell **dynamics** as **emergent** (up to $d = 3$)

One step forward

Attempt: a unifying framework

1. Minimal computational complexity:
   → What about the **scalar case** $s = 1$?
2. Diagonalization fails for arbitrary QWs:
   → And **beyond** the **abelian** case?
3. How is a QW on G related to the walk on a $N \subset G$?
   → One QW, **many representations**

One step forward

Attempt: a unifying framework

1. Minimal computational complexity:
   $\rightarrow$ What about the **scalar case** $s = 1$?
2. Diagonalization fails for arbitrary QWs:
   $\rightarrow$ And **beyond** the **abelian** case?
3. How is a QW on G related to the walk on a $N \subset G$?
   $\rightarrow$ One QW, **many representations**

Theorem

A finitely-generated group G is quasi-isometric to $\mathbb{R}^d$

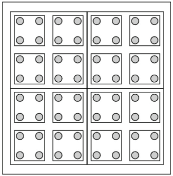
$$\Longleftrightarrow$$

G has a finite-index normal subgroup $N \cong \mathbb{Z}^d$
(i.e virtually abelian)

Tessellation of Cayley graphs

CORE IDEA: **block-spin renormalization**

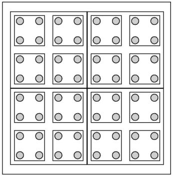
[L.P. Kadanoff. Scaling Laws for Ising Models Near T_c. *Physics*, **2**(6):63–72, 1966]

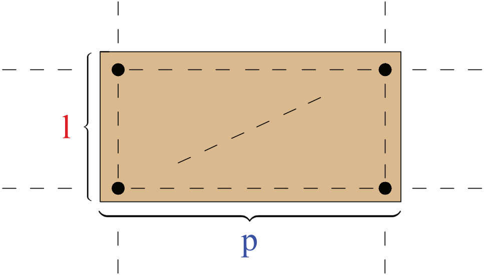


Tessellation of Cayley graphs

CORE IDEA: **block-spin renormalization**

[L.P. Kadanoff. Scaling Laws for Ising Models Near T_c. *Physics*, **2**(6):63–72, 1966]





Grouping sites in cells of $l \times p = q$ sites

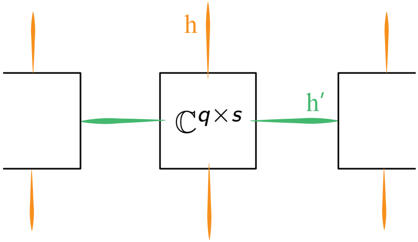
Twofold motivation

$$G = \bigcup_{j=1}^{q} c_j N \longrightarrow \text{Induces a tessellation of } \Gamma(G)$$

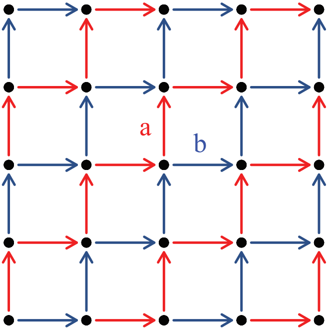
$\mathscr{F} = \ell^2(G)$ mapped to $\mathscr{K} = \ell^2(N) \otimes \mathbb{C}^q \cong \mathscr{F}$ by the unitary

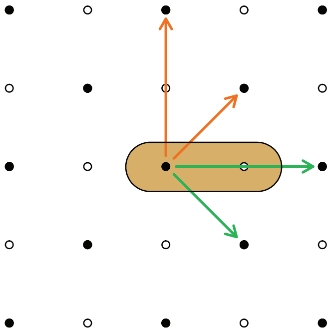
$$\begin{aligned} U_N \;:\; \mathscr{F} &\longrightarrow \mathscr{K}, \\ |c_j \mathbf{x}\rangle &\longmapsto |\mathbf{x}\rangle\,|j\rangle \end{aligned}$$

QW *represented* with an *extended coin system*

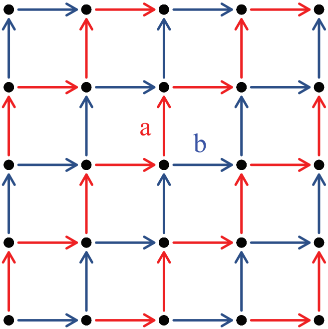


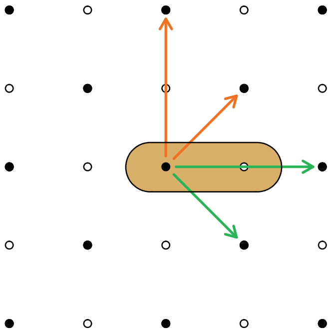
Stairs group $\langle\, a, b \mid a^2 b^{-2} \,\rangle$





Stairs group $\langle\, a, b \mid a^2 b^{-2} \,\rangle$





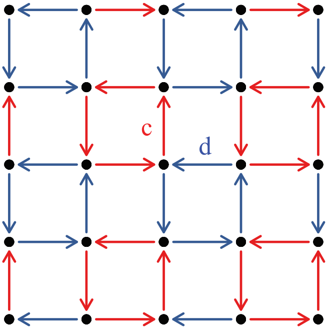
Two solutions $\mathcal{R}\left[A_1\right]_{\mathbf{k}} \equiv \mathcal{R}\left[A_2\right]^{\dagger}_{-\mathbf{k}}$

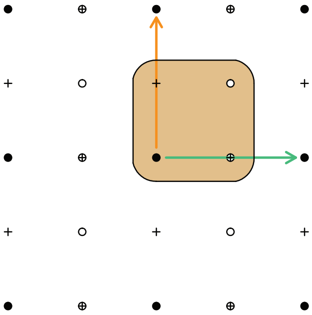
connected by PT symmetry

Analytical solution:

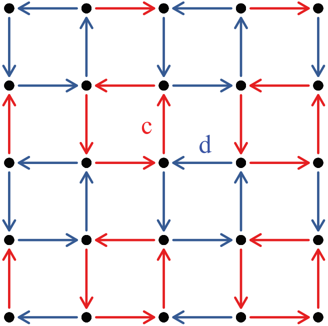
Weyl (massless)

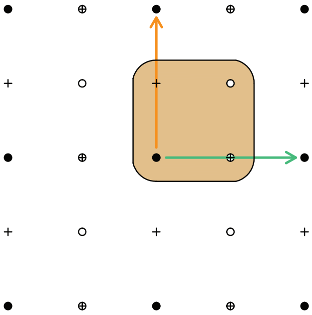
Group with cycles $\langle\, c, d \mid c^4, d^4, (cd)^2 \,\rangle$





Group with cycles $\langle\, c, d \mid c^4, d^4, (cd)^2 \,\rangle$





Equivalent families of solutions

with a mass effect

Analytical solution:

Dirac

Quantum walks without quantum coin

Theorem: Purely Abelian is Trivial

Classification of infinite abelian scalar QWs:

$$A = \int_{\mathscr{B}} \mathrm{d}\mathbf{k}\, |\mathbf{k}\rangle\langle\mathbf{k}| \otimes \bigoplus_i e^{i(\mathbf{k}\cdot\mathbf{h}_i + \theta_i)}$$

Quantum walks without quantum coin

Theorem: Purely Abelian is Trivial

Classification of infinite abelian scalar QWs:

$$A = \int_{\mathscr{B}} \mathrm{d}\mathbf{k}\, |\mathbf{k}\rangle\langle\mathbf{k}| \otimes \bigoplus_i e^{i(\mathbf{k}\cdot\mathbf{h}_i + \theta_i)}$$

Idea: Beyond Abelian

To **extend** $\mathbb{Z}^d$ as a group,

letting also **spin** degree of freedom **emerge**

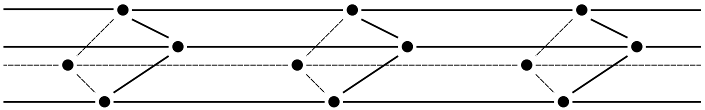
Quantum walks without quantum coin

Theorem: Purely Abelian is Trivial

Classification of infinite abelian scalar QWs:

$$A = \int_{\mathscr{B}} \mathrm{d}\mathbf{k}\, |\mathbf{k}\rangle\langle\mathbf{k}| \otimes \bigoplus_i e^{i(\mathbf{k}\cdot\mathbf{h}_i + \theta_i)}$$

Idea: Beyond Abelian



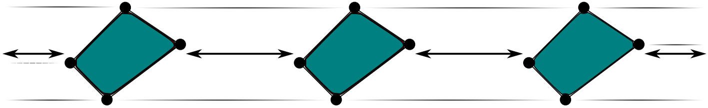
Quantum walks without quantum coin

Theorem: Purely Abelian is Trivial

Classification of infinite abelian scalar QWs:

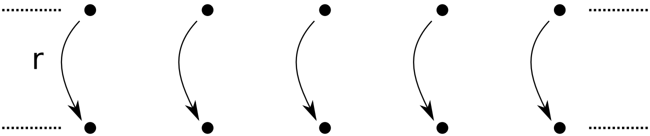
$$A = \int_{\mathscr{B}} \mathrm{d}\mathbf{k}\, |\mathbf{k}\rangle\langle\mathbf{k}| \otimes \bigoplus_i e^{i(\mathbf{k}\cdot\mathbf{h}_i + \theta_i)}$$

Idea: Beyond Abelian



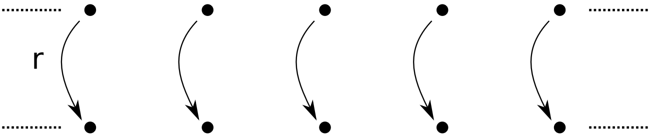
Infinite dihedral group $\langle\, a, b, c \mid b^2, c^2, a^2cb \,\rangle$

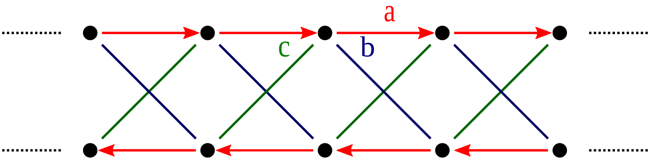
$$D_\infty = \mathbb{Z} \rtimes_\varphi \mathbb{Z}_2 = \mathbb{Z} \cup r\mathbb{Z}$$



Infinite dihedral group $\langle\, a, b, c \mid b^2, c^2, a^2cb \,\rangle$

$$D_\infty = \mathbb{Z} \rtimes_\varphi \mathbb{Z}_2 = \mathbb{Z} \cup r\mathbb{Z}$$





Dynamical behaviour

$$A = T_a z_a + T_{a^{-1}} z_{a^{-1}} + T_b z_b + T_c z_c$$

$$\downarrow$$

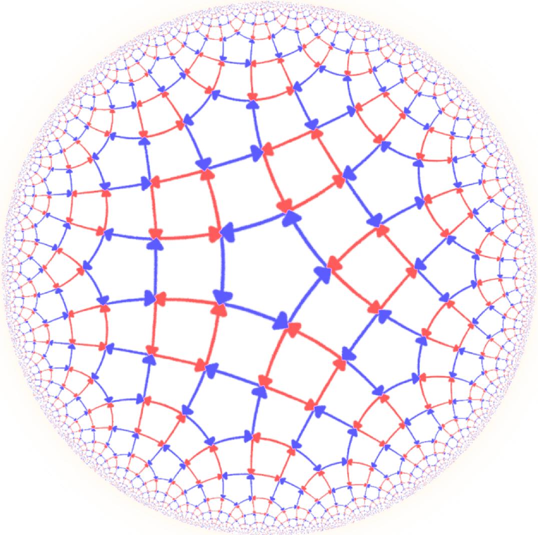
$$\mathcal{R}[A] = \tilde{T}_+ \otimes A_+ + \tilde{T}_- \otimes A_-$$

$$\Downarrow$$

$\underline{\text{1D DIRAC QW}}$: $\omega_\pm(k) = \pm \arccos\{\underbrace{(z_a + z_{a^{-1}})}_{\equiv\sqrt{1-m^2}} \cos k\}$

A posteriori interpretation of symmetries and spinorial matrices

What comes next?



- G.M. D'Ariano and P. Perinotti. Derivation of the Dirac Equation from Principles of Information Processing. *Physical Review A* **90**, 062106 (2014)
- A. Bibeau-Delisle, A. Bisio, G.M. D'Ariano, P. Perinotti, A. Tosini. Doubly special relativity from quantum cellular automata. *EPL* **109**, 5 (2015)
- A. Bisio, G.M. D'Ariano, P. Perinottim A. Tosini. Weyl, Dirac and Maxwell Quantum Cellular Automata. *Foundations of Physics*, 1–19 (2015)
- A. Bisio, G.M. D'Ariano, P. Perinotti. Lorentz symmetry for 3d Quantum Cellular Automata. *arXiv:1503.01017* [quant-ph]
- Works on virtually abelian and scalar QWs are under preparation.

Quantum Cloud

[A. Gormley, London]

JOHN TEMPLETON FOUNDATION

quantum information theory group