Università degli Studi di Milano, Istituto Nazionale di Fisica Nucleare

SINGLE-PHOTON OBSERVABLES AND PREPARATION UNCERTAINTY RELATIONS

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- $\Delta X_j \, \Delta P_j \geq \hbar/2$
- SPIN PROBABILITY DISTRIBUTION

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Historical Remarks

«*No position operator can be defined in the usual sense for particles with mass m=0*»

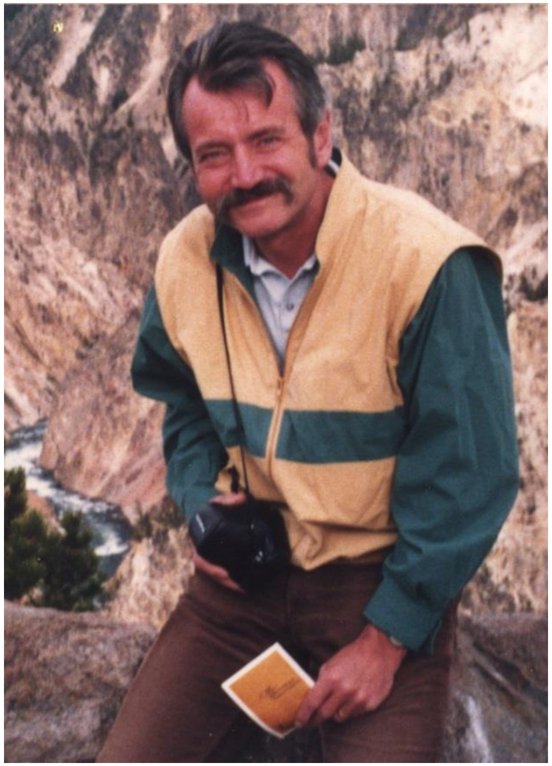
E. P. Wigner

T. D. Newton

Historical Remarks

«No position operator can be defined in the usual sense for particles with mass m=0»

K. Kraus



E. P. Wigner

T. D. Newton

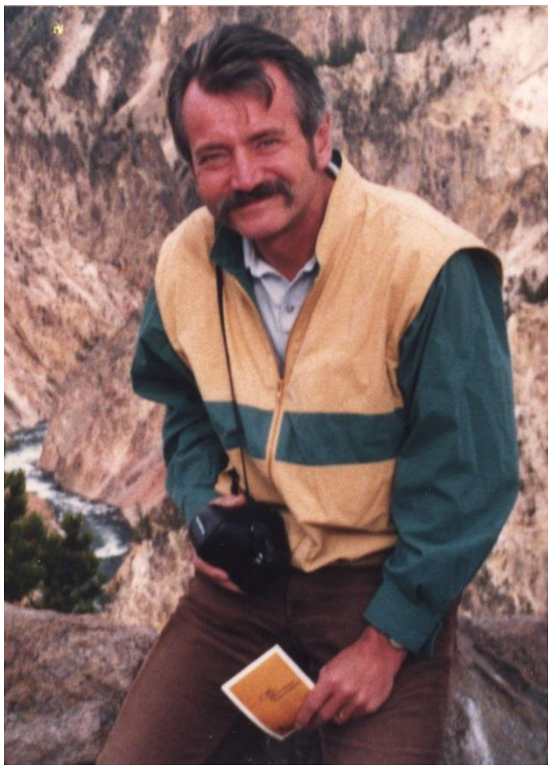
«*Position of photons can be given in terms of a POVM*»

Historical Remarks

Single photons cannot be localized with certainty in a bounded region of space

K. Kraus

E. P. Wigner



T. D. Newton

«*Position of photons can be given in terms of a POVM*»

Historical Remarks

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Single-Photon Space and States

NON – RELATIVISTIC SPIN $S = 1$ PARTICLE

↓

PHOTON

RELATIVISTIC SPIN $S = 1$ AND MASS $M = 0$ PARTICLE

Single-Photon Space and States

NON - RELATIVISTIC SPIN $S = 1$ PARTICLE

- Hilbert space $\mathscr{H}_{NR} = \mathcal{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^3$

Single-Photon Space and States

NON - RELATIVISTIC SPIN $S = 1$ PARTICLE

- Hilbert space $\mathscr{H}_{NR} = \mathcal{L}^2(\mathbb{R}^3)\otimes\mathbb{C}^3$

- Irreducible projective representation of the Galilei group (roto-translations)

$$\left(\hat{\mathcal{U}}(\mathbf{a}, R)\phi\right)^j(\mathbf{p}) = e^{-\frac{i}{\hbar}\,\mathbf{p}\cdot\mathbf{a}} \sum_{k=1}^{3} \left(e^{-\frac{i}{\hbar}\varphi\mathbf{n}\cdot\mathbf{S}}\right)^j_{\ k} \phi^k(R^{-1}\mathbf{p})$$

$\varphi =$ rotation angle, $\mathbf{n} =$ rotation axis

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \quad S_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix} \quad S_z = \hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

Single-Photon Space and States

NON - RELATIVISTIC SPIN $S = 1$ PARTICLE

KEY RELATION:

$$R = V^{\dagger} e^{-\frac{i}{\hbar}\varphi \mathbf{n}\cdot\mathbf{S}} V$$

WHICH IS A PECULIAR PREROGATIVE OF THE SPIN S=1 CASE

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \end{pmatrix}$$

Single-Photon Space and States

NON - RELATIVISTIC SPIN S = 1 PARTICLE

$$R = V^\dagger \, e^{-\frac{i}{\hbar}\varphi \mathbf{n}\cdot\mathbf{S}} \, V$$

- $\mathcal{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^3$
- $\phi(\mathbf{p})$
- $(\hat{\mathcal{U}}(\mathbf{a}, R)\phi)^j(\mathbf{p}) = e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{a}} \sum_{k=1}^{3} \left(e^{-\frac{i}{\hbar}\varphi \mathbf{n}\cdot\mathbf{S}}\right)^j{}_k \, \phi^k(R^{-1}\mathbf{p})$

Single-Photon Space and States

NON - RELATIVISTIC SPIN S = 1 PARTICLE

$$R = V^\dagger e^{-\frac{i}{\hbar}\varphi \mathbf{n}\cdot\mathbf{S}} V$$

- $\mathcal{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^3 \xrightarrow{V^\dagger} \mathcal{L}^2(\mathbb{R}^3) \otimes V^\dagger \mathbb{C}^3$
- $\phi(\mathbf{p}) \xrightarrow{V^\dagger} \psi(\mathbf{p}) \equiv V^\dagger \phi(\mathbf{p})$
- $$\left(\hat{\mathcal{U}}(\mathbf{a}, R)\phi\right)^j(\mathbf{p}) = e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{a}} \sum_{k=1}^{3} \left(e^{-\frac{i}{\hbar}\varphi \mathbf{n}\cdot\mathbf{S}}\right)^j{}_k \phi^k(R^{-1}\mathbf{p})$$

$\Big\downarrow V^\dagger$

$$\left(\hat{U}(\mathbf{a}, R)\psi\right)^j(\mathbf{p}) = e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{a}} \sum_{k=1}^{3} R^j{}_k \psi^k(R^{-1}\mathbf{p})$$

Single-Photon Space and States

NON - RELATIVISTIC SPIN $S = 1$ PARTICLE

$$R = V^\dagger e^{-\frac{i}{\hbar}\varphi \mathbf{n}\cdot\mathbf{S}} V$$

- $\mathcal{L}^2(\mathbb{R}^3)\otimes\mathbb{C}^3 \xrightarrow{V^\dagger} \mathcal{L}^2(\mathbb{R}^3)\otimes V^\dagger\mathbb{C}^3$
- $\phi(\mathbf{p}) \xrightarrow{V^\dagger} \psi(\mathbf{p}) \equiv V^\dagger\phi(\mathbf{p})$
- $(\hat{\mathcal{U}}(\mathbf{a},R)\phi)^j(\mathbf{p}) = e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{a}} \sum_{k=1}^{3} \left(e^{-\frac{i}{\hbar}\varphi\mathbf{n}\cdot\mathbf{S}}\right)^j{}_k \phi^k(R^{-1}\mathbf{p})$

$V^\dagger$ ↓

$$(\hat{U}(\mathbf{a},R)\psi)^j(\mathbf{p}) = e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{a}} \sum_{k=1}^{3} R^j{}_k\, \psi^k(R^{-1}\mathbf{p})$$

$\psi(\mathbf{p})$

Transforms as a vector field!

Single-Photon Space and States

NON - RELATIVISTIC

SPIN S = 1 PARTICLE

$$\mathscr{H}_{NR} = \mathcal{L}^2\left(\mathbb{R}^3\right) \otimes V^{\dagger}\mathbb{C}^3$$

$$\boldsymbol{\psi}(\mathbf{p})$$

VECTORS

$$\left(\hat{U}(\mathbf{a}, R)\boldsymbol{\psi}\right)^j(\mathbf{p}) = e^{-\frac{i}{\hbar}\,\mathbf{p}\cdot\mathbf{a}} \sum_{k=1}^{3} R^j{}_k\, \psi^k(R^{-1}\mathbf{p})$$

RELATIVISTIC

SPIN S = 1 PARTICLE

Single-Photon Space and States

NON – RELATIVISTIC SPIN S = 1 PARTICLE	RELATIVISTIC SPIN S = 1 PARTICLE
$\mathscr{H}_{NR} = \mathcal{L}^2\left(\mathbb{R}^3\right) \otimes V^\dagger \mathbb{C}^3$	$\mathscr{H}_{rel} = \mathcal{L}^2\left(\mathbb{R}^3, \frac{d^3\mathbf{p}}{\lvert\mathbf{p}\rvert}\right) \otimes V^\dagger \mathbb{C}^4$
$\boldsymbol{\psi}(\mathbf{p})$	$\psi(p) = \begin{pmatrix} \psi^0(p) \\ \psi^1(p) \\ \psi^2(p) \\ \psi^3(p) \end{pmatrix}$
VECTORS	FOUR-VECTORS
$\left(\hat{U}(\mathbf{a}, R)\boldsymbol{\psi}\right)^j(\mathbf{p}) = e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{a}} \sum_{k=1}^{3} R^j{}_k\, \boldsymbol{\psi}^k(R^{-1}\mathbf{p})$	$\left(\hat{U}(a, \Lambda)\psi\right)^\mu(p) = e^{\frac{i}{\hbar}a_\tau p^\tau} \Lambda^\mu{}_\nu\, \psi^\nu(\Lambda^{-1}p)$

Single-Photon Space and States

RELATIVISTIC SPIN $S = 1$ PARTICLE

- Problem: $$\mathscr{H}_{rel} = \mathcal{L}^2\left(\mathbb{R}^3, \frac{d^3\mathbf{p}}{|\mathbf{p}|}\right) \otimes V^\dagger\mathbb{C}^4$$

$$\langle\psi_A|\psi_B\rangle = -\int \frac{d^3\mathbf{p}}{|\mathbf{p}|}\,(\psi_A^\mu)^*(p)\,g_{\mu\nu}\,\psi_B^\nu(p)$$

Single-Photon Space and States

RELATIVISTIC SPIN $S=1$ PARTICLE

- Problem: $$\mathscr{H}_{rel} = \mathcal{L}^2\left(\mathbb{R}^3, \frac{d^3\mathbf{p}}{|\mathbf{p}|}\right) \otimes V^\dagger \mathbb{C}^4$$

$$\langle\psi_A|\psi_B\rangle = -\int \frac{d^3\mathbf{p}}{|\mathbf{p}|}(\psi_A^\mu)^*(p)\, g_{\mu\nu}\, \psi_B^\nu(p)$$

The scalar product is non – positive definite!

No probabilistic interpretation to the formalism can be given?!

Single-Photon Space and States

RELATIVISTIC SPIN $S = 1$ PARTICLE

Only in the case of the photon such obstacle can be overcome!

M = 0 CONDITION $\mathcal{S} \subset \mathscr{H}$

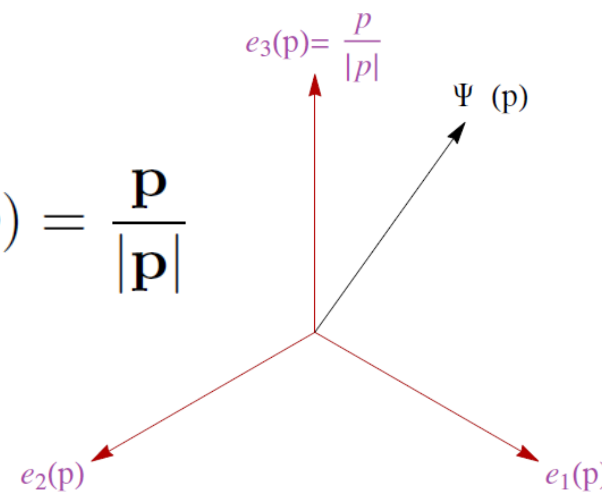
$$\mathcal{S} = \{\psi(p) : \; p^{\mu} g_{\mu\nu} \psi^{\nu}(p) = 0 \text{ for almost all } p\}$$

Properties:

- $\psi(p) \in \mathcal{S} \iff \hat{U}(a, \Lambda)\psi(p) \in \mathcal{S}$
- The restriction of the scalar product to S is non-negative!

INTRISIC FRAME

$$\tilde{\mathbf{e}}_1(p) = \frac{\mathbf{p}\times(\mathbf{m}\times\mathbf{p})}{|\mathbf{p}||\mathbf{m}\times\mathbf{p}|} \quad \tilde{\mathbf{e}}_2(p) = \frac{\mathbf{m}\times\mathbf{p}}{|\mathbf{m}\times\mathbf{p}|} \quad \tilde{\mathbf{e}}_3(p) = \frac{\mathbf{p}}{|\mathbf{p}|}$$



$$\psi^\mu(p) = \begin{pmatrix} \tilde{\psi}^0(p) \\ \tilde{\boldsymbol{\psi}}(p) \end{pmatrix} \quad \tilde{\boldsymbol{\psi}}(p) = \sum_{i=1}^{3} \tilde{\psi}^i(p)\tilde{\mathbf{e}}_i(p)$$

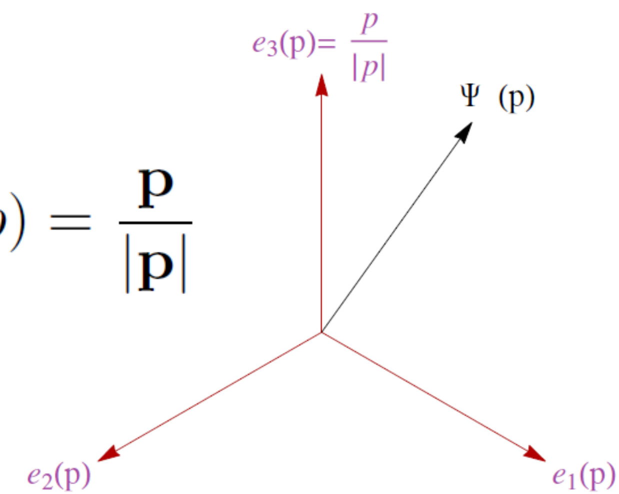
In this basis, the massless condition translates into

$$\tilde{\psi}^0(p) = \tilde{\psi}^3(p)$$

$$\langle\psi_A|\psi_B\rangle = \int \frac{d^3\mathbf{p}}{p^0}\left[(\tilde{\psi}_A^1)^*(p)\,\tilde{\psi}_B^1(p) + (\tilde{\psi}_A^2)^*(p)\,\tilde{\psi}_B^2(p)\right]$$

Single-Photon Space and States

INTRISIC FRAME



$$\tilde{\mathbf{e}}_1(p) = \frac{\mathbf{p} \times (\mathbf{m} \times \mathbf{p})}{|\mathbf{p}||\mathbf{m} \times \mathbf{p}|} \qquad \tilde{\mathbf{e}}_2(p) = \frac{\mathbf{m} \times \mathbf{p}}{|\mathbf{m} \times \mathbf{p}|} \qquad \tilde{\mathbf{e}}_3(p) = \frac{\mathbf{p}}{|\mathbf{p}|}$$

$$\psi^\mu(p) = \begin{pmatrix} \tilde{\psi}^0(p) \\ \tilde{\boldsymbol{\psi}}(p) \end{pmatrix} \qquad \tilde{\boldsymbol{\psi}}(p) = \sum_{i=1}^{3} \tilde{\psi}^i(p)\tilde{\mathbf{e}}_i(p)$$

In this basis, the massless condition translates into

$$\tilde{\psi}^0(p) = \tilde{\psi}^3(p)$$

IRRELEVANT DEGREES OF FREEDOM (gauge)

$$\langle \psi_A | \psi_B \rangle = \int \frac{d^3\mathbf{p}}{p^0} \left[(\tilde{\psi}^1_A)^*(p)\, \tilde{\psi}^1_B(p) + (\tilde{\psi}^2_A)^*(p)\, \tilde{\psi}^2_B(p) \right]$$

Single-Photon Space and States

$$\langle\psi_A|\psi_B\rangle = \int \frac{d^3\mathbf{p}}{p^0}\left[(\tilde{\psi}^1_A)^*(p)\,\tilde{\psi}^1_B(p) + (\tilde{\psi}^2_A)^*(p)\,\tilde{\psi}^2_B(p)\right]$$

Defines a seminorm on $\mathcal{S}$ → Taking the quotient $\mathcal{S}/\sim$

$$\phi \sim \psi \quad \Leftrightarrow \|\phi - \psi\| = 0$$

SINGLE - PHOTON HILBERT SPACE (K. Kraus)

$$\mathscr{H}_\mathcal{S} = \left\{\tilde{\boldsymbol{\psi}}(p) : \tilde{\boldsymbol{\psi}}(p) = \sum_{i=1}^{2} \tilde{\psi}^i(p)\tilde{\mathbf{e}}_i(\mathbf{p}), \tilde{\psi}^i(p) \in \mathcal{L}^2\left(\mathbb{R}^3, \frac{d^3\mathbf{p}}{|\mathbf{p}|}\right)\right\}$$

Contents

Single-Photon Observables

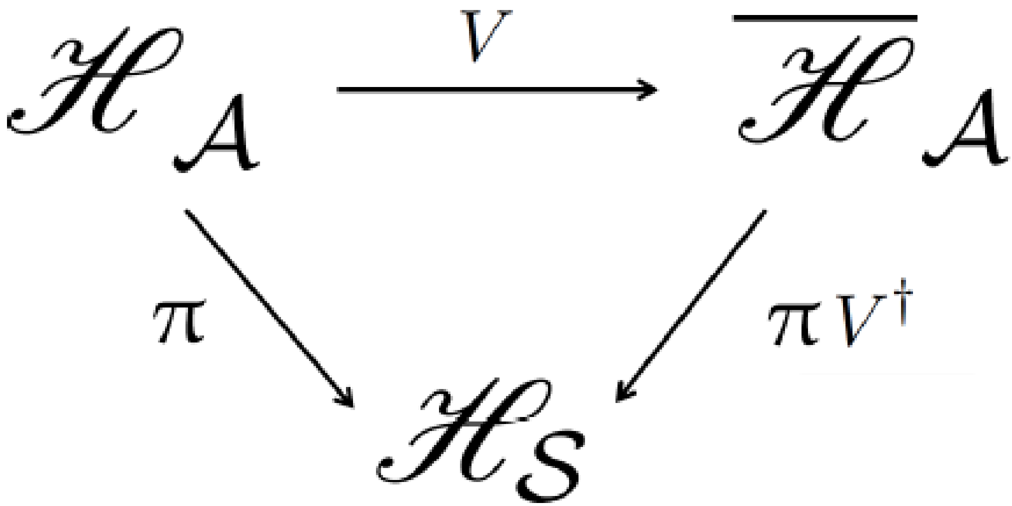
WE EMBED THE PHYSICAL HILBERT SPACE INTO

$$\mathscr{H}_A = \mathcal{L}^2\left(\mathbb{R}^3, \frac{d^3p}{|\mathbf{p}|}\right) \otimes \mathbb{C}^3 \quad \mathbf{f}_V(\mathbf{p}) = \sum_{i=1}^{3} \tilde{\psi}_V^i(\mathbf{p})\tilde{\mathbf{e}}_i(\mathbf{p})$$

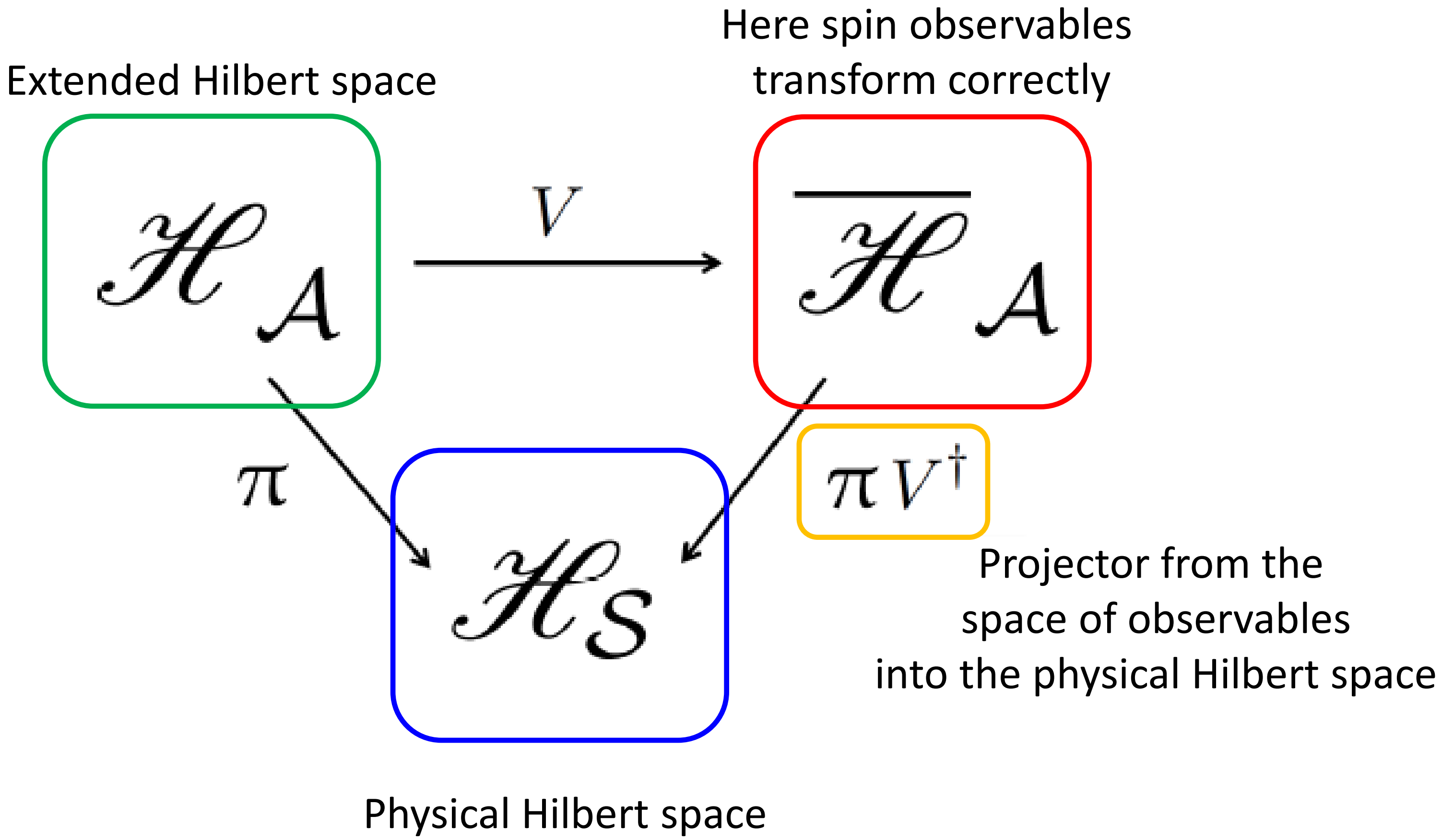
$$\pi : \mathscr{H}_A \to \mathscr{H}_S, \quad (\pi \mathbf{f}_V)^j(\mathbf{p}) = \sum_k \pi^j{}_k(\mathbf{p}) f_V^k(\mathbf{p}) \qquad \forall\, \mathbf{p} \in \mathbb{R}^3$$

$$\pi^j{}_k(\mathbf{p}) = \delta^j{}_k - \frac{p^j p_k}{|\mathbf{p}|^2} \qquad \forall\, \mathbf{p} \in \mathbb{R}^3$$

Single-Photon Observables



Single-Photon Observables



RELATIVISTIC MASSIVE SPIN $S = 1$ PARTICLE

$$(\hat{\mathbf{P}}\mathbf{f})_j(p) = \mathbf{p}\, f_j(p)$$

$$(\hat{\mathbf{S}}\mathbf{f})_j(p) = \sum_k \mathbf{S}_{jk} f_k(p)$$

Covariance under roto translations

$$(\hat{\mathbf{X}}\mathbf{f})_j(p) = i\hbar \frac{\partial f_j(p)}{\partial \mathbf{p}} - \frac{i\hbar}{2} \frac{\mathbf{p}}{(p^0)^2} f_j(p)$$

Newton – Wigner position operator

Then consider the associated PVM $\hat{E}_{O}(\mathcal{M})$

$$p(\mathcal{O} \in \mathcal{M}) = \langle \phi | \hat{E}_{\mathcal{O}}(\mathcal{M}) | \phi \rangle$$

$$\hat{E}^2_{\mathcal{O}}(\mathcal{M}) = \hat{E}_{\mathcal{O}}(\mathcal{M})$$

Single-Photon Observables

RELATIVISTIC MASSLESS SPIN $S=1$ PARTICLE

$$p^0 = |\mathbf{p}|$$

$$(\hat{\mathbf{P}}\mathbf{f})_j(p) = \mathbf{p}\, f_j(p)$$

$$(\hat{\mathbf{S}}\mathbf{f})_j(p) = \sum_k \mathbf{S}_{jk} f_k(p)$$

Covariance under roto translations

$$(\hat{\mathbf{X}}\mathbf{f})_j(p) = i\hbar \frac{\partial f_j(p)}{\partial \mathbf{p}} - \frac{i\hbar}{2}\frac{\mathbf{p}}{|\mathbf{p}|^2} f_j(p)$$

Newton – Wigner position operator

Then consider the associated PVM $\hat{E}_O(\mathcal{M})$ and project it onto the physical Hilbert space $\mathscr{H}_S$ to obtain the POVM

$$\hat{F}_O(\mathcal{M}) = (\pi V^\dagger)\, \hat{E}_O(\mathcal{M})\, (V\pi) = \hat{\Omega}_O^\dagger(\mathcal{M})\hat{\Omega}_O(\mathcal{M})$$

$$\hat{F}_{\mathcal{O}}^2(\mathcal{M}) \neq \hat{F}_{\mathcal{O}}(\mathcal{M})$$

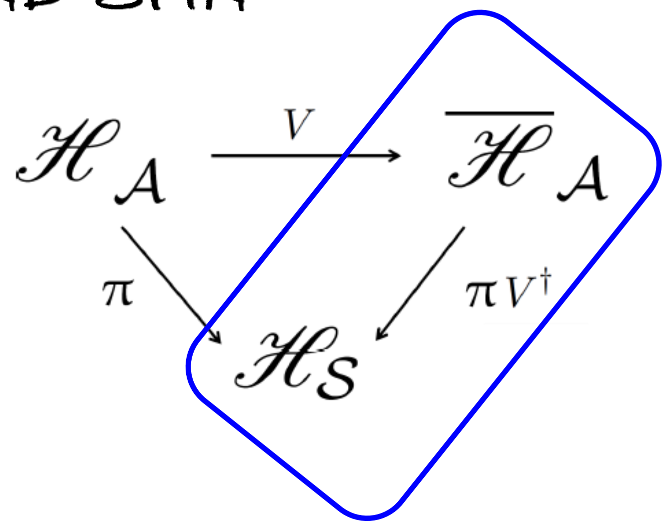
$$p(\mathcal{O} \in \mathcal{M}) = \langle \psi | \hat{F}_{\mathcal{O}}(\mathcal{M}) | \psi \rangle$$

EXAMPLES

① JOINT PROBABILITY DISTRIBUTION OF MOMENTUM AND SPIN

$$(\hat{\mathbf{P}}\mathbf{f})_j(p) = \mathbf{p}\, f_j(p)$$

$$(\hat{\mathbf{S}}\mathbf{f})_j(p) = \sum_k \mathbf{S}_{jk} f_k(p)$$



On $\overline{\mathscr{H}}_{\mathcal{A}}$ we have the joint PVM

$$(\mathcal{M}, \hbar m_s) \mapsto \left(\hat{E}_{P,S_z}(\mathcal{M}, \hbar m_s)\psi\right)_{s'}(\mathbf{p}) = 1_{\mathcal{M}}(\mathbf{p})\, \delta_{s,s'}\, \psi_s(\mathbf{p})$$

$\mathcal{M}$ Borel subset of $\mathbb{R}$

which turns into the POVM

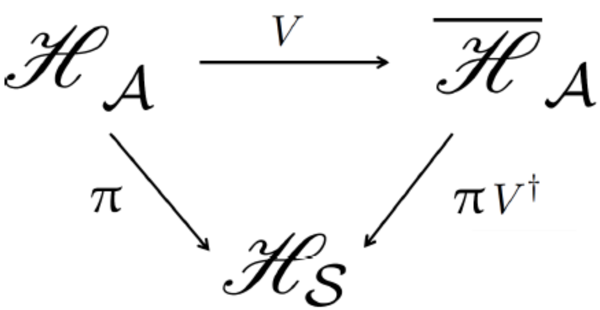
$$(\mathcal{M}, \hbar m_s) \mapsto \left(\hat{F}_{P,S_z}(\mathcal{M}, \hbar m_s)\psi\right)_{s'}(\mathbf{p}) = 1_{\mathcal{M}}(\mathbf{p})\, \delta_{s,s'} \sum_{i=1}^{2} \tilde{\psi}_V^i(\mathbf{p})\, [V\tilde{\mathbf{e}}_i(\mathbf{p})]_s$$

EXAMPLES

1 JOINT PROBABILITY DISTRIBUTION OF MOMENTUM AND SPIN

$$(\hat{\mathbf{P}}\mathbf{f})_j(p) = \mathbf{p}\, f_j(p)$$

$$(\hat{\mathbf{S}}\mathbf{f})_j(p) = \sum_k \mathbf{S}_{jk} f_k(p)$$



$$p(\mathbf{p}\in\mathcal{M}, S_z=\hbar m_s) = \langle\psi|\hat{F}_{P,S_z}(\mathcal{M},\hbar m_s)|\psi\rangle$$

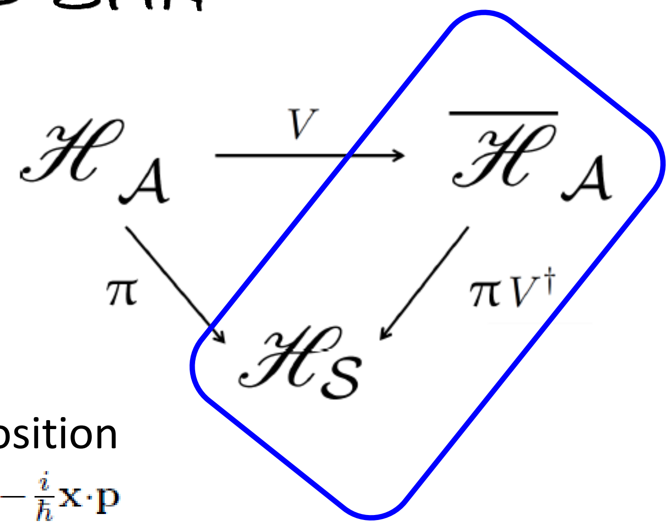
$$= \|\hat{\Omega}_{P,S_z}(\mathcal{M},\hbar m_s)\psi\|^2 = \int_{\mathcal{M}}\frac{d^3p}{|\mathbf{p}|}\left|\sum_{i=1}^{2} \tilde{\psi}_V^i(\mathbf{p})\,[V\tilde{\mathbf{e}}_i(\mathbf{p})]_s\right|^2$$

EXAMPLES

2 JOINT PROBABILITY DISTRIBUTION OF POSITION AND SPIN

$$\left(\hat{\mathbf{X}}\mathbf{f}\right)_j(p) = i\hbar \frac{\partial f_j(p)}{\partial \mathbf{p}} - \frac{i\hbar}{2}\frac{\mathbf{p}}{(p^0)^2} f_j(p)$$

$$\left(\hat{\mathbf{S}}\mathbf{f}\right)_j(p) = \sum_k \mathbf{S}_{jk} f_k(p)$$



The eigenfunctions of the Newton – Wigner position operator have the form $\mathbf{u}_{\mathbf{x},s}(\mathbf{p}) = \sqrt{|\mathbf{p}|}\,\frac{e^{-\frac{i}{\hbar}\mathbf{x}\cdot\mathbf{p}}}{(2\pi\hbar)^{\frac{3}{2}}}\,\mathbf{e}_s$

On $\overline{\mathscr{H}}_{\mathcal{A}}$ we have then the joint PVM

$$(\mathcal{M}, \hbar m_s) \mapsto \left(\hat{E}_{X,S_z}(\mathcal{M}, \hbar m_s)\psi\right)_{s'}(\mathbf{p}') = \int_{\mathcal{M}} d^3x \left[\int \frac{d^3p}{|\mathbf{p}|}\mathbf{u}^*_{\mathbf{x},s}(\mathbf{p})\cdot\psi(\mathbf{p})\right]\left[\mathbf{u}_{\mathbf{x},s}(\mathbf{p}')\right]_{s'}$$

EXAMPLES

2 JOINT PROBABILITY DISTRIBUTION OF POSITION AND SPIN

$\mathcal{M}$ Borel subset of $\mathbb{R}$

$$(\mathcal{M}, \hbar m_s) \mapsto \left(\hat{E}_{X,S_z}(\mathcal{M}, \hbar m_s)\psi\right)_{s'}(\mathbf{p}') = \int_{\mathcal{M}} d^3x \left[\int \frac{d^3p}{|\mathbf{p}|} \mathbf{u}^*_{\mathbf{x},s}(\mathbf{p}) \cdot \psi(\mathbf{p})\right] [\mathbf{u}_{\mathbf{x},s}(\mathbf{p}')]_{s'}$$

which turns into a POVM on $\mathscr{H}_S$, leading to the joint probability distribution:

$$p(\mathbf{X} \in \mathcal{M}, S_z = \hbar m_s) = \|\hat{\Omega}_{X,S_z}(\mathcal{M}, \hbar m_s)\psi\|^2 = \int_{\mathcal{M}} d^3x \left| \left[\tilde{\psi}_V(\mathbf{x})\right]_s \right|^2$$

with $$\left[\tilde{\psi}_V(\mathbf{x})\right]_s = \int \frac{d^3p}{|\mathbf{p}|} \sqrt{|\mathbf{p}|} \frac{e^{\frac{i}{\hbar}\mathbf{x}\cdot\mathbf{p}}}{(2\pi\hbar)^{\frac{3}{2}}} \sum_{i=1}^{2} \tilde{\psi}^i_V(\mathbf{p}) \left[V\tilde{\mathbf{e}}_i(\mathbf{p})\right]_s$$

Contents

Preparation Uncertainty Relations for X and P

Let's consider a Gaussian state with definite polarization, i.e. a state in $\mathscr{H}_S$ of the form

$$\begin{pmatrix} \tilde{\psi}_V^1(\mathbf{p}) \\ \tilde{\psi}_V^2(\mathbf{p}) \end{pmatrix} = \sqrt{|\mathbf{p}|}\, \frac{e^{-\frac{|\mathbf{p}-\mathbf{p}_0|^2}{8ap_0^2}}}{\left(4\pi a p_0^2\right)^{\frac{3}{4}}}\, e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}_0} \begin{pmatrix} \gamma^1 \\ \gamma^2 \end{pmatrix}$$

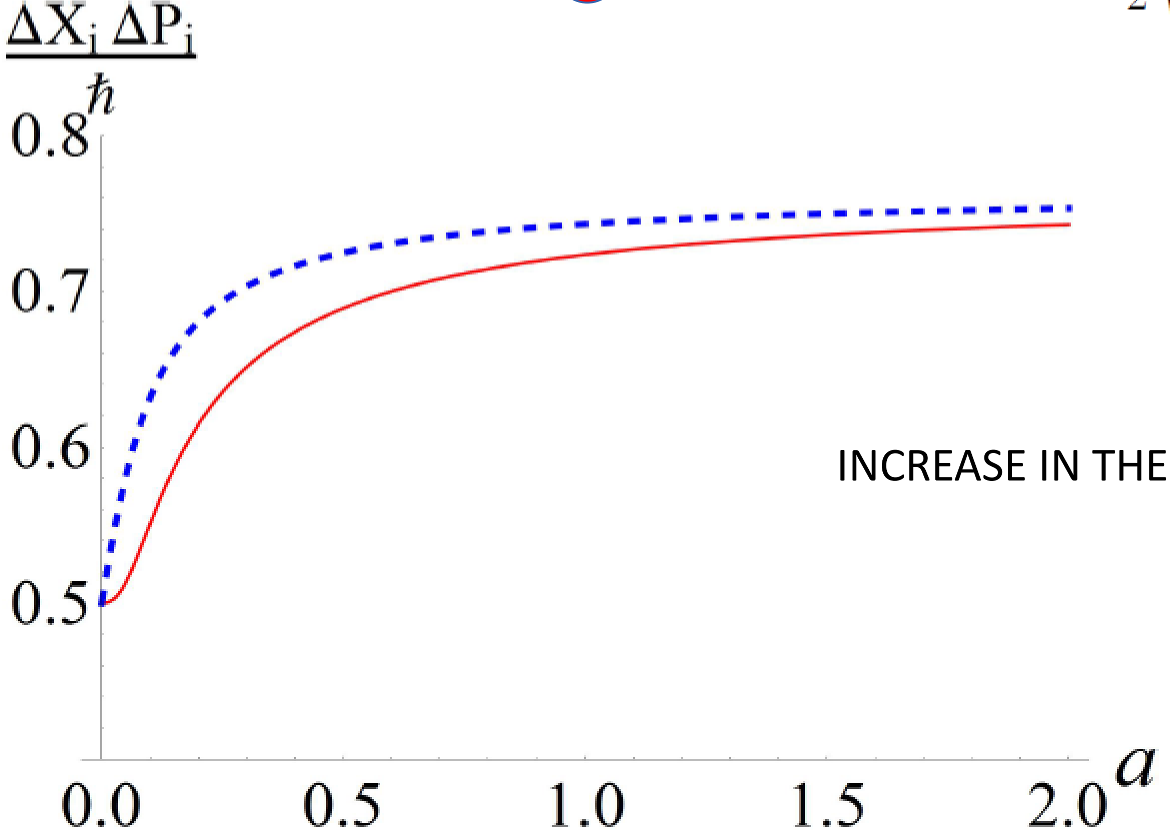
- $\mathbf{x}_0 = \langle \hat{\mathbf{X}} \rangle,\ \sum_{i=1}^{2} |\gamma^i|^2 = 1,\ \mathbf{p}_0 = \langle \hat{\mathbf{P}} \rangle = |\mathbf{p}_0| e_z$
- $a = \dfrac{(\Delta p)^2}{2p_0^2}$ Wavefunction's width in the momentum space

Preparation Uncertainty Relations for X and P

Results

$$\Delta Z \Delta P_z = \frac{\hbar}{2}\sqrt{1 - 4a + 4\sqrt{a}\,(1+2a)\,\mathcal{D}\left(\frac{1}{2\sqrt{a}}\right)}$$

$$\Delta X \Delta P_x = \Delta Y \Delta P_y = \frac{\hbar}{2}\sqrt{1 + 8a - 16a^{3/2}\mathcal{D}\left(\frac{1}{2\sqrt{a}}\right)}$$



INCREASE IN THE STATISTICAL CHARACTER

Contents

Spin probability distribution

- Consider the projection of the spin along a certain desired direction $S_n = \mathbf{S} \cdot \mathbf{n}$ which in $\mathbb{C}^3$ admits the spectral decomposition $\mathbf{S} \cdot \mathbf{n} = \sum_{m_s} \hbar m_s \left|\phi_{\mathbf{n},m_s}\right\rangle \left\langle\phi_{\mathbf{n},m_s}\right|$

- The probability distribution of such observable can be calculated projecting the PVM onto the physical Hilbert space $\mathscr{H}_S$, this leading to

$$p(S_n = \hbar m_s) = \int_{\mathbb{R}^3} \frac{d^3 p}{|\mathbf{p}|} \left| \sum_{i=1}^{2} \tilde{\psi}_V^i(\mathbf{p}) \; \phi^*_{\mathbf{n},m_s} \cdot V \tilde{\mathbf{e}}_i(\mathbf{p}) \right|^2$$

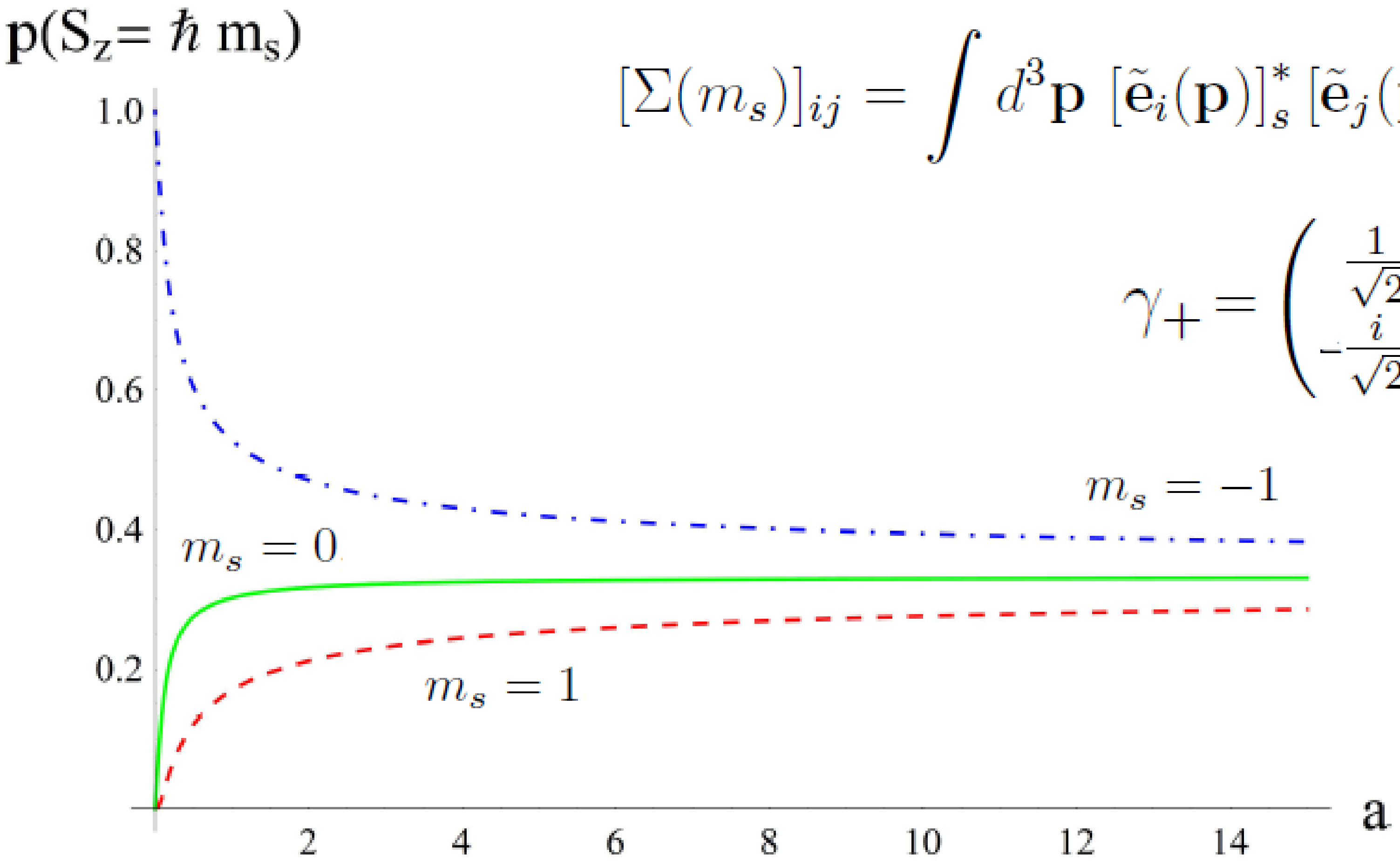
- For example let's calculate the probability distribution of S_z along physical states with definite polarization

Spin probability distribution

$$p(S_z = \hbar m_s) = \sum_{ij} (\gamma^i)^* \, [\Sigma(m_s)]_{ij} \, \gamma^j \qquad\qquad m_s = 2 - s$$

$$[\Sigma(m_s)]_{ij} = \int d^3\mathbf{p} \, [\tilde{\mathbf{e}}_i(\mathbf{p})]_s^* \, [\tilde{\mathbf{e}}_j(\mathbf{p})]_s \; g(\mathbf{p})$$

$$\gamma_+ = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \tilde{\psi}_V(p)$$

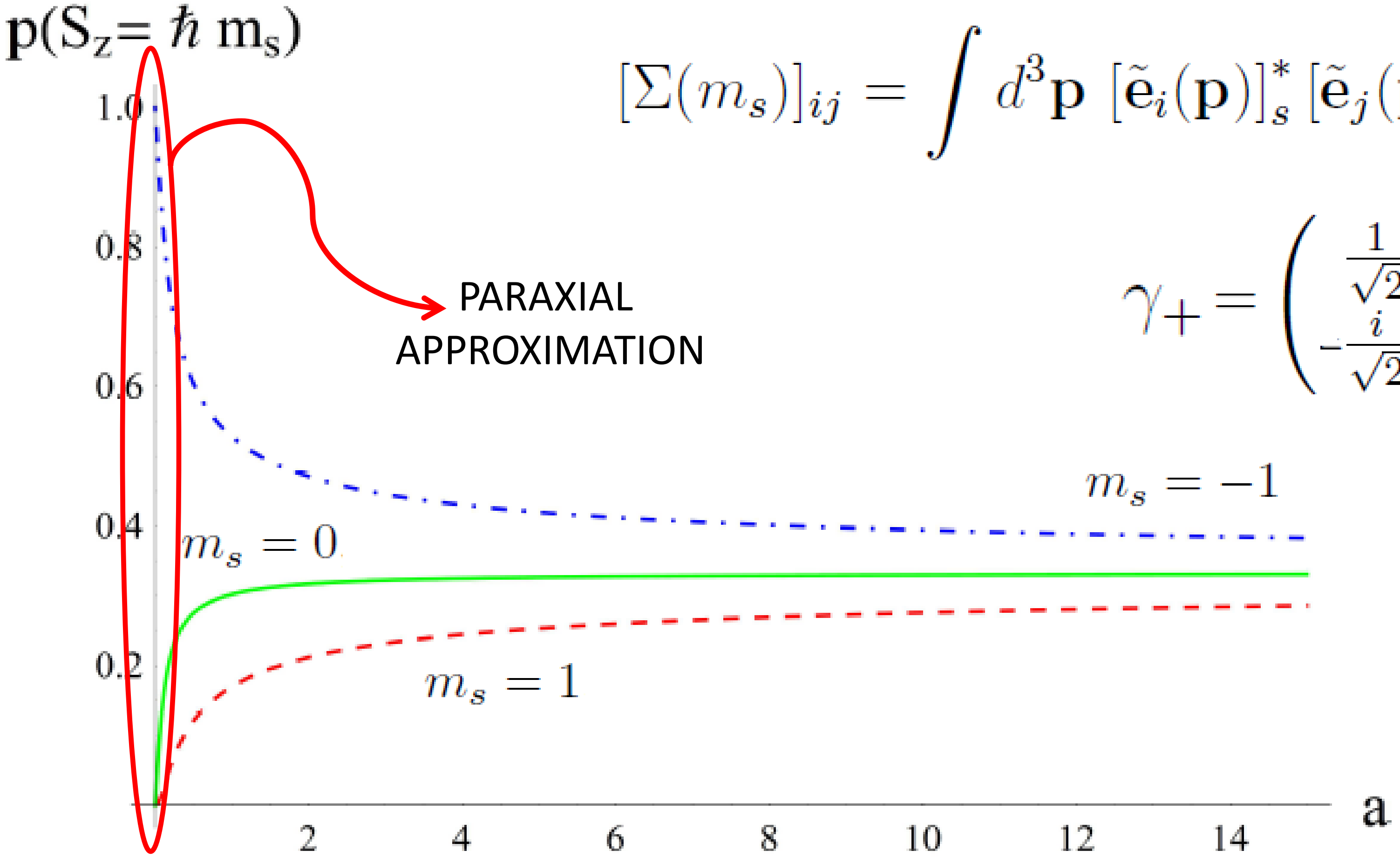


Spin probability distribution

$$p(S_z = \hbar m_s) = \sum_{ij} (\gamma^i)^* \, [\Sigma(m_s)]_{ij} \, \gamma^j \qquad m_s = 2 - s$$

$$[\Sigma(m_s)]_{ij} = \int d^3\mathbf{p} \, [\tilde{\mathbf{e}}_i(\mathbf{p})]_s^* \, [\tilde{\mathbf{e}}_j(\mathbf{p})]_s \;\; g(\mathbf{p})$$

$$\gamma_+ = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \tilde{\psi}_V(p)$$



Conclusions

1 The projection from the extended Hilbert space onto the physical one, naturally determined by the suppression of the longitudinal component of the photon wavefunction, leads to a systematic increment in the statistical character of the theory.

2 While the momentum operator commutes with the projection, the position operator does not, leading to the well – known result that a single photon cannot be localized with certainty in a certain bounded region of space. This is reflected in the uncertainty relations $\Delta X_j \, \Delta P_j \geq \hbar/2$, which in fact grows with the width of the wavefunction in the momentum space.

3 Also the spin of a single photon does not commute with the projection, this allowing to recover the result that a photon cannot be prepared with probability 1 in any eigenstate of spin. Our formalism allows however to calculate the probability distribution of such observable.

4 All these quantities could be in principle be measured in laboratory.

G. G. , M. Motta, L. Lanz, J. Phys. A: Math. Theor. **48** (2015) 265302

Summary

1 We have seen how to introduce a well-defined Hilbert space for a single photon only by requiring to deal with an irreducible representation of the Poincaré group for spin s=1 and mass m=0 particles and retrieving Kraus notorious result.

2 We have shown how the suppression of the zero-helicity component of the single-photon wavefunction, which through the isomorphism V corresponds to the suppression of the longitudinal component, can be viewed as a projection from an extended Hilbert space into the physical one.

3 We have then shown how this construction naturally brings along the notion of POVM, and we gave given a general formula, valid for any observable mutuated from the theory of relativistic spin s=1 massive particles, that allows to calculate ANY probability distribution

4 We have applied this formalism to evaluate the preparation uncertainty relations and spin probability distribution for a large class of physically relevant states, namely gaussian states with definite polarization. Results show an increment of the statistical character of the theory, mirrored by the enlargement of the PUR and of the probability distribution of spin, which ceases to be a definite quantity for single photons.

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THANK YOU
FOR YOUR ATTENTION

Single-Photon Space and States

POINT OF THE SITUATION

START:

$$\mathcal{L}^2(\mathbb{R}^3)\otimes\mathbb{C}^3 \xrightarrow{V^\dagger} \mathcal{L}^2(\mathbb{R}^3)\otimes V^\dagger\mathbb{C}^3$$

Non – relativistic massive spin s=1 particles

Isomorphism

Transformation law as a vector field

$$\rightarrow \mathcal{L}^2\left(\mathbb{R}^3,\frac{d^3\mathbf{p}}{|\mathbf{p}|}\right)\otimes V^\dagger\mathbb{C}^4 \rightarrow \mathcal{S}$$

Passage to the relativistic case

Transformation law as a four - vector field

Massless condition m=0

Non – negative seminorm

$$\xrightarrow{\mathcal{S}/\sim} \mathscr{H}_{\mathcal{S}}=\left\{\tilde{\boldsymbol{\psi}}(p):\tilde{\boldsymbol{\psi}}(p)=\sum_{i=1}^{2}\tilde{\psi}^i(p)\tilde{\mathbf{e}}_i(\mathbf{p}),\tilde{\psi}^i(p)\in\mathcal{L}^2\left(\mathbb{R}^3,\frac{d^3\mathbf{p}}{|\mathbf{p}|}\right)\right\}$$

Single-Photon Space and States

More familiar description of the single – photon states as the ones with non – zero helicity?

NO PROBLEM!

POINT OF THE SITUATION

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$$\mathcal{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^3$$

Non – relativistic massive spin s=1 particles

$V^\dagger$ → Isomorphism

$$\mathcal{L}^2(\mathbb{R}^3) \otimes V^\dagger \mathbb{C}^3$$

Transformation law as a vector field

→ Passage to the relativistic case

$$\mathcal{L}^2\left(\mathbb{R}^3, \frac{d^3\mathbf{p}}{|\mathbf{p}|}\right) \otimes V^\dagger \mathbb{C}^4$$

Transformation law as a four - vector field

→ Massless condition m=0

$$\mathcal{S}$$

Non – negative seminorm

$\mathcal{S}/\sim$ →

$$\mathscr{H}_S = \left\{ \tilde{\psi}(p) : \tilde{\psi}(p) = \sum_{i=1}^{2} \tilde{\psi}^i(p) \tilde{\mathbf{e}}_i(\mathbf{p}), \tilde{\psi}^i(p) \in \mathcal{L}^2\left(\mathbb{R}^3, \frac{d^3\mathbf{p}}{|\mathbf{p}|}\right) \right\} \simeq \mathcal{L}^2\left(\mathbb{R}^3, \frac{d^3\mathbf{p}}{|\mathbf{p}|}\right) \otimes V^\dagger \mathbb{C}^2$$

V ↓ Isomorphism

$$\overline{\mathscr{H}}_S \simeq \mathcal{L}^2\left(\mathbb{R}^3, \frac{d^3\mathbf{p}}{|\mathbf{p}|}\right) \otimes \mathbb{C}^2$$

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V

Isomorphism

$$\overline{\mathscr{H}}_S \simeq \mathcal{L}^2\left(\mathbb{R}^3, \frac{d^3\mathbf{p}}{|\mathbf{p}|}\right) \otimes \mathbb{C}^2$$

NO PROBLEM!

Helicity Operator

$$\epsilon = \frac{1}{\hbar}\left(\mathbf{S} \cdot \frac{\mathbf{p}}{|\mathbf{p}|}\right)$$

$$\chi_0(\mathbf{p}) \equiv V \tilde{\mathbf{e}}_3(\mathbf{p})$$

$$\chi_\pm(\mathbf{p}) \equiv V \tilde{\mathbf{e}}_\pm(\mathbf{p})$$