2d Quantum Gravity with matter: a statistical mechanical approach

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in collaboration with

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Outline

Euclidean Quantum Gravity

In the spirit of QFT, one would like to define a gravitational path-integral

$$\mathcal{Z}(\Lambda, G) = \int_{Geom(M)} \mathscr{D}[g] \; e^{-\mathcal{S}_{\Lambda,G}[g]}$$

where $Geom(M) = Metric(M)/Diff(M)$, and

$$\mathcal{S}_{\Lambda,G}[g] = \frac{1}{16\pi G} \int_M \mathrm{d}^d x \sqrt{|\det g|}(-R + 2\Lambda)$$

is the Euclidean Einstein-Hilbert action.

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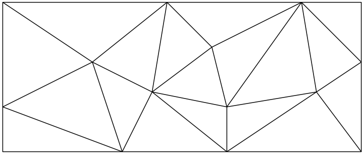

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- Dynamical Triangulation (DT): all edge lengths fixed to a (cut-off parameter), variable incidence matrix (geometry encoded in the connectivity). Emergence of causality violating geometries.

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- Causal Dynamical (Lorentzian) Triangulation (CDT): causality imposed from the start by choosing DTs that can be sliced perpendicularly to the time-like direction and space-like subgraphs with fixed topology.

2d discrete Quantum Gravity

In 2 dimensions the curvature term in the Einstein-Hilbert action is purely topological (Gauss-Bonnet theorem)

$$\begin{aligned}\mathcal{S}_{\Lambda,G}[g] &= \frac{1}{16\pi G}\int_M \mathrm{d}^2x\sqrt{|\det g|}(-R+2\Lambda)\\ &= -\frac{\chi(h)}{4G}+\frac{\Lambda}{8\pi G}V_g\end{aligned}$$

$\chi(h)$ is the Euler characteristic of M and V_g its volume for a given (diffeomorphism class of) metric g.

2d discrete Quantum Gravity

Fixing the topology and noting that for a given triangulation T $V_g \propto |F(T)|$, where $|F(T)|$ is the number of triangles in T, the 2-dimensional discrete action can be defined as

$$S_\mu(T) = \mu |F(T)|$$

and the gravitational path-integral reduces to

$$Z(\mu) = \sum_{T \in \mathcal{T}} e^{-\mu |F(T)|},$$

where the sum is intended over inequivalent triangulations.
The original path-integral problem is now reduced to a combinatorial one and the continuum can be restored by a proper scaling limit procedure.

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- Give to each $G \in \mathcal{G}_N$ a weight $w(G)$ and define a probability measure on $\mathcal{G}_N$

$$p_N(G) = \frac{w(G)}{Z_N}, \quad Z_N = \sum_{G \in \mathcal{G}_N} w(G)$$

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Examples: Uniform Infinite Planar Triangulation (O. Angel, O. Schramm, 2002), Infinite Random Trees (B. Durhuus, T. Jonsson, J.F. Wheater 2006), Infinite Causal Triangulation (B. Durhuus, T. Jonsson, J.F. Wheater 2010)

2d Quantum Gravity as a theory of random geometry

Alternatively, one can study scaling limits of such random graphs. In analogy with random walk and the related Brownian motion, one can see the scaling limit of random triangulations as some sort of *Brownian surface*. Indeed, it has been proved (J.F. Le Gall, 2011 and G. Miermont, 2011) that a (unif. distrib.) p-angulation of $\mathbb{S}^2$ converges, in some sense, to a unique compact metric space homeomorphic to $\mathbb{S}^2$ (the so-called *Brownian map*).

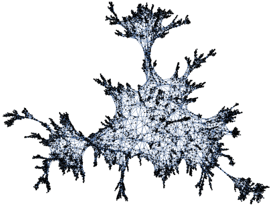

Figure: Triangulation of a sphere (J.F. Le Gall, Proceedings of ICM2014, Seoul).

The 2d CDT model with matter

Some useful notions

For a given graph $G = (V, E)$, define

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$$\gamma = \{(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)\}$$

- *Graph distance* $d_g(v, v')$, size of the shortest path between v and v'

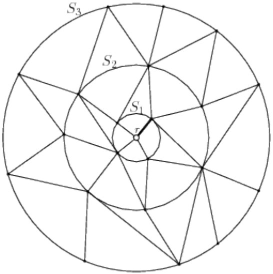
$$d_g(v, v') = \min \left\{ |\gamma| \;\middle|\; \gamma \text{ has endpoints } v,\ v' \right\}$$

Connected graph: there exists a path between any two vertices.

Lorentzian Triangulation

A rooted planar locally finite connected graph $T = (V(T), E(T))$ such that

1. The set of vertices at graph distance i from the root, together with the edges connecting them form a cycle denoted by S_i
2. All internal faces of the graph are triangles
3. One edge attached to the root vertex r is marked



$$\mathcal{T}_N = \{\text{Lorentzian triangulations with } N \text{ layers}\}$$

Pure CDT (no matter)

Define a probability measure on the set of Lorentzian triangulations $\mathcal{T}_N$ (note that this is a countable set)

$$p_{N,\mu}(T) = \frac{e^{-\mu|F(T)|}}{Z_N(\mu)},$$

where the partition function is

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$\mathcal{T}_N$ can be partitioned by fixing the number $\mathbb{k} = (k_1, \ldots, k_N)$ of vertices on each layer, i.e. $\mathcal{T}_N = \bigcup_{\mathbb{k}} \mathcal{T}_{\mathbb{k},N}$. $|\mathcal{T}_{\mathbb{k},N}|$ is known and $|F(T)| = 2\sum_{i=1}^{N} k_i - k_N$.

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The partition function can be computed exactly!
It is critical for any N at $\mu = \mu^{cr} = \log 2$.

Pure CDT. Some results.

Theorem (V. Malyshev, A. Yambartsev and Zamyatin, 2001)

If $\mu > \mu^{cr}$ (subcritical regime) the partition function

$$Z_{N,l}(\mu) = \sum_{T \in \mathcal{T}_{N,l}} e^{-\mu F(T)}, \quad \mathcal{T}_{N,l} = \{T \in \mathcal{T}_N : |S_N| = l\},$$

is finite for all N and its asymptotics as $N \to \infty$ is

$$Z_{N,l}(\mu) \sim (1 - \lambda(\mu)^2)^2 \, (\lambda(\mu))^{2N-l},$$

where $\lambda(\mu) = \frac{2e^{-\mu}}{1+\sqrt{1-4e^{-2\mu}}} < 1$.

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$$\mathbb{E}_{2N}|S_N| < f(\mu),\ \forall N \in \mathbb{N}$$

- At criticality, $\mu = \mu^{cr}$, it looks like a paraboloid of revolution.

$$\mathbb{E}_{2N}|S_N| \sim N, \text{ as } N \to \infty$$

The infinite triangulation has Hausdorff dimension $d_H = 2$ a.s.

Inserting matter

Matter can be inserted in the model by considering statistical mechanical systems (e.g. an Ising model) running on the random graphs. There are two ways to do that:

- Quenched coupling: first sample a triangulation with a probability measure p_μ, then run an Ising model on it. The Ising model is known to undergo a phase transition on the infinite *critical* Lorentzian triangulation (M. Krikun and A. Yambartsev, 2012).

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  The Ising model is known to undergo a phase transition on the infinite *critical* Lorentzian triangulation (M. Krikun and A. Yambartsev, 2012).
- Annealed coupling: triangulation and Ising spin configuration sampled together. E.g. DT (V.A. Kazakov, 1986) and random trees (GMN and B. Durhuus, 2012).

Annealed coupling. Spin-graph

Let Λ_N denote the set of graphs on finite triangulations of height N, together with spin configurations on them,

$$\Lambda_N = \left\{ (T, \sigma(T)) : T \in \mathcal{T}_N,\, \sigma(T) \in \{+1, -1\}^{V(T)} \right\}.$$

We call a *spin-graph* an element of space Λ_N. Notice that any element of $\mathcal{T}_N$ is a finite graph, therefore Λ_N is a set of finite spin-graphs.

We denote by $\Lambda_{N,l}$ the set of spin-graphs with fixed number of vertices on the N-th layer,

$$\Lambda_{N,l} = \left\{ (T, \sigma(T)) : T \in \mathcal{T}_N,\, |S_N| = l,\, \sigma(T) \in \{+1, -1\}^{V(T)} \right\}.$$

Gibbs distribution

A *Gibbs* family on the space of finite spin-graphs Λ_N is a probability measure defined by

$$p_{N,\beta,\mu}(T,\sigma) = \frac{e^{-\beta H(T,\sigma)-\mu F(T)}}{Z_N(\beta,\mu)},$$

$\beta, \mu \geq 0$, $F(T)$ is the number of triangles in T and

$$Z_N(\beta,\mu) = \sum_{(T,\sigma)\in\Lambda_N} e^{-\beta H(T,\sigma)-\mu F(T)}.$$

The interaction energy between spins is described by the Hamiltonian (nearest neighbor)

$$H(T,\sigma) = -\sum_{(u,v)\in E(T)} \sigma_u \sigma_v.$$

Gibbs distribution

Similarly on $\Lambda_{N,l}$ we define the probability measure

$$p_{N,l,\beta,\mu}(T,\sigma) = \frac{e^{-\beta H(T,\sigma)-\mu F(T)}}{Z_{N,l}(\beta,\mu)},$$

with

$$Z_{N,l}(\beta,\mu) = \sum_{(T,\sigma)\in\Lambda_{N,l}} e^{-\beta H(T,\sigma)-\mu F(T)}.$$

Critical parameters

Define for any fixed N, β and l critical values $\mu_N^{cr}(\beta)$ and $\mu_{N,l}^{cr}(\beta)$ so that

$$Z_N(\beta,\mu) < \infty, \text{ if } \mu > \mu_N^{cr}(\beta),$$

and

$$Z_N(\beta,\mu) = \infty, \text{ if } \mu < \mu_N^{cr}(\beta),$$

and similar

$$Z_{N,l}(\beta,\mu) < \infty, \text{ if } \mu > \mu_{N,l}^{cr}(\beta),$$

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$$Z_{N,l}(\beta,\mu) = \infty, \text{ if } \mu < \mu_{N,l}^{cr}(\beta).$$

Existence of a critical curve

Since

$$Z_N(\beta,\mu) = \sum_l Z_{N,l}(\beta,\mu)$$

we have for any l

$$\mu^{cr}_{N,l}(\beta) \le \mu^{cr}_N(\beta).$$

Proposition (GMN and T. Turova, 2015)

For any fixed l there exists $\mu^{cr} = \mu^{cr}(\beta)$ such that

$$\lim_{N\to\infty} \mu^{cr}_{N,l} = \mu^{cr}(\beta).$$

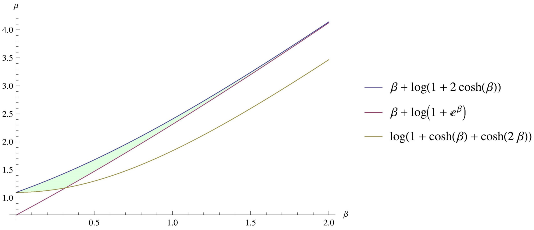
Theorem (GMN and T. Turova, 2015)

The partition function $Z_N(\beta, \mu)$ is finite for all $N \in \mathbb{N}$ in the region of the (β, μ)-plane defined by

$$\Delta_f = \{(\beta, \mu) \in \mathbb{R}^2 : \beta \geq 0, \mu > \beta + \log(1 + 2\cosh\beta)\}.$$

Moreover, if $Z_N(\beta, \mu)$ is finite for all N, then we necessarily have

$$\mu > \max\{\log(1 + \cosh\beta + \cosh(2\beta)), \beta + \log(1 + e^{\beta})\}.$$



Asymptotics

Proposition (GMN and T. Turova, 2015)

At least for all $\mu > \log 2 + \beta$ and such that $\log Z_{N,l}(\beta, \mu)$ is defined

$$\lim_{N\to\infty} \frac{\log Z_{N,l}(\beta,\mu)}{N} = \alpha(\beta,\mu,l),$$

where for all $\mu > \mu^{cr}(\beta)$

$$\alpha(\beta,\mu,l) < 0.$$

Spin-graph with boundary conditions

Given a triangulation $T \in \mathcal{T}_{N,l}$, a spin configuration on T with *boundary conditions* $\tilde{\sigma} \in \{+1,-1\}^l$ is an element of the set

$$\Omega^{\tilde{\sigma}}(T) = \{\sigma \in \Omega(T) : \sigma_v = \tilde{\sigma}_v, v \in V(S_N)\}$$

and a spin-graph (T,σ) of height N with $(l,\tilde{\sigma})$-boundary conditions is an element of

$$\Lambda^{\tilde{\sigma}}_{N,l} = \left\{(T,\sigma(T)) : T \in \mathcal{T}_{N,l}, \quad \sigma(T) \in \Omega^{\tilde{\sigma}}(T)\right\}.$$

Spin-graph with boundary conditions

A *Gibbs distribution* on the space of finite spin-graphs $\Lambda_{N,l}^{\tilde{\sigma}}$ is a probability measure defined by

$$p_{N,l,\beta,\mu}^{\tilde{\sigma}}(T,\sigma) = \frac{e^{-\beta H(T,\sigma)-\mu|F(T)|}}{Z_{N,l}^{\tilde{\sigma}}(\beta,\mu)}, \quad (T,\sigma) \in \Lambda_{N,l}^{\tilde{\sigma}},$$

where

$$Z_{N,l}^{\tilde{\sigma}}(\beta,\mu) = \sum_{(T,\sigma)\in\Lambda_{N,l}^{\tilde{\sigma}}} e^{-\beta H(T,\sigma)-\mu|F(T)|}.$$

Magnetization of central spin

Theorem (GMN and T. Turova, 2015)

For β small enough and $\mu > \frac{3}{2}\log(\cosh\beta) + 3\log 2$, the mean magnetization of the central spin of spin-graphs with $(I, -)$- as well as with $(I, +)$-boundary conditions converges to 0 as N goes to infinity:

$$\lim_{N\to\infty} \langle\sigma_0\rangle^{+}_{N,I,\beta,\mu} = 0 = \lim_{N\to\infty} \langle\sigma_0\rangle^{-}_{N,I,\beta,\mu}.$$

However, for any finite N and any $(\beta, \mu) \in \Delta_f$ one has

$$\langle\sigma_0\rangle^{-}_{N,I,\beta,\mu} < 0 < \langle\sigma_0\rangle^{+}_{N,I,\beta,\mu}.$$

Conjectures

- Above the critical line the average surface is expected to be 1-dimensional. If so, no phase transition for the spin configuration should occur.
- At criticality, i.e. for $\mu = \mu^{cr}(\beta)$, the average surface should behave as a 2-dimensional one, therefore the spin system is expected to be critical at $\beta = \beta^{cr}$.

Note that to obtain these results we do not need to know the exact partition function, but "only" its asympotics as $N \to \infty$.

- Critical exponents identical to Onsager values (showed by computer simulations J. Ambørn, K.N. Anagnostopoulos and R. Loll, 1999).

Thank you!