QCD tree level amplitudes and scattering equations

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Outline

Tree level QCD primitive amplitudes

The scattering equations

The CHY (Cachazo-He-Yuan) representation of tree-level primitive QCD amplitudes

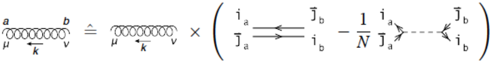
Conclusions and discussions

Introduction: Color flow decomposition

- Multi-parton QCD amplitudes have a complicated structure.
  - Factorization of color and kinematic information: $\mathcal{A} = \sum_i C_i A_i$.
  - C_i = Products of $\in \{c_{\mathrm{open}}(q_i, g, \ldots, g, \bar{q}_{\bar{j}}), c_{\mathrm{closed}}(g, \ldots, g)\}$ (phase space independent)
  - "Partial amplitudes": gauge invariant set of color stripped diagrams (phase space dependent but independent of the color flow). **Color ordering**
- **Color flow basis:** $SU(N) \to U(N)/U(1)$

$$c_{\mathrm{closed}}(g_1, \ldots, g_n) = \delta_{1\bar{2}}\delta_{2\bar{3}} \ldots \delta_{n\bar{1}}$$

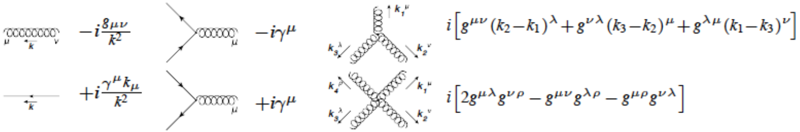
$$c_{\mathrm{open}}(q_i, g_1, \ldots, g_n, \bar{q}_{\bar{j}}) = \delta_{i_q\bar{1}}\delta_{1\bar{2}} \ldots \delta_{n\bar{j}_q}$$



U(1) gluons: Carry no color informations; couple only between quark lines, not among each others and not to $U(N)$ gluons.

Introduction:Primite amplitudes

- Color stripped Feynman rules (antysymmetric wrt exchanging legs)



- Partial amplitudes are in general not cyclic ordered !
  - Not-all diagrams in the sub-amplitudes have the same ordering of the external legs.
  - Express partial amplitudes A_i further through cyclic ordered primitive amplitudes P_j

$$\mathcal{A}_n = \sum_i C_i A_i = \sum_i C_i \sum_j F_{ij} P_j$$

Introduction:Examples

- Primitive amplitudes are obtained from gauge invariant set of diagrams with a fixed cyclic ordering.
- Simple example at tree level
  - Gluon Born Amplitude

$$\mathcal{A}^{\text{tree}} = \left(\frac{g_s}{\sqrt{2}}\right)^{n-2} \sum_{P(1,\ldots,n-1)} c_{\text{closed}}(g_1,\ldots,g_n) \times A_n^{(0)}(1,2,\ldots,n)$$

Here $A_n^{(0)}(1,2,\ldots,n) = P_n^{(0)}(1,2,\ldots,n)$

Born amplitude with one quark pair

$$\mathcal{A}_n^{\text{tree}} = \left(\frac{g_s}{\sqrt{2}}\right)^{n-2} \sum_{P(2,\ldots,n-1)} c_{\text{open}}(q_1,g_2,\ldots,g_{n-1},\bar{q}_1) \times$$

$$A_n^{(0)}(q_1,2,\ldots,n-1,\bar{q}_1)$$

Here $A_n^{(0)}(q_1,2,\ldots,n-1,\bar{q}_1) = P_n^{(0)}(q_1,2,\ldots,n-1,\bar{q}_1)$.

Introduction:Multiple quark pairs (tree-level)

- Amplitudes with distinct n_q quark pairs

$$\hat{\mathcal{A}}_n^{\text{tree}} = \left(\frac{g_s}{\sqrt{2}}\right)^{n-2} \sum_{\pi \in S_{n_q}} \sum_{\substack{i_1, i_2, \ldots, i_{n_q} \geq 0 \\ i_1 + i_2 + \cdots + i_{n_q} = n}} \sum_{\sigma \in S_{n_g}}$$

$$c_{\text{open}}(q_1, g_{\sigma_1}, \ldots, g_{\sigma_{i_1}}, \bar{q}_{\pi(1)}) c_{\text{open}}(q_2, g_{\sigma_{i_1+1}}, \ldots, g_{\sigma_{i_1+i_2}}, \bar{q}_{\pi(2)})$$

$$\ldots c_{\text{open}}(q_{n_q}, g_{\sigma_{i_1+\cdots+i_{n_q-1}+1}} + \cdots + g_{\sigma_{1+\ldots n_q}} + \bar{q}_{\pi_{(n_q)}})$$

$$A_n^0(q_1, g_{\sigma_1}, \ldots, g_{\sigma_{i_1}}, \bar{q}_{\pi(1)}, q_2, \ldots, g_{\sigma_{i_1+\cdots+i_{n_q}}}, \bar{q}_{\pi_{(n_q)}})$$

- All distinct colorflow combinations by summing all distinct permutations of color indices:
  - Permutations π of n_q antiquarks color indices
  - Partitions/Distribution of $\{i_1, \ldots, i_{n_q}\}$ of n_g gluons among n_q open color strings.
  - Permutations σ of external gluon indices.

Primitive amplitudes and cyclicity

Sneak: Each partial amplitude can be associated to a a set $\{u_i\}$ of r cyclic words! ($\pi \Rightarrow \{u_i\}$).
Two cyclic words are connected by a U(1) gluon
$\rightarrow r-1\ U(1)-\mathrm{gluons.}$
Shuffles operations $U(u_1, \ldots, u_r)$ among the r cyclic words yeld all-cyclic orderings:

$$A_n^{(0)} = \left(-\frac{1}{N}\right)^{r-1} \sum_{w \in U(u_1,\ldots,u_r)} P_n^{(0)}(w)$$

Linearity on permutations space

Alphabet $A = \{\ell_i\} = \{q_1, \ldots, q_{n_q}, \bar{q}_1, \ldots, \bar{q}_{n_q}, g_1, \ldots, g_{n_g}\}$
Word $w = \ell_1 \ldots \ell_n$ are ordered sequences of $n = n_g + 2n_q$ letters.
Primitive amplitudes P(w) - Linear operators on the vector space of cyclic words w:

$$\sum_{w \in \lambda_1 w_1 + \lambda_2 w_2} P(w) = \lambda_1 P(w_1) + \lambda_2 P(w_2)$$

Reflection identity: $P(w) = (-1)^n P(w^T)$, where
$w^T : w = \ell_1 \ldots \ell_n \to \ell_n \ldots \ell_1$.
Partial reflection, as e.g. $P(q_1, \bar{q}_1, q_2, \bar{q}_2) = -P(q_1, \bar{q}_1, \bar{q}_2, q_2)$
(antisymmetry of vertices)

Relations among primitive amplitudes

1. Kleiss-Kuijf relations Given the subwords $w_1 = \ell_{\alpha_1} \dots \ell_{\alpha_j}$ and $w_2 = \ell_{\beta_1} \dots \ell_{\beta_{n-2-j}}$ such that

   $$\{\ell_1\} \cup \{\ell_{\alpha_1}, \dots, \ell_{\alpha_j}\} \cup \{\ell_{\beta_1}, \dots, \ell_{\beta_{n-2-j}}\} \cup \{\ell_n\} = \{\ell_1, \dots, \ell_n\}$$

   $$P_n^{(0)}(\ell_1, \ell_{\alpha_1}, \dots \ell_{\alpha_j} \ell_n \ell_{\beta_1} \dots \ell_{\beta_{n-2-j}}) = (-1)^{n-2-j} P_n^{(0)}(\ell_1 (w_1 ⧢ w_2^T) \ell_n)$$

   Shuffle product

   $$w_1 ⧢ w_2 = \sum_{\text{shuffles } \sigma} \ell_{\sigma(1)} \dots \ell_{\sigma(r)}$$

   - All permutations, which preserve the relative order $\ell_1 \dots \ell_k$ and $\ell_{k+1} \dots \ell_r$.
   - Exclude permutations where crossing of fermions lines cannot be avoided.

2. Bern-Carrasco-Johansson (BCJ) relations

   $$\sum_{i=2}^{n-1} \left(\sum_{j=i+1}^{n} 2p_i \cdot p_j \right) P_n^{(0)}(\ell_1 \ell_3 \dots \ell_i \ell_2 \ell_{i+1} \dots \ell_{n-1} \ell_n) = 0$$

The amplitude basis

The relations among tree-level primitive QCD amplitudes allow to express all amplitudes for a given set of external particles in terms of a set of basis amplitudes. The size of the basis is

$$N_{\rm basis} = \begin{cases} (n-3)! & n_q \in \{0,1\}, \\ (n-3)!\,\frac{2(n_q-1)}{n_q!} & n_q \geq 2. \end{cases}$$

Primitive amplitudes with no crossed fermion lines may be described by generalised Dyck words. Consider an alphabet consisting of n_q distinct opening brackets "$(_i$" for q_i and n_q corresponding closing brackets "$)_i$" for $\bar{q}_i$. A generalised Dyck word is any word from this alphabet with properly matched brackets.

$$N_{\rm Dyck} = \frac{(2n_q)!}{(n_q+1)!}$$

is the number of generalized Dyck's words with length $2n_q$.

The Amplitude basis

Let us now describe the amplitude basis for the various cases. For $n_q = 0$ the set of words corresponding to a possible basis is given by

$$B = \{\, \ell_1\ell_2...\ell_n \in P(1,2,\ldots,n) \,|\, \ell_1 = g_1,\, \ell_{n-1} = g_{n-1},\, \ell_n = g_n \,\};$$

for $n_q = 1$

$$B = \{\, \ell_1\ell_2...\ell_n \in P(1,2,\ldots,n) \,|\, \ell_1 = q_1,\; \ell_2 = g_1,\; \ell_n = \bar{q}_1 \;\};$$

for $n_q \geq 2$

$$B = \left\{\, \ell_1\ell_2...\ell_n \in \mathrm{Dyck}_{n_q} \mid \ell_1 = q_1,\; \ell_2 \in \{q_2, ..., q_{n_q}\},\; \ell_n = \bar{q}_1 \;\right\}.$$

An arbitrary primitive amplitude $P_n(w)$ with $w \in P(1,2,\ldots,n)$ is a linear combination of primitive amplitudes $P_n(w_j)$ with $w_j \in B$.

Review of scattering equations

Scattering equations (Cachazo, He, Yuan, 2013) (Naculich 2014)
Momentum configuration space of n external massless particles (straigthfordwally extended to the massive case)

$$\Phi_n = \{(p_1, p_2, \ldots, p_n) \in (CM)^n | p_1 + \cdots + p_n = 0, p_{g_j}^2 = 0 = p_{q_j}^2 = p_{\bar{q}_j}^2\}$$

$$f_i(z,p) = \sum_{j=1, j\neq i} \frac{2p_i \cdot p_j}{z_i - z_j} = 0 \quad z \in (CP^1)^n$$

If $\sum_{j\neq i} p_i \cdot p_j = 0$ $\;PSL(2,C) = SL(2,C)/Z_2$ invariance

$$z_i \to \frac{az_i + b}{cz_i + d} \quad ad - bc \neq 0$$

$n-3$ independent solutions modulo $SL(2,C)$ have $(n-3)!$ distinct solutions

Example: n=4, fix $(z_1, z_2, z_3) = (0, 1, \infty)$

$$f_1 = \frac{2p_1 \cdot p_2}{0-1} + \frac{2p_1 \cdot p_3}{0-\infty} + \frac{2p_1 \cdot p_4}{0 - z_4} = 0 \iff z_4 = -\frac{p_1 \cdot p_4}{p_1 \cdot p_2}$$

The CHY representation of tree-level primitive QCD amplitudes

All tree-level primitive QCD amplitudes have a representation in the form

$$P_n(w, p, \varepsilon) = \frac{i}{(2\pi i)^{n-3}} \int \frac{d^n z}{\mathrm{Vol}(PSL(2,C))} \times$$

$$\prod{}' \delta(f_a(z,p)) \; \hat{C}(w,z) \; \hat{E}(z,p,\varepsilon)$$

the primed product of delta functions stands for

$$\prod{}' \delta(f_a(z,p)) = (-1)^{i+j+k} (z_i - z_j)(z_j - z_k)(z_k - z_i) \prod_{a \neq i,j,k} \delta(f_a(z,p))$$

taking into account that only $(n-3)$ scattering equations are independent.

Short-hand notation

Define a $n!$-dimensional vector P_w with components

$$P_w = P_n\left(w, p, \varepsilon\right)$$

a $n! \times (n-3)!$-dimensional matrix

$$\hat{M}_{wj} \equiv J\left(z^{(j)}, p\right)\ \hat{C}\left(w, z^{(j)}\right)$$

and a $(n-3)!$-dimensional vector

$$\hat{E}_j \equiv \hat{E}\left(z^{(j)}, p, \varepsilon\right).$$

The CHY representation amounts to

$$P_w = i\,\hat{M}_{wj}\hat{E}_j,$$

where a sum over j is understood.

The Parke-Taylor factor

We label the external particles of a primitive amplitude P_n by 1, ..., n and the associated complex variables z_j occurring in the scattering equations by z_1, ..., z_n, such that the complex variable z_j corresponds to particle j.

- We define the standard cyclic factor $C(w, z)$ for $w = \ell_1 \ell_2 \dots \ell_n$ by

$$C\left(\ell_1 \ell_2 \dots \ell_n, z\right) = \frac{1}{\left(z_{\ell_1} - z_{\ell_2}\right)\left(z_{\ell_2} - z_{\ell_3}\right) \dots \left(z_{\ell_n} - z_{\ell_1}\right)}$$

- The standard cyclic factor $C(w, z)$, for z a solution of the scattering equations, satisfies all the relations of the pure gluonic primitive tree amplitudes.
- View $C(w, z)$ and $\hat{C}(w, z)$ as linear operators on the words vector space with basis on the permutations $P(1, \dots, n)$

$$\begin{aligned} C\left(\lambda_1 w_1 + \lambda_2 w_2, z\right) &= \lambda_1 C\left(w_1, z\right) + \lambda_2 C\left(w_2, z\right), \\ \hat{C}\left(\lambda_1 w_1 + \lambda_2 w_2, z\right) &= \lambda_1 \hat{C}\left(w_1, z\right) + \lambda_2 \hat{C}\left(w_2, z\right). \end{aligned}$$

Definition of the generalized Parke-Taylor factor $\hat{C}$

- For words with the first and the last letter fixed, with no crossed fermionic lines and standard orientation of the lines
$$\hat{C}(w,z) = C(w,z).$$
- For words with crossed fermionic lines
$$\hat{C}(w,z) = 0.$$
- The generalized cyclic factor $\hat{C}(w,z)$for words with non-standard orientation of the fermionic lines can be recursively expressed in terms of generalized cyclic factors for word with standard orientation of the fermionic lines.
- $$\hat{C}\left(\ell_1 w_1 \ell_n w_2, z\right) = (-1)^{|w_2|}\, \hat{C}\left(\ell_1 \left(w_1 \sqcup\!\sqcup w_2^T\right) \ell_n, z\right).$$
  (Kleiss-Kuijf relation)
- $$\hat{C}\left(w_1 \ell_1 w_2, z\right) = \hat{C}\left(\ell_1 w_2 w_1, z\right).$$
  (Cyclic invariance)

On the generalized permutational invariant function $\hat{E}$

We defined a $n! \times (n-3)!$-dimensional matrix $\hat{M}_{wj}$ such that

$$P_w = i \, \hat{M}_{wj} \hat{E}_j$$

by restricting for $w \in B$ we get a $N_{\mathrm{basis}} \times (n-3)!$ matrix.

- For $w \in B$, $\hat{M}_{wj}$ depends on the standard Parke-Taylor factor.
- We first establish that the matrix $\hat{M}^{\mathrm{red}}_{wj}$ has full row rank:

$$\mathrm{rank}\, \hat{M}^{\mathrm{red}}_{wj} = N_{\mathrm{basis}}.$$

  If $\hat{M}^{\mathrm{red}}_{wj}$ has full row rank, a right-inverse $\hat{N}^{\mathrm{red}}_{jw}$ exists. The right-inverse might not be unique.
- Conjecture: the external orderings of a minimal amplitude basis for $n_q > 0$ remain linearly independent, when viewed as the external ordering of pure gluonic amplitudes. We have verified this conjecture for all amplitudes up to 10 points.

Conclusions and discussions

- In most phenomenological applications one usually just wants the amplitude computed from a given set of external four-momenta. The best way to do this numerically are Berends-Giele recursion relations, or Britto-Cachazo-Feng-Witten recusion relations. This does not involve the scattering equations.
- Sometimes it is useful to have compact analytical formulae for the scattering amplitudes. Using spinor techniques this can be done for every specific helicity configuration and every specific external ordering. Doing this for every helicity configuration and every external ordering does not really give you a clue how the results change as you change the helicity configuration or the external ordering.
- Here the approach based on the scattering equations is useful: It gives you a formula, which allows you to switch a single helicity or to swap the order of two external particles. From an aestetic point of view this is nice to know.