DUALS OF LATTICE QCD IN THE PRESENCE OF DYNAMICAL FERMIONS

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I. Duals of pure gauge lattice models

Dual representations and their significance:

- Map strong coupling (disordered) phase to weak coupling (ordered) phase

- Allow to establish in many cases relevant configurations responsible for phenomena like confinement, mass gap generation, etc.

- Have been used to establish many rigorous results in spin and gauge models

Conventional dual transformations can be defined as a sequence of transformations consisting of the following steps:

1. Fourier expansion of the Boltzmann weight $\exp(H)$. This is an essentially character expansion on the group.

2. Exact integration over original degrees of freedom. As a result one obtains a set of constraints on the summation variables (which label representations of the group G and matrix elements of $U(x)$) in the character expansion.

3. Solution of the constraints in terms of new dual variables.

Other approaches: plaquette formulation, n-link action.

LGT in $(d+1)$-dimensions

$\Lambda = N_t \times L^d$ periodic lattice

$$Z_\Lambda = \int \prod_l dU_l \ \exp \sum_{x;\mu<\nu} \beta_{\mu\nu} \, \mathrm{Re}\, \chi_{\mathrm{f}}(\mathrm{U_p}) \ , \ p = (x; \mu, \nu)$$

$$U_p \ = \ U_\mu(x) U_\nu(x+\mu) U_\mu^\dagger(x+\nu) U_\nu^\dagger(x) \ , \ U_l \equiv U_\mu(x) \in U(N_c), SU(N_c)$$

$\chi_f(U)$ is the character of the fundamental representation and

$$\beta_{\mu\nu} \ = \ \beta \begin{cases} \xi, \ \mu \ , \nu \ = \ 1, \cdots, d \ , \\ \xi^{-1}, \ \text{if one of } \mu \ , \nu \ = \ 0 \ . \end{cases}$$

β is a dimensionless lattice coupling in $(d+1)$-dimensions

$$\beta \ = \ \frac{a_s^{d+1-4}}{g^2} \ , \ \xi = a_t / a_s$$

Temperature is defined as $T = (a_t N_t)^{-1}$.

Duals of abelian models

1. Fourier expansion

$$\prod_p \exp \beta_{\mu\nu} \cos \phi_p = \prod_p \sum_{r_p=-\infty}^{\infty} I_{r_p}(\beta_{\mu\nu}) e^{i r_p \phi_p}$$

2. Integration over gauge fields generates constraints on every link

$$\sum_{p \in l} \tilde{r}_p = 0, \tilde{r}_p = \pm r_p$$

3. Solution is given in terms of dual variables and depends on lattice dimension and abelian group. E.g., for $d = 3$ and $U(1)$ group: $r_p = r_x - r_{x+e_n}$, where x are centers of original cubes.

$$Z = \sum_{r_x=-\infty}^{\infty} \prod_l I_{r_x - r_{x+e_n}}(\beta_{\mu\nu})$$

Duals of non-abelian models

1. Fourier expansion

$$\prod_p \exp \beta_{\mu\nu} \, \mathrm{Re}\, \chi_{\mathrm{f}}(\mathrm{U_p}) = \prod_p \sum_{r_p} C_{r_p}(\beta_{\mu\nu}) \chi_{r_p}(U_p)$$

2. Integration over gauge fields generates a set of non-linear constraints on every link encoded in the Clebsch-Gordan coefficients of the non-abelian group which are triangular conditions on the representations and restrictions on allowed magnetic numbers (matrix elements).

3. Summation over matrix elements can be performed explicitly, and this leads to a represetation of the partition function in terms of gauge-invariant $6j$-symbols of the group. A solution of triangular conditions remains unknown.

$$Z = \sum_{r_p, r_l} \prod_p C_{r_p}(\beta_{\mu\nu}) \prod_x (6j \text{ links}) \prod_{\mathrm{c}} (\text{6j cubes})$$

II. Duals in the strong coupling limit

The Boltzmann weight for fermions

$$B_F[U_l; \overline{\psi^i_f}(x), \psi^i_f(x)] \;=\; \prod_{f=1}^{N_f} \exp\left[S_F + S_m\right] \;,$$

where the Kogut-Susskind action is used. S_m is the massive term

$$S_m \;=\; \frac{m_f}{d+1}\, \overline{\psi}_f(x)\psi_f(x)$$

and S_F describes interaction (colour indices are suppressed)

$$S_F = \gamma\, \frac{\eta_\nu(x)}{2}\, \left(y_\nu\, a(l)\, U_\nu(x) + y_\nu^{-1}\, b(l)\, U_\nu^\dagger(x)\right)$$

$$a(l) \;=\; \overline{\psi}_f(x)\psi_f(x+e_\nu)\;,\; b(l) \;=\; \psi_f(x)\overline{\psi}_f(x+e_\nu)\;.$$

m_f is a mass of the fermion field, N_f is the number of flavours and

$$\eta_0(x) \;=\; \xi^{-1}\;;\; \eta_\nu(x) \;=\; (-1)^{x_0+x_1+\ldots+x_{\nu-1}}\;,\; \nu \neq 0\;.$$

The chemical potential μ is introduced via y_ν

$$y_\nu \;=\; \begin{cases} \exp(\mu),\; \nu = 0\;, \\ 1,\; \nu = 1, \cdots, d\;. \end{cases}$$

Integration over fermions and gauge fields

$$\int \prod_l B_F[U_l; \overline{\psi^i_f}(x), \psi^i_f(x)] \prod_x \prod_{f,i} d\overline{\psi^i_f}(x) d\psi^i_f(x) \prod_l dU_l$$

produces what is called sometime dual of the strong coupling QCD. For $N_f = 1$

$$Z = \sum_{\{k(l)\}=0}^{N_c} \sum_{\{n(l)\}=0}^{N_c} \sum_{\{s(x)\}=0}^{N_c} \prod_l \left[\gamma \, \frac{\eta_\nu(x)}{2}\right]^{k(l)+n(l)}$$
$$\prod_l [y_\nu]^{k(l)-n(l)} \prod_x m^{s(x)} \, Q[k(l), n(l), s(x); N_c]$$

A set of constraints generated by integration

$$k_l - n_l = 0(\text{mod } N_c), \text{for } SU(N_c); k_l - n_l = 0, \text{for } U(N_c)$$

$$s(x)+k(x) = N_c, s(x)+n(x) = N_c; \; k(x) \; = \; \sum_{\nu=1}^{d} [k_\nu(x)+n_\nu(x-e_\nu)]$$

defines monomer-dimer model for $U(N_c)$ and monomer-dimer-closed baryon loop model for $SU(N_c)$ LGT.

III. Full QCD and its dual: problems and motivations

$$Z_\Lambda = \int \exp \sum_{x;\mu<\nu} \beta_{\mu\nu} \, \mathrm{Re} \, \chi_{\mathrm{f}}(\mathrm{U_p})$$
$$\prod_l B_F[U_l; \overline{\psi^i_f}(x), \psi^i_f(x)] \prod_x \prod_{f,i} d\overline{\psi^i_f}(x) d\psi^i_f(x) \prod_l dU_l$$

Formally, the dual of a full QCD is given by

$$Z = \sum_{\{r(p)\}} \sum_{\{k(l)\}=0}^{N_c} \sum_{\{n(l)\}=0}^{N_c} \sum_{\{s(x)\}=0}^{N_c} \prod_p d(r(p)) C[r(p)]$$
$$\prod_l \left[\gamma \, \frac{\eta_\nu(x)}{2}\right]^{k(l)+n(l)} \prod_l [y_\nu]^{k(l)-n(l)} \prod_x m^{s(x)} \, Q[r(p); k(l), n(l), s(x)] \; ,$$

$$Q[r; k, n, s] \; = \; \sum_{i_1 \cdots i_{2k}} \sum_{j_1 \cdots j_{2n}} F^{i_1 \cdots i_{2k}}_{j_1 \cdots j_{2n}}[k, n, s] \, \Xi^{i_1 \cdots i_{2k}}_{j_1 \cdots j_{2n}}[r] \; .$$

$F^{i_1 \cdots i_{2k}}_{j_1 \cdots j_{2n}}[k,n,s] = F$ is the integral over fermion fields

$$F = \int \prod_x \prod_{i=1}^{N_c} d\psi^i(x) d\overline{\psi^i}(x) \prod_x (\sigma(x))^{s(x)} \prod_l [a^{i_1 i_2}(l)]^{k(l)} \, [b^{j_1 j_2}(l)]^{n(l)}$$

and $\Xi^{i_1 \cdots i_{2k}}_{j_1 \cdots j_{2n}}[r] = \Xi$ is the integral over gauge fields

$$\Xi \; = \; \int \prod_l dU_l \; \prod_p \; \chi_{r(p)}(U_p) \; \prod_l [U_l^{i_1 i_2}]^{k(l)} [U_l^{j_1 j_2, \dagger}]^{n(l)} \; .$$

Convention is

$$[P^{i_1 i_2}]^k \; = \; P^{i_1 i_2} P^{i_3 i_4} \ldots P^{i_{2k-1} i_{2k}} \; .$$

Rewriting the character as a sum over matrix elements one obtains

$$\Xi \; = \; \left(\prod_p \sum_{m_i} \right) \prod_l \; G(l) \; , \; i = 1, ..., 4 \; \; ,$$

where the one-link integral is given by

$$G(l) \; = \; \int dU \; U_{r_1}^{m_1 n_1} \ldots U_{r_{2(d-1)}}^{m_{2(d-1)} n_{2(d-1)} \dagger} \; [U^{i_1 i_2}]^{k(l)} [U^{j_1 j_2, \dagger}]^{n(l)} \; .$$

- Can one perform summations over all matrix (colour) indices locally and express partition function in terms of gauge invariant objects? Equivalently: is a dual representation a theory with a local interaction between gauge invariant objects?

- Can one replace, and if yes - how, the complicated $6j$-symbols with something more manegable? Equivalently: can a dual representation be really useful?

- In case of positive answers to above questions: what kind of non-perturbative physics can be studied using dual form of QCD?

- Sign problem at $\mu \neq 0$: can it be made more treatable or even solved within some version of the dual formulation?

IV. Dual transformations for $U(N)$ LGT

The main idea is to replace the conventional character expansion of the Boltzmann weight with the Taylor expansion

$$\exp \beta \,\mathrm{Re}\, \chi_{\mathrm{f}}(\mathrm{U_p}) \;=\; \sum_{r=0}^{\infty} \frac{\beta^r}{r!} \left(\mathrm{Re}\, \chi_{\mathrm{f}}(\mathrm{U_p})\right)^{\mathrm{r}}$$

All previous formulae remain valid but one link integral becomes

$$G(l) \;=\; \int dU \; U^{m_1 n_1} \ldots U^{m_r n_r \dagger} \; [U^{i_1 i_2}]^{k(l)} [U^{j_1 j_2, \dagger}]^{n(l)} \; .$$

Only matrices in fundamental representation appear in the integrand.

$$\int dU \prod_{k=1}^{r_1} U^{i_k j_k} \prod_{n=1}^{r_2} U^{m_n l_n *} = \delta_{r_1, r_2} \sum_{\tau, \sigma \in S_r} Wg^N(\tau^{-1}\sigma) \prod_{k=1}^{r} \delta_{i_k, m_{\sigma(k)}} \delta_{j_k, l_{\tau(k)}} .$$

S_r - group of permutations of r elements. $Wg^N(\sigma)$ - Weingarten function which depends only on the length of cycles of σ.

$$Wg^N(\sigma) = \frac{1}{r!} \sum_{\lambda} \frac{d(\lambda)}{J^1_\lambda(1^N)} . \chi_\lambda(\sigma)$$

$d(\lambda), \chi_\lambda(\sigma)$ - dimension and character of S_r.

Example: Simple integral

$$\int dU \, (TrU)^s \, (TrU^*)^q \; = \; \delta_{s,q} \sum_{\lambda} d^2(\lambda) \; .$$

Example: $2D$ model in the large volume limit

$$Z = [z_0]^{N_p} \; , \; z_0 = \mathrm{det} \; I_{i-j}(\beta), \; 1 \leq i,j \leq N \; .$$

$$z_0 = \sum_{k=0}^{\infty} \frac{(\frac{\beta}{2})^{2k}}{(k!)^2} \sum_{\lambda} d^2(\lambda), \; \lambda_1 \geq \cdots \geq \lambda_N, \; \sum_{i=1}^{N} \lambda_i = k$$

Example: One-link integral in the strong coupling limit

$$\sum_{k=0}^{N} (\eta_\nu/2)^{2k} \frac{1}{(k!)^2} (\sigma_x \sigma_{x+e_\nu})^k \; \sum_{\tau \in S_k} Wg^N(\tau)$$

Application of the Weingarten formula

1. Unlike nj-symbols of $U(N)$, properties of $Wg^N(\sigma)$ (general form, recurrence relations, asymptotics) are known.

2. In abelian case one recovers the conventional dual model.

3. The constraint $r_1 = r_2$ is essentialy abelian one. One solves the constraint by introducing genuine dual variables, like in $U(1)$ model. No other constraints are generated.

4. For $U(N)$, in any dimension, summation over group (matrix) indices is factorized in every lattice site and can be done locally. Dual theory is a theory with only local interaction.

5. Last property holds also in the presence of fermions.

Partition function of pure gauge $U(N)$ theory

$3D$ model

$$Z_\Lambda = \sum_{r_x, k_l} \prod_l \frac{\beta^{2k_l + |r_x - r_{x+e_n}|}}{k_l!(k_l + |r_x - r_{x+e_n}|)!}$$

$$\sum_{\tau_p, \sigma_p} \prod_p Wg^N(\tau_p^{-1}\sigma_p) \prod_c (\text{group factor})$$

$$\text{group factor} = \text{symmetric function } P_\mu(1) = N^{|\mu|}$$

$|\mu|$ - the number of cycles in combined permutations $\sigma_p, \tau_p, p \in c$.

$4D$ model

$$Z_\Lambda = \sum_{r_l, k_p} \prod_p \frac{\beta^{2k_l + |r_{l_1} + r_{l_2} - r_{l_3} - r_{l_4}|}}{k_p!(k_p + |r_{l_1} + r_{l_2} - r_{l_3} - r_{l_4}|)!}$$

$$\sum_{\tau_c, \sigma_c} \prod_c Wg^N(\tau_c^{-1}\sigma_c) \prod_h (\text{group factor})$$

$2D\ U(N)$ LGT with fermions

Partition function of two-dimensional $U(N)$ LGT with one flavor of the Kogut-Susskind fermions at finite temperature and non-zero chemical potential takes the form

$$Z = \sum_{\{r(p)\}=-\infty}^{\infty} \sum_{\{t(p)\}=0}^{\infty} \sum_{\{k(l)\}=0}^{N_c} \sum_{\{n(l)\}=0}^{N_c} \sum_{\{s(x)\}=0}^{N_c} \sum_{\{\tau_l,\sigma_l\}\in S_z(l)}$$

$$\prod_p \frac{(\beta/2)^{2t(p)+|r(p)|}}{t_p!(t_p+|r(p)|)!} \prod_l \left[\frac{\eta_\nu(x)}{2}\right]^{k(l)+n(l)} [y_\nu]^{k(l)-n(l)}\ Wg^N(\tau_l^{-1}\sigma_l)$$

$$\prod_x m^{s(x)} \times (\text{constraints}) \times \prod_x (\text{group factor}).$$

- Gauge (fermion) integration produces constraint on every link (site)

$$\prod_l \delta\left(r(p)-r(p')+k(l)-n(l)\right) \ p,p' \text{ have common link } l,$$
$$\prod_x \delta\left(s(x)+k(x)-N\right) \ \delta\left(s(x)+n(x)-N\right) \ ,$$
$$k(x)=\sum_{\nu=1}^{d}\left[k_\nu(x)+n_\nu(x-e_\nu)\right], n(x)=\sum_{\nu=1}^{d}\left[n_\nu(x)+k_\nu(x-e_\nu)\right] .$$

- Permutation group $S_z(l)$ is fixed by

$$z(l)=t(p)+t(p')+\frac{1}{2}\ \left(|r(p)|+r(p)+|r(p')|-r(p')\right)+k(l) \ .$$

- Group factor arises after summation over all matrix (colour) indices

$$\text{Group factor} = N^{|\mu|}$$

with $|\mu|$ - the number of cycles in combined permutations $\sigma_l, \tau_l, l \in x$.

V. Problems and Perspectives

- LGT at large N: large N asymptotics of Weingarten function is known.
- Confinement problem and mass gap generation: expectation values.
- Group factor in the theory with dynamical fermions.
- Dual weight with non-zero chemical potential.
- Extension to $SU(N)$ models. One link integrals include terms proportional to unit anti-symmetric tensor on the group (= baryon states).
- Possibility of numerical simulations.