Infinite order phase transitions in two-dimensional $U(N)$ and $SU(N)$ models.

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OUTLINE

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I. Definitions of the models.

Consider a $2d$ lattice $\Lambda = L \times L$ with periodic boundary conditions. W_x spin variables on lattice sites are the elements of $U(N)$ or $SU(N)$ group.

$$Z(\beta) = \int \prod_x dW(x) \exp\left(\frac{\beta}{N^2} S_{\mathrm{f}} + \frac{\lambda}{N^2} S_{\mathrm{adj}}\right)$$

$$S_{\mathrm{f}} = \sum_{x,n} \mathrm{Re}\, \chi_{\mathrm{f}} W_x \, \chi_{\mathrm{f}} W^*_{x+e_n} \, , \; S_{\mathrm{adj}} = \sum_{x,n} \chi_{\mathrm{adj}} W_x \, \chi_{\mathrm{adj}} W_{x+e_n}$$

$$\chi_{\mathrm{f}} W = \mathrm{Tr} W \, , \; \chi_{\mathrm{adj}} W = \mathrm{Tr} W \; \mathrm{Tr} W^*$$

Parametrizing W by its eigenvalues $e^{i\omega_1}$, $e^{i\omega_2} \ldots e^{i\omega_N}$ we can write the invariant integration over $U(N)$ group as

$$\int dW \ F \ = \int_0^{2\pi} \prod_{j=1}^{N} \frac{d\omega_k}{2\pi} \prod_{k<l} |e^{i\omega_k} - e^{i\omega_l}|^2 \ F$$

In case of $SU(N)$ an extra condition $\prod_k \exp i\omega_k = 1$ is needed.

II. Effective model in terms of trace variables.

$$\mathrm{Tr}W = N\,\rho\,e^{i\omega}$$

$$Z(\beta) = \prod_x \int_0^1 \rho_x d\rho_x \int_0^{2\pi} d\omega_x M(\rho_x, \omega_x)$$
$$\exp \sum_{x,n} \left(\beta \rho_x \rho_{x+e_n} \cos\left(\omega_x - \omega_{x+e_n}\right) + \lambda N^2 \rho_x^2 \rho_{x+e_n}^2\right)$$

$M(\rho, \omega)$ can be considered invariant measure for trace and is found as

$$M(\rho, \omega) = \sum_{n=-\infty}^{\infty} \frac{N^2}{N!} \epsilon_{i_1, i_2, \ldots i_N} \epsilon_{j_1, j_2, \ldots j_N}$$
$$\int_0^{2\pi} \prod_{k=1}^{N} \frac{d\omega_k}{2\pi} \exp\left[i\omega_k(n + i_k - j_k)\right]$$
$$\delta\left(N\rho \cos\omega - \sum_{k=1}^{N} \cos\omega_k\right) \delta\left(N\rho \sin\omega - \sum_{k=1}^{N} \sin\omega_k\right)$$

Integrating over ω_k one gets

$$M(\rho,\omega) \;=\; N^2 \sum_{n=-\infty}^{\infty} e^{inN\omega}\, \Phi_n(\rho)$$

$$\Phi_n(\rho) \;=\; \int_0^1 r J_n(Nnr) \mathrm{det} J_{n+i-j}(r) dr$$

For $U(1)$ $\Phi(\rho) = \delta(1-\rho^2)$

For $U(2)$ $\Phi(\rho) = \frac{4}{\pi\rho}\sqrt{1-\rho^2}$

So the partition function for $U(N)$ can be written as

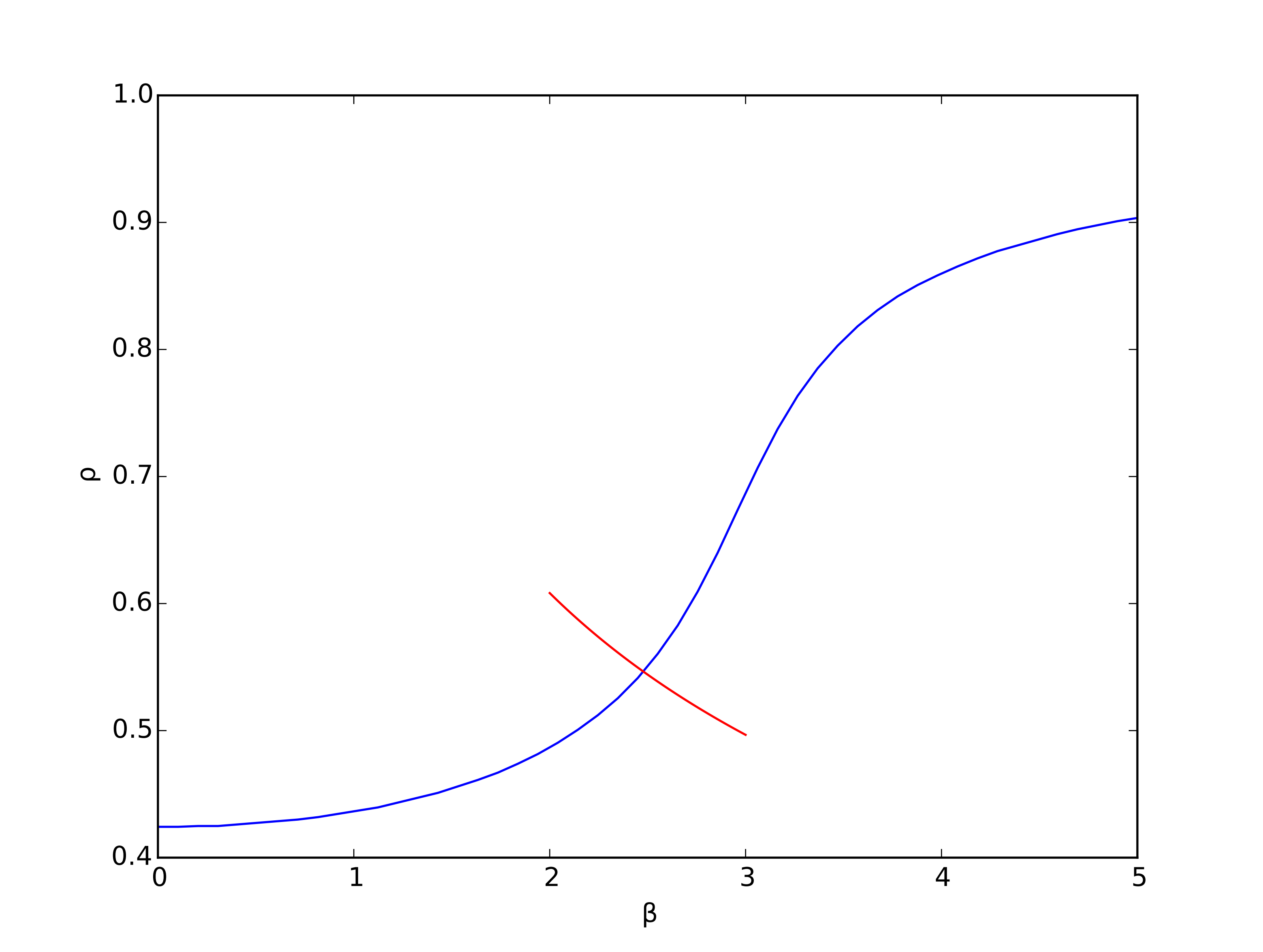
$$Z_{U(N)}(\beta) = \prod_x \int_0^1 \rho_x d\rho_x \int_0^{2\pi} d\omega_x \Phi(\rho_x) \\ \exp \sum_{x,n} \left(\beta \rho_x \rho_{x+e_n} \cos\left(\omega_x - \omega_{x+e_n}\right)\right)$$

Dual formulation

$$Z_{U(N)}(\beta) = \prod_x \int_0^1 \rho_x d\rho_x \Phi(\rho_x) \sum_{r_x} \prod_{x,n} I_{r^*_x - r^*_{x+e_n}}(\beta \rho_x \rho_{x+e_n})$$

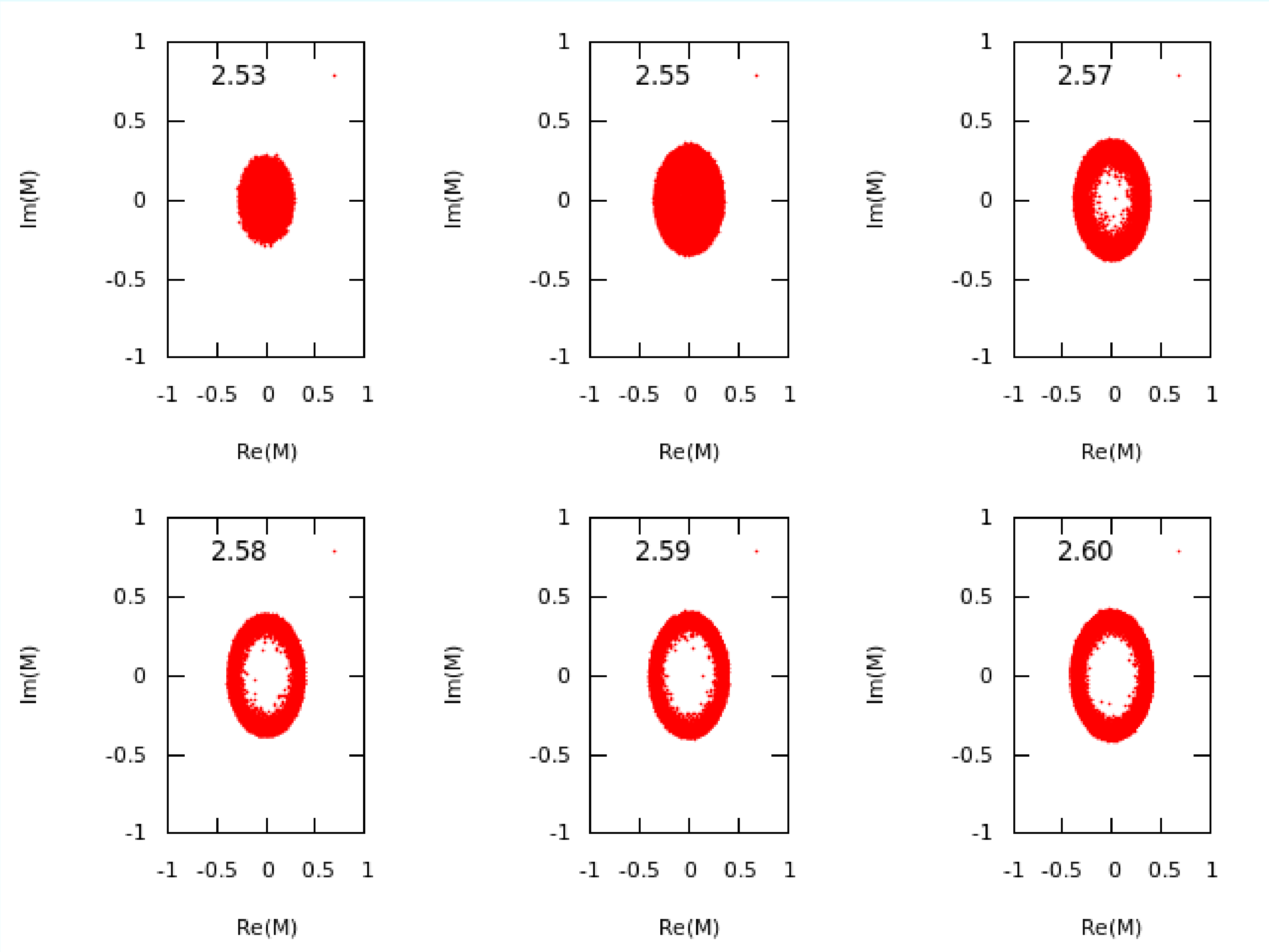
Now we can apply mean field approach for ρ variables.

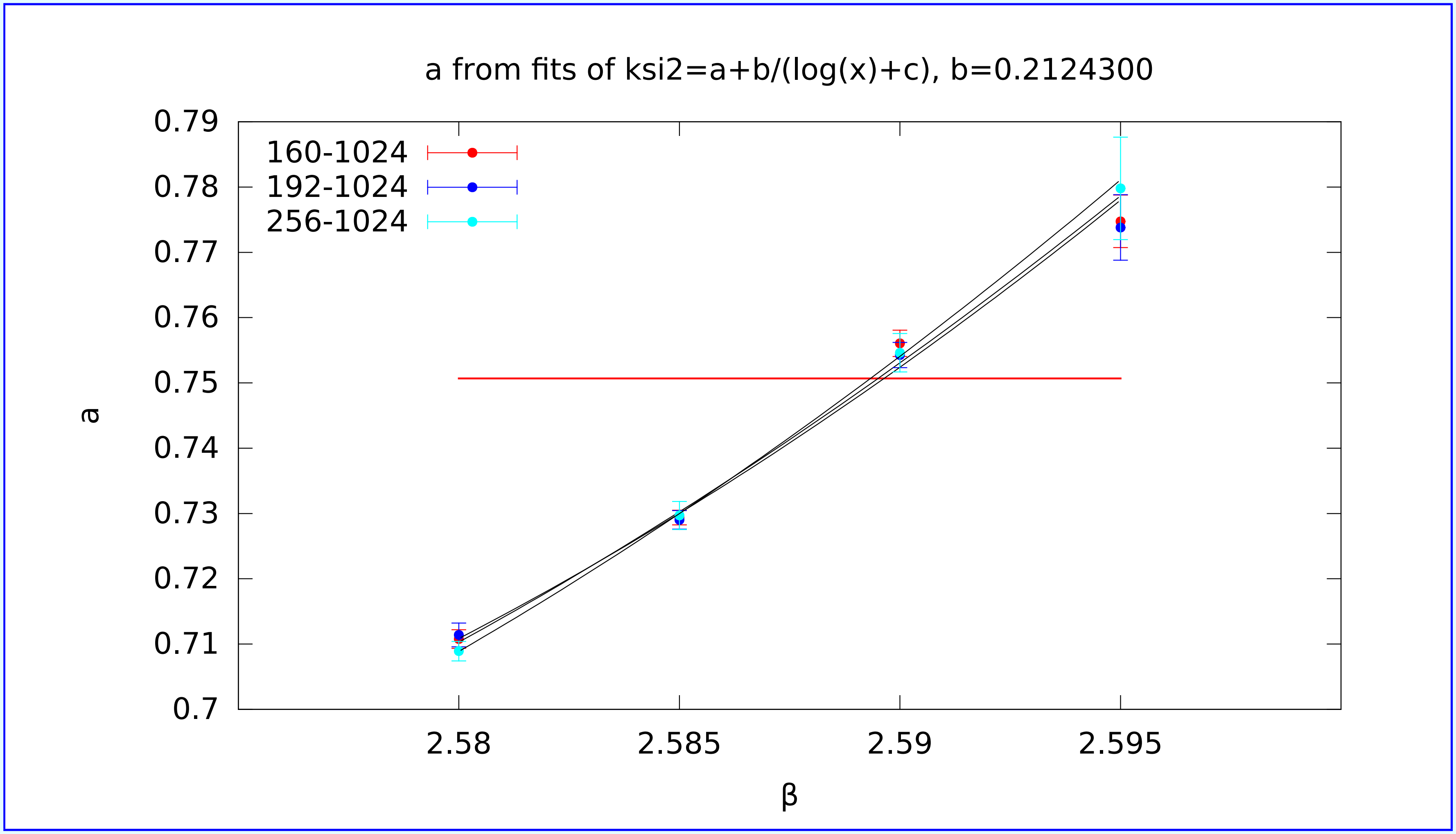
$$
\begin{aligned}
Z_{U(N)}(\beta) &= Z_{U(1)}\left(\beta u^2\right) Z_{mf}\,(\beta, u) \\
u &= \frac{\int_0^1 \rho^2\, \Phi(\rho)\, e^{4\beta u\rho}\, d\rho}{\int_0^1 \rho\, \Phi(\rho)\, e^{4\beta u\rho}\, d\rho}
\end{aligned}
$$



Mean-field ρ dependance on β for $U(2)$.

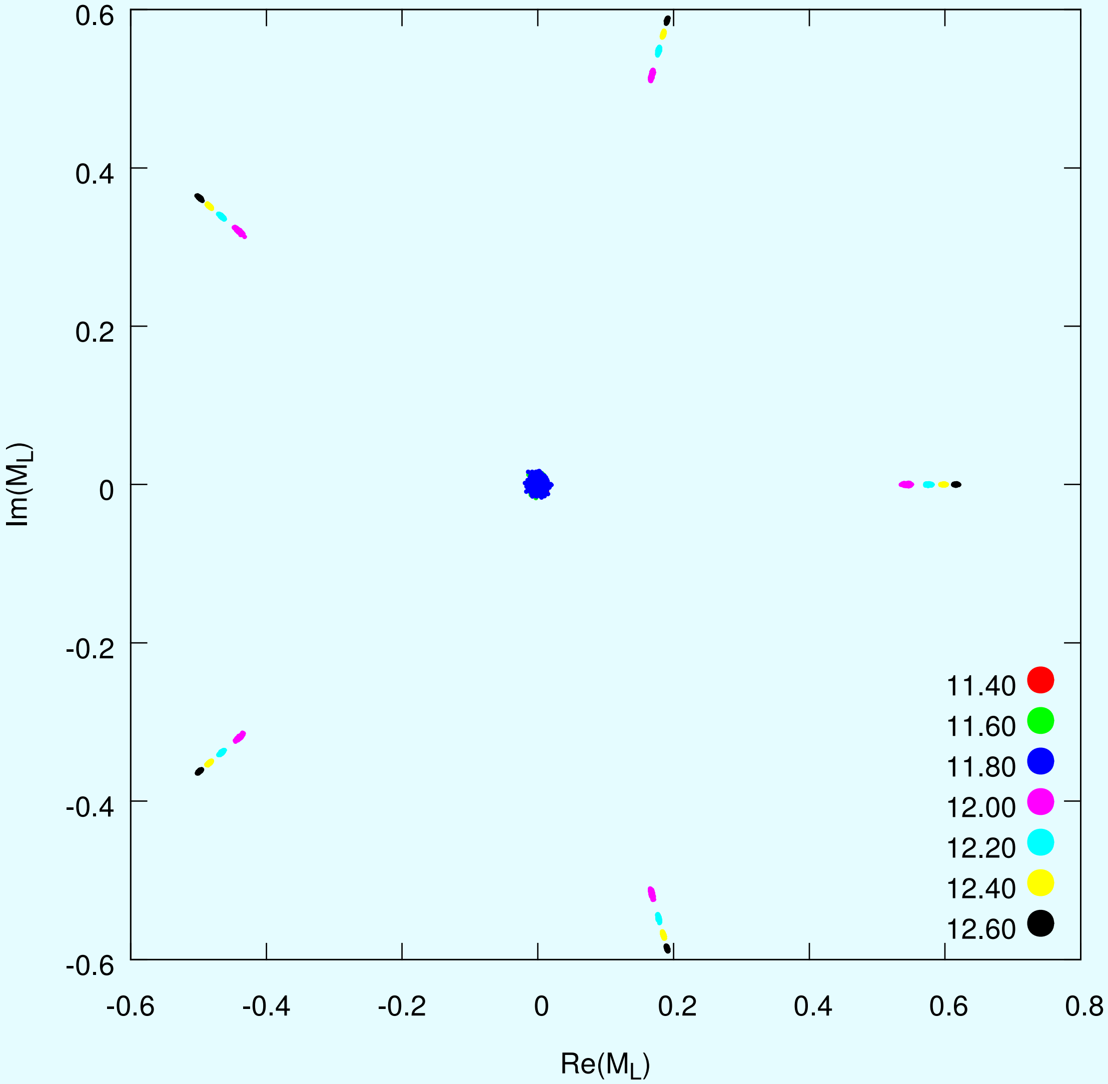
III. Numeric simulations of U(2) model.



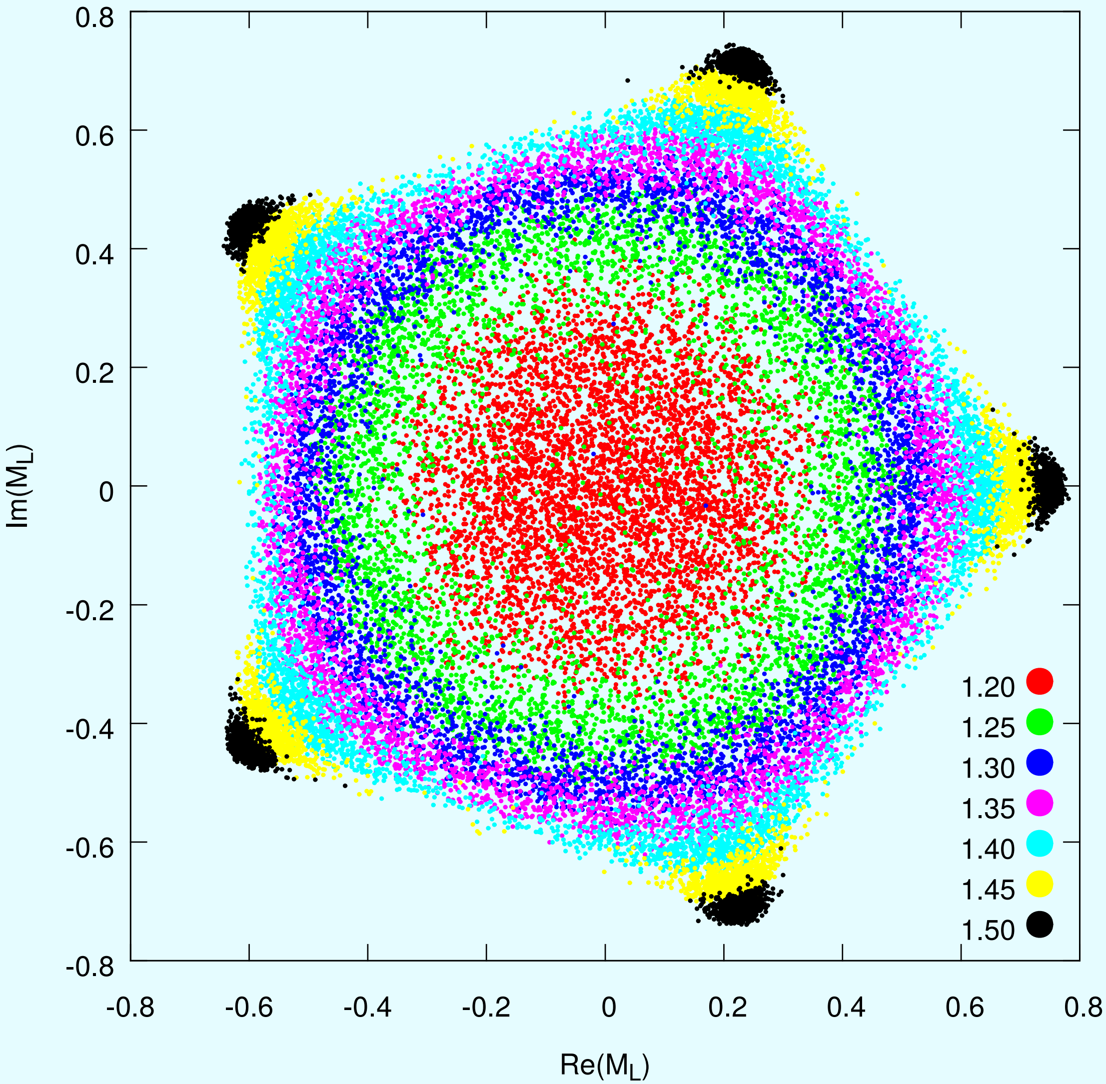


a from fits of ksi2=a+b/(log(x)+c), b=0.2124300
0.79
0.78
0.77
0.76
0.75
0.74
0.73
0.72
0.71
0.7
a
160-1024
192-1024
256-1024
2.58
2.585
2.59
2.595
β

IV. Numeric simulations of SU(N) model.

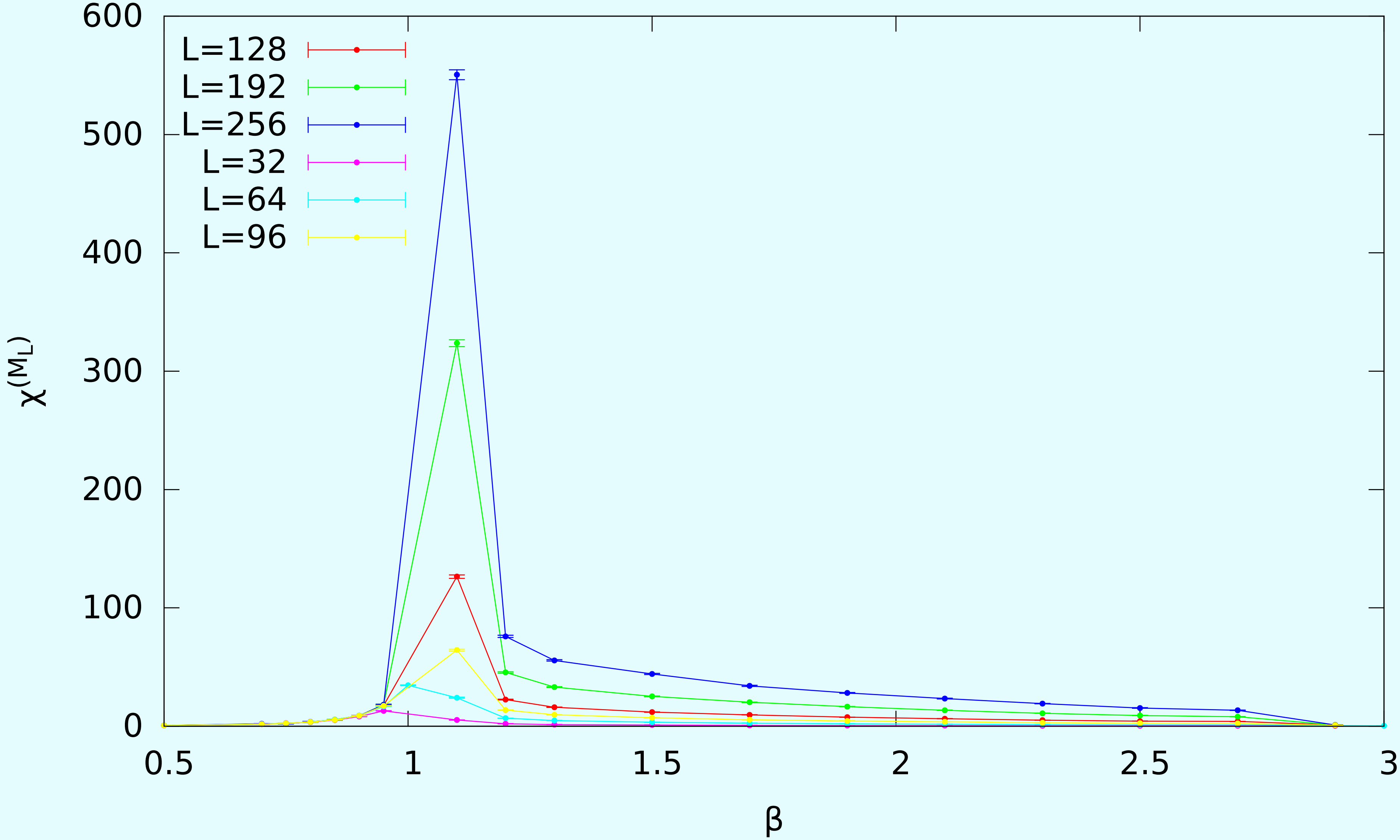


$$\lambda = 0.$$

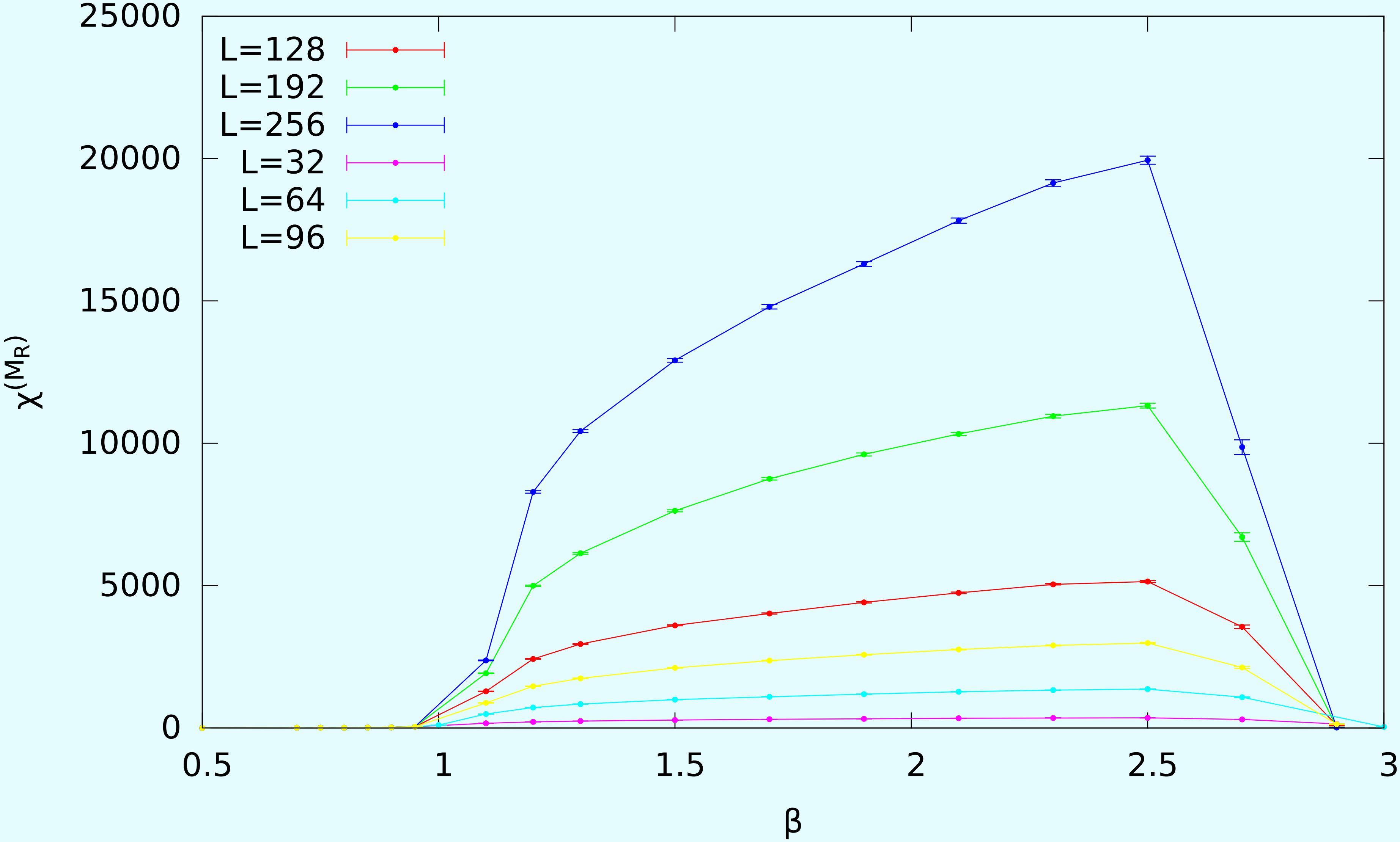


0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
Im(M_L)
Re(M_L)
1.20
1.25
1.30
1.35
1.40
1.45
1.50

$\lambda = 2.0$



Magnetization susceptibility in $SU(8)$ model, $\lambda = 2.0$



Rotated magnetization susceptibility in $SU(8)$ model, $\lambda = 2.0$

Results

Infinite order phase transition appears in some non-abelian two dimensional spin theories:

- Numeric and analytical proof of existence of a phase transition of the universality class of 2d XY model in $U(2)$ model.

- Numeric simulations show that in $SU(N)$ models with adjoint action term two infinite-order transitions appear when λ is large enough.